On generalized plastic structures

Adara M. Blaga and Antonella Nannicini

Abstract

We introduce the concept of generalized almost plastic structure, and, on a pseudo-Riemannian manifold endowed with two $(1, 1)$ -tensor fields satisfying some compatibility conditions, we construct a family of generalized almost plastic structures and characterize their integrability with respect to a given affine connection on the manifold.

1 Introduction

Let M be a smooth manifold and let E be a fiber bundle over M . Let us denote by $\Gamma^{\infty}(E)$ the set of smooth sections of E. A polynomial structure on E is a morphism $J : \Gamma^{\infty}(E) \to \Gamma^{\infty}(E)$ such that there exists a polynomial P with $P(J) = 0$. Almost complex structures, almost product structures, and metallic structures are examples of polynomial structures. A morphism J such that $J^3 - pJ - qI = 0$, where p, q are positive integers and I is the identity operator on $\Gamma^{\infty}(E)$, is called an *almost nylon structure*, and, if $p = q = 1$, then, it is called an *almost plastic structure*. The name plastic goes back to Hans van der Laan (1924) and G. Cordonnier (1928) who discovered architectural proportions based on the unique real solution of the cubic equation $x^3 - x - 1 = 0$, given by $\rho = \sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}}$ and called it, the *plastic number*, [\[6\]](#page-12-0).

The present paper aims to focus on almost plastic structures on the generalized tangent bundle $E = TM \oplus T^*M$. In this spirit, we will define a *generalized almost plastic structure* on M as being an almost plastic structure on E . Given a pseudo-Riemannian metric on

²⁰¹⁰ Mathematics Subject Classification. 53C15; 53B05; 53C50

Key words and phrases. generalized structure; plastic structure.

M, we construct a family of generalized almost plastic structures induced by two $(1, 1)$ tensor fields on M satisfying some compatibility conditions, and we characterize their integrability with respect to a given affine connection on M . On the other hand, we define a generalized almost plastic structure by means of a $(1, 1)$ -tensor field on M satisfying the cubic equation $x^3 - x + 1 = 0$ and we establish sufficient integrability conditions in terms of quasi-statistical structures. We remark that this construction gives rise to a duality between the two equations $x^3 - x - 1 = 0$ and $x^3 - x + 1 = 0$ in terms of associated generalized structures.

2 Preliminaries

2.1 Plastic matrices

DEFINITION 2.1. Let $\mathbb{R}(n)$ be the set of real square matrices of order n and let $A \in \mathbb{R}(n)$. Then, A is called a *plastic matrix* if A satisfies the equation:

$$
A^3 - A - I = 0,
$$

where I is the identity matrix.

We determine the expressions of the plastic matrices of order 2, as follows.

PROPOSITION 2.2. Let $A \in \mathbb{R}(2)$ be a plastic matrix. Then, A has one of the following forms:

$$
(1) \t\t A = \rho I,
$$

where ρ is the plastic number and I is the identity matrix, or

(2)
$$
A = \begin{pmatrix} a_{11} & \frac{1 - a_{11}a_{22} - a_{11}^2 - a_{22}^2}{a_{21}} \\ a_{21} & a_{22} \end{pmatrix}
$$

with $a_{11}, a_{22} \in \mathbb{R}$, $a_{21} \in \mathbb{R} \setminus \{0\}$ and $(trA)^3 - trA + 1 = 0$, where $trA := a_{11} + a_{22}$.

Proof. Let

$$
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.
$$

We have:

$$
A^{3} = \begin{pmatrix} a_{11}^{3} + 2a_{11}a_{12}a_{21} + a_{12}a_{21}a_{22} & a_{11}^{2}a_{12} + a_{12}^{2}a_{21} + a_{11}a_{12}a_{22} + a_{12}a_{22}^{2} \\ a_{11}^{2}a_{21} + a_{22}^{2}a_{21} + a_{11}a_{21}a_{22} + a_{12}a_{21}^{2} & a_{22}^{3} + 2a_{22}a_{12}a_{21} + a_{12}a_{21}a_{11} \end{pmatrix}
$$

