On generalized plastic structures

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Abstract

We introduce the concept of generalized almost plastic structure, and, on a pseudo-Riemannian manifold endowed with two (1, 1)-tensor fields satisfying some compatibility conditions, we construct a family of generalized almost plastic structures and characterize their integrability with respect to a given affine connection on the manifold.

1 Introduction

Let M be a smooth manifold and let E be a fiber bundle over M. Let us denote by $\Gamma^{\infty}(E)$ the set of smooth sections of E. A polynomial structure on E is a morphism $J : \Gamma^{\infty}(E) \to \Gamma^{\infty}(E)$ such that there exists a polynomial P with P(J) = 0. Almost complex structures, almost product structures, and metallic structures are examples of polynomial structures. A morphism J such that $J^3 - pJ - qI = 0$, where p, q are positive integers and I is the identity operator on $\Gamma^{\infty}(E)$, is called an *almost nylon structure*, and, if p = q = 1, then, it is called an *almost plastic structure*. The name plastic goes back to Hans van der Laan (1924) and G. Cordonnier (1928) who discovered architectural proportions based on the unique real solution of the cubic equation $x^3 - x - 1 = 0$, given by $\rho = \sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}}$ and called it, the *plastic number*, [6].

The present paper aims to focus on almost plastic structures on the generalized tangent bundle $E = TM \oplus T^*M$. In this spirit, we will define a *generalized almost plastic structure* on M as being an almost plastic structure on E. Given a pseudo-Riemannian metric on

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M, we construct a family of generalized almost plastic structures induced by two (1, 1)tensor fields on M satisfying some compatibility conditions, and we characterize their
integrability with respect to a given affine connection on M. On the other hand, we define
a generalized almost plastic structure by means of a (1, 1)-tensor field on M satisfying the
cubic equation $x^3 - x + 1 = 0$ and we establish sufficient integrability conditions in terms
of quasi-statistical structures. We remark that this construction gives rise to a **duality**between the two equations $x^3 - x - 1 = 0$ and $x^3 - x + 1 = 0$ in terms of associated
generalized structures.

2 Preliminaries

2.1 Plastic matrices

DEFINITION 2.1. Let $\mathbb{R}(n)$ be the set of real square matrices of order n and let $A \in \mathbb{R}(n)$. Then, A is called a *plastic matrix* if A satisfies the equation:

$$A^3 - A - I = 0.$$

where I is the identity matrix.

We determine the expressions of the plastic matrices of order 2, as follows.

PROPOSITION 2.2. Let $A \in \mathbb{R}(2)$ be a plastic matrix. Then, A has one of the following forms:

(1)
$$A = \rho I$$

where ρ is the plastic number and I is the identity matrix, or

(2)
$$A = \begin{pmatrix} a_{11} & \frac{1 - a_{11}a_{22} - a_{11}^2 - a_{22}^2}{a_{21}} \\ a_{21} & a_{22} \end{pmatrix}$$

with $a_{11}, a_{22} \in \mathbb{R}, a_{21} \in \mathbb{R} \setminus \{0\}$ and $(trA)^3 - trA + 1 = 0$, where $trA := a_{11} + a_{22}$.

PROOF. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

We have:

$$A^{3} = \begin{pmatrix} a_{11}^{3} + 2a_{11}a_{12}a_{21} + a_{12}a_{21}a_{22} & a_{11}^{2}a_{12} + a_{12}^{2}a_{21} + a_{11}a_{12}a_{22} + a_{12}a_{22}^{2} \\ a_{11}^{2}a_{21} + a_{22}^{2}a_{21} + a_{11}a_{21}a_{22} + a_{12}a_{21}^{2} & a_{22}^{3} + 2a_{22}a_{12}a_{21} + a_{12}a_{21}a_{11} \end{pmatrix}$$

Then, A is a plastic matrix if and only if the following system is satisfied:

$$\begin{cases} a_{11}^3 + 2a_{11}a_{12}a_{21} + a_{12}a_{21}a_{22} = a_{11} + 1\\ a_{12}(a_{11}^2 + a_{12}a_{21} + a_{11}a_{22} + a_{22}^2 - 1) = 0\\ a_{21}(a_{11}^2 + a_{12}a_{21} + a_{11}a_{22} + a_{22}^2 - 1) = 0\\ a_{22}^3 + 2a_{22}a_{12}a_{21} + a_{12}a_{21}a_{11} = a_{22} + 1 \end{cases}$$

If we suppose $a_{21} = 0$ (respectively, $a_{12} = 0$), then we get that $a_{11} = a_{22}$ is the plastic number and furthermore, also that $a_{12} = 0$ (respectively, also that $a_{21} = 0$), hence, we have (1).