Then, A is a plastic matrix if and only if the following system is satisfied:

$$
\begin{cases}\na_{11}^3 + 2a_{11}a_{12}a_{21} + a_{12}a_{21}a_{22} = a_{11} + 1 \\
a_{12}(a_{11}^2 + a_{12}a_{21} + a_{11}a_{22} + a_{22}^2 - 1) = 0 \\
a_{21}(a_{11}^2 + a_{12}a_{21} + a_{11}a_{22} + a_{22}^2 - 1) = 0 \\
a_{22}^3 + 2a_{22}a_{12}a_{21} + a_{12}a_{21}a_{11} = a_{22} + 1\n\end{cases}
$$

.

.

If we suppose $a_{21} = 0$ (respectively, $a_{12} = 0$), then we get that $a_{11} = a_{22}$ is the plastic number and furthermore, also that $a_{12} = 0$ (respectively, also that $a_{21} = 0$), hence, we have (1) .

Now, if we suppose $a_{12}a_{21} \neq 0$, then we get the following conditions:

$$
\begin{cases}\na_{11}^3 + 2a_{11}a_{12}a_{21} + a_{12}a_{21}a_{22} = a_{11} + 1 \\
a_{11}^2 + a_{12}a_{21} + a_{11}a_{22} + a_{22}^2 = 1 \\
a_{22}^3 + 2a_{22}a_{12}a_{21} + a_{12}a_{21}a_{11} = a_{22} + 1\n\end{cases}
$$

In particular, we get:

$$
\begin{cases}\n a_{12}a_{21} = 1 - a_{11}a_{22} - a_{11}^2 - a_{22}^2 \\
 (a_{11} + a_{22})^3 - (a_{11} + a_{22}) + 1 = 0\n\end{cases}
$$

hence, we have [\(2\)](#page-1-1), and the proof is complete.

COROLLARY 2.3. Let $A \in \mathbb{R}(2)$ be a plastic matrix. Then, $A = \rho I$, where ρ is the plastic number and I is the identity matrix, or there exists an invertible matrix $C \in \mathbb{R}(2)$, such that $A = C^{-1}BC$, where

(3)
$$
B = \begin{pmatrix} \alpha & 1 - \alpha^2 \\ 1 & 0 \end{pmatrix},
$$

for α the unique real solution of the equation $x^3 - x + 1 = 0$.

PROOF. Let $A \in \mathbb{R}(2)$ be a plastic matrix. If A has a real eigenvalue, then, it has multiplicity two and $A = \rho I$. If A has not real eigenvalues, then, it has two complex

.

conjugate eigenvalues and, as a consequence of the previous proposition, their sum is the real solution, α , of the equation $x^3 - x + 1 = 0$. In particular, if

$$
A = \begin{pmatrix} a_{11} & \frac{1 - a_{11}a_{22} - a_{11}^2 - a_{22}^2}{a_{21}} \\ a_{21} & a_{22} \end{pmatrix},
$$

then we can choose

$$
C = \begin{pmatrix} a_{21} & a_{22} \\ 0 & 1 \end{pmatrix}
$$

and, as $\alpha = a_{11} + a_{22}$, we get the statement.

2.2 Almost plastic structures

DEFINITION 2.4. Let M be a smooth manifold. An *almost plastic structure* on M is a tensor field J , of type $(1, 1)$, on M , which satisfies the equation:

$$
J^3 - J - I = 0.
$$

If J is an almost plastic structure on M, then, (M, J) is called an *almost plastic manifold*.

DEFINITION 2.5. Let (M, g) be a pseudo-Riemannian manifold. An *almost plastic* pseudo-Riemannian structure on (M, g) is an almost plastic structure J on M such that $g(JX,Y) = g(X,JY)$ for all $X,Y \in \Gamma^\infty(TM)$. If J is an almost plastic pseudo-Riemannian structure on (M, g) , then, (M, g, J) is called an *almost plastic pseudo-Rieman*nian manifold.

Let $N(J)$ be the Nijenhuis tensor field of J, defined as:

$$
N(J)(X,Y) := [JX, JY] - J[JX, Y] - J[X, JY] + J^{2}[X, Y]
$$

for all $X, Y \in \Gamma^\infty(TM)$.

According to [\[8\]](#page-12-1), we say that an almost plastic structure J is *integrable* if $N(J) = 0$. We will call *plastic structure* an integrable almost plastic structure. An almost plastic manifold (M, J) with J integrable will be called a *plastic manifold*.

EXAMPLE 2.6. Let $\rho = \sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}}$ be the plastic number. Then, $J = \rho I$ is a pseudo-Riemannian plastic structure on any pseudo-Riemannian manifold (M, g) which is called the trivial pseudo-Riemannian plastic structure.

REMARK 2.7. We remark that the concept of "almost plastic Riemannian structure" has no sense because, excepting the trivial case $J = \rho I$, a plastic structure is not real diagonalizable and then it cannot be symmetric with respect to a positive definite scalar product.

REMARK 2.8. We remark that an almost plastic structure J is invertible and the inverse tensor field is given by $J^{-1} = J^2 - I$. Indeed, $J(J^2 - I) = J^3 - J = I$. However, J^{-1} is not an almost plastic structure because $(J^{-1})^3 - J^{-1} - I = -J$.

REMARK 2.9. A *metallic structure* on M, [\[4\]](#page-12-2), is a tensor field J, of type $(1, 1)$, on M, which satisfies the equation:

$$
J^2 - pJ - qI = 0,
$$

where p, q are positive integers. We remark that, if J is a metallic structure, then, J is an almost plastic structure if and only if

$$
\begin{cases} p^3 - p + 1 = 0 \\ q = 1 - p^2 \end{cases}
$$
 or
$$
\begin{cases} q \neq 1 - p^2 \\ J = \frac{1 - pq}{p^2 + q - 1} I \end{cases}
$$

respectively, J satisfies a **dual** equation, $J^3 - J + I = 0$, if and only if

$$
\begin{cases}\n p = \rho \text{ is the plastic number} \\
 q = 1 - \rho^2\n\end{cases}\n\quad \text{or} \quad\n\begin{cases}\n q \neq 1 - p^2 \\
 J = -\frac{1 + pq}{p^2 + q - 1}I\n\end{cases}.
$$

3 Plastic structures in generalized geometry

3.1 Geometrical properties of the generalized tangent bundle

Let $TM \oplus T^*M$ be the generalized tangent bundle of M. On $TM \oplus T^*M$, we consider the natural indefinite metric

$$
\langle X + \eta, Y + \beta \rangle := -\frac{1}{2}(\eta(Y) + \beta(X))
$$

and the natural symplectic structure

$$
(X + \eta, Y + \beta) := -\frac{1}{2}(\eta(Y) - \beta(X))
$$

for all $X, Y \in \Gamma^{\infty}(TM)$ and $\eta, \beta \in \Gamma^{\infty}(T^{*}M)$.

If g is a non-degenerate $(0, 2)$ -tensor field on M, we will denote by g the flat isomorphism $\not g: \Gamma^\infty(TM) \to \Gamma^\infty(T^*M)$, $\not g(X) := i_X g$, and by g^{-1} its inverse, and we define the bilinear form, \check{g} , on $TM \oplus T^*M$ by:

$$
\check{g}(X + \eta, Y + \beta) := g(X, Y) + g(g^{-1}(\eta), g^{-1}(\beta))
$$

for all $X, Y \in \Gamma^{\infty}(TM)$ and $\eta, \beta \in \Gamma^{\infty}(T^{*}M)$.

Furthermore, given an affine connection ∇ on M, we define the affine connections $\hat{\nabla}$ and $\check{\nabla}$ on $TM \oplus T^*M$ by:

$$
\hat{\nabla}_{X+\eta}(Y+\beta) := \nabla_X Y + g(\nabla_X(g^{-1}(\beta)))
$$

and

$$
\check{\nabla}_{X+\eta}(Y+\beta) := \nabla_X Y + \nabla_X \beta
$$

for all $X, Y \in \Gamma^\infty(TM)$ and $\eta, \beta \in \Gamma^\infty(T^*M)$. We remark that $\hat{\nabla} = \check{\nabla}$ if and only if $\nabla g = 0.$

Finally, we define the bracket $[\cdot, \cdot]_{\nabla}$:

$$
[X + \eta, Y + \beta]_{\nabla} := [X, Y] + \nabla_X \beta - \nabla_Y \eta
$$

for all $X, Y \in \Gamma^{\infty}(TM)$ and $\eta, \beta \in \Gamma^{\infty}(T^{*}M)$.