Now, if we suppose $a_{12}a_{21} \neq 0$, then we get the following conditions:

$$\begin{array}{l}
a_{11}^{3} + 2a_{11}a_{12}a_{21} + a_{12}a_{21}a_{22} = a_{11} + 1 \\
a_{11}^{2} + a_{12}a_{21} + a_{11}a_{22} + a_{22}^{2} = 1 \\
a_{22}^{3} + 2a_{22}a_{12}a_{21} + a_{12}a_{21}a_{11} = a_{22} + 1
\end{array}$$

In particular, we get:

$$\begin{cases} a_{12}a_{21} = 1 - a_{11}a_{22} - a_{11}^2 - a_{22}^2 \\ (a_{11} + a_{22})^3 - (a_{11} + a_{22}) + 1 = 0 \end{cases}$$

,

hence, we have (2), and the proof is complete.

COROLLARY 2.3. Let $A \in \mathbb{R}(2)$ be a plastic matrix. Then, $A = \rho I$, where ρ is the plastic number and I is the identity matrix, or there exists an invertible matrix $C \in \mathbb{R}(2)$, such that $A = C^{-1}BC$, where

(3)
$$B = \begin{pmatrix} \alpha & 1 - \alpha^2 \\ 1 & 0 \end{pmatrix},$$

for α the unique real solution of the equation $x^3 - x + 1 = 0$.

PROOF. Let $A \in \mathbb{R}(2)$ be a plastic matrix. If A has a real eigenvalue, then, it has multiplicity two and $A = \rho I$. If A has not real eigenvalues, then, it has two complex

conjugate eigenvalues and, as a consequence of the previous proposition, their sum is the real solution, α , of the equation $x^3 - x + 1 = 0$. In particular, if

$$A = \begin{pmatrix} a_{11} & \frac{1 - a_{11}a_{22} - a_{11}^2 - a_{22}^2}{a_{21}} \\ a_{21} & a_{22} \end{pmatrix},$$

then we can choose

$$C = \begin{pmatrix} a_{21} & a_{22} \\ 0 & 1 \end{pmatrix}$$

and, as $\alpha = a_{11} + a_{22}$, we get the statement.

2.2 Almost plastic structures

DEFINITION 2.4. Let M be a smooth manifold. An *almost plastic structure* on M is a tensor field J, of type (1, 1), on M, which satisfies the equation:

$$J^3 - J - I = 0$$

If J is an almost plastic structure on M, then, (M, J) is called an *almost plastic manifold*.

DEFINITION 2.5. Let (M, g) be a pseudo-Riemannian manifold. An almost plastic pseudo-Riemannian structure on (M, g) is an almost plastic structure J on M such that g(JX, Y) = g(X, JY) for all $X, Y \in \Gamma^{\infty}(TM)$. If J is an almost plastic pseudo-Riemannian structure on (M, g), then, (M, g, J) is called an almost plastic pseudo-Riemannian manifold.

Let N(J) be the Nijenhuis tensor field of J, defined as:

$$N(J)(X,Y) := [JX, JY] - J[JX,Y] - J[X, JY] + J^{2}[X,Y]$$

for all $X, Y \in \Gamma^{\infty}(TM)$.

According to [8], we say that an almost plastic structure J is *integrable* if N(J) = 0. We will call *plastic structure* an integrable almost plastic structure. An almost plastic manifold (M, J) with J integrable will be called a *plastic manifold*.

EXAMPLE 2.6. Let $\rho = \sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}}$ be the plastic number. Then, $J = \rho I$ is a pseudo-Riemannian plastic structure on any pseudo-Riemannian manifold (M, g) which is called the *trivial pseudo-Riemannian plastic structure*.

 \Box

REMARK 2.7. We remark that the concept of "almost plastic Riemannian structure" has no sense because, excepting the trivial case $J = \rho I$, a plastic structure is not real diagonalizable and then it cannot be symmetric with respect to a positive definite scalar product.