3.2 Generalized almost plastic structures

Any morphism \hat{J} : $\Gamma^{\infty}(TM \oplus T^*M) \to \Gamma^{\infty}(TM \oplus T^*M)$ will be called a *generalized* structure on M.

DEFINITION 3.1. Let \hat{J} : $\Gamma^{\infty}(TM \oplus T^{*}M) \rightarrow \Gamma^{\infty}(TM \oplus T^{*}M)$ be a generalized structure on M. If \hat{J} satisfies the equation

$$
\hat{J}^3 - \hat{J} - I = 0,
$$

then, \hat{J} is called a *generalized almost plastic structure on* M .

EXAMPLE 3.2. Let M be a smooth manifold and let J_1, J_2 be almost plastic structures on M. Then, the generalized structure

(4)
$$
\hat{J} := \begin{pmatrix} J_1 & 0 \\ 0 & J_2^* \end{pmatrix},
$$

where $J_2^*: \Gamma^\infty(T^*M) \to \Gamma^\infty(T^*M)$ is defined by $J_2^*(\eta)(X) := \eta(J_2X)$ for all $\eta \in \Gamma^\infty(T^*M)$ and $X \in \Gamma^\infty(TM)$, is a generalized almost plastic structure on M. For $J_1 = J_2 = J$, we will denote the induced generalized almost plastic structure by:

(5)
$$
\hat{J} := \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix}.
$$

PROPOSITION 3.3. Let \hat{J} be the generalized almost plastic structure defined by [\(5\)](#page-6-0). Then,

$$
<\hat{J}(X+\eta), Y+\beta>=
$$
;

moreover, if g is a pseudo-Riemannian metric and $(M, g, J_1), (M, g, J_2)$ are almost plastic pseudo-Riemannian manifolds, then, the structure \hat{J} defined by [\(4\)](#page-5-0) satisfies:

$$
\check{g}(\hat{J}(X+\eta), Y+\beta) = \check{g}(X+\eta, \hat{J}(Y+\beta))
$$

for all $X, Y \in \Gamma^{\infty}(TM)$ and $\eta, \beta \in \Gamma^{\infty}(T^{*}M)$.

PROOF. We have:

$$
<\hat{J}(X+\eta), Y+\beta> = = -\frac{1}{2}(\eta(JY)+\beta(JX));
$$

on the other hand,

$$
\langle X + \eta, \hat{J}(Y + \beta) \rangle = \langle X + \eta, JY + J^*(\beta) \rangle = -\frac{1}{2}(\eta(JY) + \beta(JX)),
$$

and the first statement is proved. Moreover, \hat{J} defined by [\(4\)](#page-5-0) satisfies:

$$
\check{g}(\hat{J}(X+\eta), Y+\beta) = \check{g}(J_1X + J_2^*(\eta), Y+\beta) = g(J_1X, Y) + g(g^{-1}(J_2^*(\eta)), g^{-1}(\beta))
$$

\n
$$
= g(X, J_1Y) + g(J_2(g^{-1}(\eta)), g^{-1}(\beta))
$$

\n
$$
= g(X, J_1Y) + g(g^{-1}(\eta), J_2(g^{-1}(\beta)))
$$

\n
$$
= g(X, J_1Y) + g(g^{-1}(\eta), g^{-1}(J_2^*(\beta)))
$$

\n
$$
= \check{g}(X+\eta, \hat{J}(Y+\beta)),
$$

and the proof is complete.

PROPOSITION 3.4. Let \hat{J} be the generalized almost plastic structure defined by [\(4\)](#page-5-0). Then, $\hat{\nabla} \hat{J} = 0$ if and only if

$$
\begin{cases} \nabla J_1 = 0 \\ \nabla J_2 = 0 \end{cases}.
$$

Moreover, $\check{\nabla} \hat{J} = 0$ if and only if

$$
\begin{cases} \nabla J_1 = 0 \\ \nabla J_2 = 0 \end{cases}.
$$

PROOF. A direct computation gives:

$$
(\hat{\nabla}\hat{J})_{X+\eta}(Y+\beta) = (\nabla_X J_1)Y + g((\nabla_X J_2)(g^{-1}(\beta)))
$$

for all $X, Y \in \Gamma^\infty(TM)$ and $\eta, \beta \in \Gamma^\infty(T^*M)$, and the first statement is proved. Moreover, a direct computation gives:

$$
(\check{\nabla}\hat{J})_{X+\eta}(Y+\beta) = (\nabla_X J_1)Y + (\nabla_X J_2^*)\beta
$$

for all $X, Y \in \Gamma^{\infty}(TM)$ and $\eta, \beta \in \Gamma^{\infty}(T^{*}M)$, and the proof is complete.

3.3 Generalized almost plastic structures defined by two $(1, 1)$ tensor fields

Taking inspiration from the form of order two plastic matrices, we now construct a family of generalized almost plastic structures on a pseudo-Riemannian manifold.

PROPOSITION 3.5. Let (M, g) be a pseudo-Riemannian manifold and let J_1 , J_2 be two $(1, 1)$ -tensor fields on M satisfying the following conditions: (i) $J_1J_2 = J_2J_1$ (ii) $(J_1 + J_2)^3 - (J_1 + J_2) + I = 0$ (iii) $g(J_1X, Y) = g(X, J_1Y)$, for all $X, Y \in \Gamma^\infty(TM)$ (iv) $g(J_2X, Y) = g(X, J_2Y)$, for all $X, Y \in \Gamma^\infty(TM)$. Then,

(6)
$$
\hat{J} := \begin{pmatrix} J_1 & (I - J_1 J_2 - J_1^2 - J_2^2)g^{-1} \\ g & J_2^* \end{pmatrix}
$$

is a generalized almost plastic structure on M.

PROOF. From

$$
\hat{J}(X+\eta) = (J_1X + (I-J_1J_2 - J_1^2 - J_2^2)g^{-1}(\eta)) + (g(X) + J_2^*(\eta)),
$$

by using the properties (i) , (iii) , (iv) , we get:

$$
\hat{J}^{2}(X+\eta) = (J_{1}^{2}X + (I - J_{1}J_{2} - J_{1}^{2} - J_{2}^{2})J_{1}g^{-1}(\eta) + (I - J_{1}J_{2} - J_{1}^{2} - J_{2}^{2})(X + J_{2}g^{-1}(\eta)))
$$

+
$$
(g(J_{1}X) + (I - J_{1}J_{2} - J_{1}^{2} - J_{2}^{2})^{*}(\eta) + g(J_{2}X) + (J_{2}^{*})^{2}(\eta))
$$

=
$$
((I - J_{1}J_{2} - J_{2}^{2})X + (J_{1} + J_{2} - 2J_{1}^{2}J_{2} - 2J_{2}^{2}J_{1} - J_{1}^{3} - J_{2}^{3})g^{-1}(\eta))
$$

+
$$
(g(J_{1}X) + (I - J_{1}J_{2} - J_{1}^{2})^{*}(\eta) + g(J_{2}X));
$$

moreover,

$$
\hat{J}^3(X + \eta) = ((J_1 + (J_1 + J_2) - (J_1 + J_2)^3)X + (I - J_1J_2 - J_1^2 - J_2^2)g^{-1}(\eta))
$$

+
$$
(g(X) + (J_2^* + (J_1^* + J_2^*) - (J_1^* + J_2^*)^3(\eta)),
$$

and the proof is complete.

REMARK 3.6. From the proof of the previous proposition, we get that, if (J_1+J_2) is an almost plastic structure on M, then, the generalized structure \hat{J} defined by [\(6\)](#page-7-0) satisfies the condition $\hat{J}^3 - \hat{J} + I = 0$. Therefore, this construction of a generalized structure defines a **duality** between the two equations $x^3 - x - 1 = 0$ and $x^3 - x + 1 = 0$.