REMARK 2.8. We remark that an almost plastic structure J is invertible and the inverse tensor field is given by $J^{-1} = J^2 - I$. Indeed, $J(J^2 - I) = J^3 - J = I$. However, J^{-1} is not an almost plastic structure because $(J^{-1})^3 - J^{-1} - I = -J$.

REMARK 2.9. A metallic structure on M, [4], is a tensor field J, of type (1, 1), on M, which satisfies the equation:

$$J^2 - pJ - qI = 0,$$

where p, q are positive integers. We remark that, if J is a metallic structure, then, J is an almost plastic structure if and only if

$$\begin{cases} p^3 - p + 1 = 0 \\ q = 1 - p^2 \end{cases} \quad \text{or} \quad \begin{cases} q \neq 1 - p^2 \\ J = \frac{1 - pq}{p^2 + q - 1}I \end{cases},$$

respectively, J satisfies a **dual** equation, $J^3 - J + I = 0$, if and only if

$$\left\{ \begin{array}{ll} p=\rho \ \, \mbox{is the plastic number} \\ q=1-\rho^2 \end{array} \right. \ \, \mbox{or} \ \, \left\{ \begin{array}{l} q\neq 1-p^2 \\ J=-\frac{1+pq}{p^2+q-1}I \end{array} \right. . \label{eq:prod}$$

3 Plastic structures in generalized geometry

3.1 Geometrical properties of the generalized tangent bundle

Let $TM \oplus T^*M$ be the generalized tangent bundle of M. On $TM \oplus T^*M$, we consider the natural indefinite metric

$$< X + \eta, Y + \beta > := -\frac{1}{2}(\eta(Y) + \beta(X))$$

and the natural symplectic structure

$$(X+\eta,Y+\beta) := -\frac{1}{2}(\eta(Y)-\beta(X))$$

for all $X, Y \in \Gamma^{\infty}(TM)$ and $\eta, \beta \in \Gamma^{\infty}(T^*M)$.

If g is a non-degenerate (0, 2)-tensor field on M, we will denote by g the flat isomorphism $\flat_g : \Gamma^{\infty}(TM) \to \Gamma^{\infty}(T^*M), \, \flat_g(X) := i_X g$, and by g^{-1} its inverse, and we define the bilinear form, \check{g} , on $TM \oplus T^*M$ by:

$$\check{g}(X+\eta, Y+\beta) := g(X,Y) + g(g^{-1}(\eta), g^{-1}(\beta))$$

for all $X, Y \in \Gamma^{\infty}(TM)$ and $\eta, \beta \in \Gamma^{\infty}(T^*M)$.

Furthermore, given an affine connection ∇ on M, we define the affine connections $\hat{\nabla}$ and $\check{\nabla}$ on $TM \oplus T^*M$ by:

$$\hat{\nabla}_{X+\eta}(Y+\beta) := \nabla_X Y + g(\nabla_X(g^{-1}(\beta)))$$

and

$$\check{\nabla}_{X+\eta}(Y+\beta) := \nabla_X Y + \nabla_X \beta$$

for all $X, Y \in \Gamma^{\infty}(TM)$ and $\eta, \beta \in \Gamma^{\infty}(T^*M)$. We remark that $\hat{\nabla} = \check{\nabla}$ if and only if $\nabla g = 0$.

Finally, we define the bracket $[\cdot, \cdot]_{\nabla}$:

$$[X + \eta, Y + \beta]_{\nabla} := [X, Y] + \nabla_X \beta - \nabla_Y \eta$$

for all $X, Y \in \Gamma^{\infty}(TM)$ and $\eta, \beta \in \Gamma^{\infty}(T^*M)$.

3.2 Generalized almost plastic structures

Any morphism $\hat{J} : \Gamma^{\infty}(TM \oplus T^*M) \to \Gamma^{\infty}(TM \oplus T^*M)$ will be called a generalized structure on M.

DEFINITION 3.1. Let $\hat{J} : \Gamma^{\infty}(TM \oplus T^*M) \to \Gamma^{\infty}(TM \oplus T^*M)$ be a generalized structure on M. If \hat{J} satisfies the equation

$$\hat{J}^3 - \hat{J} - I = 0,$$

then, \hat{J} is called a generalized almost plastic structure on M.