PROPOSITION 3.7. Let \hat{J} be the generalized almost plastic structure defined by [\(6\)](#page-7-0). Then, $\hat{\nabla} \hat{J} = 0$ if and only if

$$
\begin{cases} \nabla J_1 = 0 \\ \nabla J_2 = 0 \end{cases}.
$$

Moreover, $\check{\nabla} \hat{J} = 0$ if and only if

$$
\begin{cases} \nabla J_1 = 0 \\ \nabla J_2 = 0 \\ \nabla g = 0 \end{cases}
$$

.

PROOF. A direct computation gives:

$$
(\hat{\nabla}\hat{J})_{X+\eta}(Y+\beta) = (\nabla_X J_1)Y + (\nabla_X (I - J_1 J_2 - J_1^2 - J_2^2))(g^{-1}(\beta)) + g((\nabla_X J_2)(g^{-1}(\beta)))
$$

for all $X, Y \in \Gamma^\infty(TM)$ and $\eta, \beta \in \Gamma^\infty(T^*M)$, and the first statement is proved. Moreover, a direct computation gives:

$$
(\check{\nabla}\hat{J})_{X+\eta}(Y+\beta) = (\nabla_X J_1)Y + (\nabla_X (I - J_1 J_2 - J_1^2 - J_2^2)g^{-1})(\beta) + (\nabla_X g)Y + (\nabla_X J_2^*)\beta
$$

for all $X, Y \in \Gamma^{\infty}(TM)$ and $\eta, \beta \in \Gamma^{\infty}(T^{*}M)$, and the proof is complete.

 \Box

4 Integrability of some generalized almost plastic structures

Let M be a smooth manifold and let ∇ be an affine connection on M .

DEFINITION 4.1. A generalized structure \hat{J} on M is called ∇ -integrable if its Nijenhuis tensor field $N^{\nabla}(\hat{J})$ with respect to ∇ :

$$
N^{\nabla}(\hat{J})(\sigma,\tau) := [\hat{J}\sigma,\hat{J}\tau]_{\nabla} - \hat{J}[\hat{J}\sigma,\tau]_{\nabla} - \hat{J}[\sigma,\hat{J}\tau]_{\nabla} + \hat{J}^2[\sigma,\tau]_{\nabla}
$$

vanishes for all $\sigma, \tau \in \Gamma^{\infty}(TM \oplus T^{*}M)$.

We shall study the integrability of some generalized almost plastic structures, characterizing it also in terms of quasi-statistical structures. We recall the following definition.

DEFINITION 4.2. [\[5,](#page-12-3) [7\]](#page-12-4) Let g be a non-degenerate $(0, 2)$ -tensor field on M and let ∇ be an affine connection on M with torsion operator T^{∇} . Then, (g, ∇) is called a quasi-statistical structure on M if

$$
(\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) + g(T^{\nabla}(X,Y),Z) = 0
$$

for all $X, Y, Z \in \Gamma^\infty(TM)$, where T^{∇} is the torsion of the given connection ∇ . If (g, ∇) is a quasi-statistical structure on M, then, (M, g, ∇) is called a *quasi-statistical manifold*.

For generalized quasi-statistical structures, see [\[1,](#page-11-0) [2,](#page-12-5) [3\]](#page-12-6).

4.1 Integrability of generalized almost plastic structures induced by two almost plastic structures on M

PROPOSITION 4.3. Let M be a smooth manifold, let ∇ be an affine connection on M, and let J_1, J_2 be two almost plastic structures on M. Then, the generalized almost plastic structure defined by

$$
\hat{J} := \begin{pmatrix} J_1 & 0 \\ 0 & J_2^* \end{pmatrix},
$$

is ∇ -integrable if and only if the following conditions are satisfied:

(7)
$$
\begin{cases} N(J_1) = 0 \\ \nabla_{J_1 X} J_2 = J_2(\nabla_X J_2) \end{cases}
$$

for all $X \in \Gamma^\infty(TM)$.

PROOF. A direct computation gives:

$$
N^{\nabla}(\hat{J})(X + \eta, Y + \beta) = N(J_1)(X, Y)
$$

+
$$
((\nabla_{J_1X}J_2^*) - J_2^*(\nabla_XJ_2^*))(\beta) - ((\nabla_{J_1Y}J_2^*) - J_2^*(\nabla_YJ_2^*))(\eta)
$$

for all $X, Y \in \Gamma^{\infty}(TM)$ and $\eta, \beta \in \Gamma^{\infty}(T^{*}M)$. Moreover, for any $X, Y \in \Gamma^{\infty}(TM)$ and for any $\eta \in \Gamma^\infty(T^*M)$:

$$
((\nabla_{J_1X} J_2^*)(\eta))(Y) = \eta((\nabla_{J_1X} J_2)Y)
$$

and

$$
((J_2^*(\nabla_X J_2^*))(\eta))(Y) = \eta(J_2(\nabla_X J_2)Y),
$$

and the proof is complete.

REMARK 4.4. We remark that if $J_1 = J_2 = J$, then [\(7\)](#page-9-0) becomes:

$$
\begin{cases}\nT^{\nabla}(JX,JY) - JT^{\nabla}(JX,Y) - JT^{\nabla}(X,JY) + J^2T^{\nabla}(X,Y) = 0 \\
\nabla_{JX}J = J(\nabla_X J)\n\end{cases}
$$

for all $X, Y \in \Gamma^\infty(TM)$. In particular, for a torsion-free affine connection, the integrability condition becomes:

$$
\nabla_{JX}J=J(\nabla_XJ)
$$

for all $X \in \Gamma^\infty(TM)$.

4.2 Integrability of generalized almost plastic structures induced by a polynomial structure J on M such that $J^3 - J + I = 0$

Taking inspiration from [\(3\)](#page-2-0), we consider the generalized almost plastic structure defined by:

(8)
$$
\hat{J} := \begin{pmatrix} J & (I - J^2)g^{-1} \\ g & 0 \end{pmatrix},
$$

where J is a polynomial structure on M such that $J^3 - J + I = 0$.

PROPOSITION 4.5. Let g be a pseudo-Riemannian metric and let ∇ be an affine connection on M. If J is integrable, $\nabla J = 0$, and (M, g, ∇) is a quasi-statistical manifold, then \hat{J} given by [\(8\)](#page-10-0) is ∇ -integrable.

PROOF. A direct computation gives:

$$
N^{\nabla}(\hat{J})(X,Y) = N(J)(X,Y) + (I - J^2)g^{-1}((\nabla_Y g)X - (\nabla_X g)Y + g(T^{\nabla}(Y,X)))
$$

+
$$
(\nabla_{JX} g)Y - (\nabla_Y g)JX + g(T^{\nabla}(JX,Y))
$$

+
$$
(\nabla_X g)JY - (\nabla_{JY} g)X + g(T^{\nabla}(X,JY))
$$

+
$$
(\nabla_Y g)X - (\nabla_X g)Y + g(T^{\nabla}(Y,X)) + (\nabla_Y J^*)g(X) - (\nabla_X J^*)g(Y)
$$

+
$$
J^*((\nabla_Y g)X - (\nabla_X g)Y + g(T^{\nabla}(Y,X)))
$$

for all $X, Y \in \Gamma^\infty(TM)$;

$$
N^{\nabla}(\hat{J})(X,g(Z)) = -(\nabla_{JX}J^{2})Z - (\nabla_{(I-J^{2})Z}J)X + J(\nabla_{X}J^{2})Z
$$

\n
$$
-T^{\nabla}(JX,(I-J^{2})Z) + JT^{\nabla}(X,(I-J^{2})Z)
$$

\n
$$
-(I-J^{2})g^{-1}((\nabla_{JX}g)Z) + J(I-J^{2})(g^{-1}(\nabla_{X}g)Z)
$$

\n
$$
+(\nabla_{X}g)Z - (\nabla_{Z}g)X + g(T^{\nabla}(X,Z))
$$

\n
$$
- (J^{*})^{2}((\nabla_{X}g)Z) + (\nabla_{J^{2}Z}g)X - g(T^{\nabla}(X,J^{2}Z))
$$