EXAMPLE 3.2. Let M be a smooth manifold and let J_1, J_2 be almost plastic structures on M. Then, the generalized structure

(4)
$$\hat{J} := \begin{pmatrix} J_1 & 0\\ 0 & J_2^* \end{pmatrix},$$

where $J_2^* : \Gamma^{\infty}(T^*M) \to \Gamma^{\infty}(T^*M)$ is defined by $J_2^*(\eta)(X) := \eta(J_2X)$ for all $\eta \in \Gamma^{\infty}(T^*M)$ and $X \in \Gamma^{\infty}(TM)$, is a generalized almost plastic structure on M. For $J_1 = J_2 = J$, we will denote the induced generalized almost plastic structure by:

(5)
$$\hat{J} := \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix}.$$

PROPOSITION 3.3. Let \hat{J} be the generalized almost plastic structure defined by (5). Then,

$$\langle \hat{J}(X+\eta), Y+\beta \rangle = \langle X+\eta, \hat{J}(Y+\beta) \rangle;$$

moreover, if g is a pseudo-Riemannian metric and $(M, g, J_1), (M, g, J_2)$ are almost plastic pseudo-Riemannian manifolds, then, the structure \hat{J} defined by (4) satisfies:

$$\check{g}(\hat{J}(X+\eta),Y+\beta) = \check{g}(X+\eta,\hat{J}(Y+\beta))$$

for all $X, Y \in \Gamma^{\infty}(TM)$ and $\eta, \beta \in \Gamma^{\infty}(T^*M)$.

PROOF. We have:

$$<\hat{J}(X+\eta), Y+\beta>==-\frac{1}{2}(\eta(JY)+\beta(JX));$$

on the other hand,

$$< X + \eta, \hat{J}(Y + \beta) > = < X + \eta, JY + J^{*}(\beta) > = -\frac{1}{2}(\eta(JY) + \beta(JX)),$$

and the first statement is proved. Moreover, \hat{J} defined by (4) satisfies:

$$\begin{split} \check{g}(\hat{J}(X+\eta),Y+\beta) &= \check{g}(J_1X+J_2^*(\eta),Y+\beta) = g(J_1X,Y) + g(g^{-1}(J_2^*(\eta)),g^{-1}(\beta)) \\ &= g(X,J_1Y) + g(J_2(g^{-1}(\eta)),g^{-1}(\beta)) \\ &= g(X,J_1Y) + g(g^{-1}(\eta),J_2(g^{-1}(\beta))) \\ &= g(X,J_1Y) + g(g^{-1}(\eta),g^{-1}(J_2^*(\beta))) \\ &= \check{g}(X+\eta,\hat{J}(Y+\beta)), \end{split}$$

and the proof is complete.

PROPOSITION 3.4. Let \hat{J} be the generalized almost plastic structure defined by (4). Then, $\hat{\nabla}\hat{J} = 0$ if and only if

$$\begin{cases} \nabla J_1 = 0\\ \nabla J_2 = 0 \end{cases}$$

Moreover, $\check{\nabla} \hat{J} = 0$ if and only if

$$\begin{cases} \nabla J_1 = 0 \\ \nabla J_2 = 0 \end{cases}$$

PROOF. A direct computation gives:

$$(\hat{\nabla}\hat{J})_{X+\eta}(Y+\beta) = (\nabla_X J_1)Y + g((\nabla_X J_2)(g^{-1}(\beta)))$$

for all $X, Y \in \Gamma^{\infty}(TM)$ and $\eta, \beta \in \Gamma^{\infty}(T^*M)$, and the first statement is proved. Moreover, a direct computation gives:

$$(\check{\nabla}\tilde{J})_{X+\eta}(Y+\beta) = (\nabla_X J_1)Y + (\nabla_X J_2^*)\beta$$

for all $X, Y \in \Gamma^{\infty}(TM)$ and $\eta, \beta \in \Gamma^{\infty}(T^*M)$, and the proof is complete.

3.3 Generalized almost plastic structures defined by two (1,1)tensor fields

Taking inspiration from the form of order two plastic matrices, we now construct a family of generalized almost plastic structures on a pseudo-Riemannian manifold.