\n
$$
= -g^{-1}((\nabla_{JX}g)(I-J^{2})Z)) - (\nabla_{(I-J^{2})Z}g)JX + g(T^{\nabla}(JX,(I-J^{2})Z)))
$$

\n
$$
+ Jg^{-1}((\nabla_{X}g)(I-J^{2})Z) - (\nabla_{(I-J^{2})Z}g)X + g(T^{\nabla}(X,(I-J^{2})Z)))
$$

\n
$$
+ (\nabla_{X}g)Z - (\nabla_{Z}g)X + g(T^{\nabla}(X,Z))
$$

\n
$$
- (\nabla_{X}g)J^{2}Z + (\nabla_{J^{2}Z}g)X - g(T^{\nabla}(X,J^{2}Z))
$$

for all $X, Z \in \Gamma^\infty(TM)$;

$$
N^{\nabla}(\hat{J})(g(Z), g(W)) = g^{-1}((\nabla_{W}g)Z - (\nabla_{Z}g)W + g(T^{\nabla}(W, Z)))
$$

+ $g^{-1}((\nabla_{J^{2}W}g)J^{2}Z - (\nabla_{J^{2}Z}g)J^{2}W + g(T^{\nabla}(J^{2}W, J^{2}Z)))$
+ $g^{-1}((\nabla_{Z}g)J^{2}W - (\nabla_{J^{2}W}g)Z + g(T^{\nabla}(Z, J^{2}W)))$
+ $g^{-1}((\nabla_{J^{2}Z}g)W - (\nabla_{W}g)J^{2}W + g(T^{\nabla}(J^{2}Z, W)))$

for all $Z, W \in \Gamma^{\infty}(TM)$, and we get the conclusion.

\Box

References

[1] A.M. Blaga, A. Nannicini, Statistical structures, α -connections and generalized geometry, Rivista di Matematica della Universita di Parma, 13(2), (2022); Proceedings of the meeting Cohomology of Complex Manifolds and Special Structures II, Levico Terme (Trento), July 5-9, 2021.

- [2] A.M. Blaga, A. Nannicini, α -connections in generalized geometry, J. Geom. Phys., 165, 104225 (2021). https://doi.org/10.1016/j.geomphys.2021.104225
- [3] A.M. Blaga, A. Nannicini, Generalized quasi-statistical structures, Bulletin of the Belgian Mathematical Society – Simon Stevin, 27(5), 731–754 (2020). https://doi.org/10.36045/j.bbms.191023
- [4] C.E. Hretcanu, M. Crasmareanu, Metallic structures on Riemannian manifolds, Rev. Un. Mat. Argentina 54(2), 15–27 (2013).
- [5] T. Kurose, Statistical Manifolds Admitting Torsion, Geometry and Something; Fukuoka Univ.: Fukuoka-shi (In Japanese) (2007).
- [6] L. Marohnić, T. Strmečki, Plastic number: construction and applications, Advanced research in scientific areas, international virtual conference, December, 3-7: 1523– 1528 (2012).
- [7] H. Matsuzoe, Quasi-statistical manifolds and geometry of affine distributions, Pure and Applied Differential Geometry 2012: In Memory of Franki Dillen, Berichte aus der Mathematik, ed. Joeri Van der Veken, Ignace Van de Woestyne, Leopold Verstraelen, Luc Vrancken, Shaker Verlag (2013).
- [8] Vanzura, J. Integrability conditions for polynomial structures. Kodai Math. Sem. Rep. 27 (1-2), 42-50 (1976)

Adara M. BLAGA,

Department of Mathematics, Faculty of Mathematics and Computer Science, West University of Timisoara, Bld. V. Pârvan 4, 300223, Timișoara, Romania, Email: adarablaga@yahoo.com

Antonella NANNICINI, Department of Mathematics and Informatics "U. Dini", University of Florence, Viale Morgagni 67/a, 50134, Firenze, Italy, Email: antonella.nannicini@unifi.it