PROPOSITION 3.5. Let (M, g) be a pseudo-Riemannian manifold and let J_1 , J_2 be two (1, 1)-tensor fields on M satisfying the following conditions: (i) $J_1J_2 = J_2J_1$ (ii) $(J_1 + J_2)^3 - (J_1 + J_2) + I = 0$ (iii) $g(J_1X, Y) = g(X, J_1Y)$, for all $X, Y \in \Gamma^{\infty}(TM)$ (iv) $g(J_2X, Y) = g(X, J_2Y)$, for all $X, Y \in \Gamma^{\infty}(TM)$. Then,

(6)
$$\hat{J} := \begin{pmatrix} J_1 & (I - J_1 J_2 - J_1^2 - J_2^2)g^{-1} \\ g & J_2^* \end{pmatrix}$$

is a generalized almost plastic structure on M.

PROOF. From

$$\hat{J}(X+\eta) = \left(J_1X + (I - J_1J_2 - J_1^2 - J_2^2)g^{-1}(\eta)\right) + \left(g(X) + J_2^*(\eta)\right),$$

by using the properties (i), (iii), (iv), we get:

$$\begin{aligned} \hat{J}^2(X+\eta) &= \left(J_1^2 X + (I - J_1 J_2 - J_1^2 - J_2^2) J_1 g^{-1}(\eta) + (I - J_1 J_2 - J_1^2 - J_2^2) (X + J_2 g^{-1}(\eta))\right) \\ &+ \left(g(J_1 X) + (I - J_1 J_2 - J_1^2 - J_2^2)^*(\eta) + g(J_2 X) + (J_2^*)^2(\eta)\right) \\ &= \left((I - J_1 J_2 - J_2^2) X + (J_1 + J_2 - 2J_1^2 J_2 - 2J_2^2 J_1 - J_1^3 - J_2^3) g^{-1}(\eta)\right) \\ &+ \left(g(J_1 X) + (I - J_1 J_2 - J_1^2)^*(\eta) + g(J_2 X)\right);\end{aligned}$$

moreover,

$$\hat{J}^{3}(X+\eta) = \left((J_{1} + (J_{1} + J_{2}) - (J_{1} + J_{2})^{3})X + (I - J_{1}J_{2} - J_{1}^{2} - J_{2}^{2})g^{-1}(\eta) \right) + \left(g(X) + (J_{2}^{*} + (J_{1}^{*} + J_{2}^{*}) - (J_{1}^{*} + J_{2}^{*})^{3}(\eta) \right),$$

and the proof is complete.

REMARK 3.6. From the proof of the previous proposition, we get that, if $(J_1 + J_2)$ is an almost plastic structure on M, then, the generalized structure \hat{J} defined by (6) satisfies the condition $\hat{J}^3 - \hat{J} + I = 0$. Therefore, this construction of a generalized structure defines a **duality** between the two equations $x^3 - x - 1 = 0$ and $x^3 - x + 1 = 0$.

PROPOSITION 3.7. Let \hat{J} be the generalized almost plastic structure defined by (6). Then, $\hat{\nabla}\hat{J} = 0$ if and only if

$$\begin{cases} \nabla J_1 = 0 \\ \nabla J_2 = 0 \end{cases}$$

Moreover, $\check{\nabla} \hat{J} = 0$ if and only if

$$\begin{cases} \nabla J_1 = 0\\ \nabla J_2 = 0\\ \nabla g = 0 \end{cases}.$$

PROOF. A direct computation gives:

$$(\hat{\nabla}\hat{J})_{X+\eta}(Y+\beta) = (\nabla_X J_1)Y + (\nabla_X (I-J_1J_2-J_1^2-J_2^2))(g^{-1}(\beta)) + g((\nabla_X J_2)(g^{-1}(\beta)))$$

for all $X, Y \in \Gamma^{\infty}(TM)$ and $\eta, \beta \in \Gamma^{\infty}(T^*M)$, and the first statement is proved. Moreover, a direct computation gives:

$$(\check{\nabla}\hat{J})_{X+\eta}(Y+\beta) = (\nabla_X J_1)Y + (\nabla_X (I-J_1J_2-J_1^2-J_2^2)g^{-1})(\beta) + (\nabla_X g)Y + (\nabla_X J_2^*)\beta$$

for all $X, Y \in \Gamma^{\infty}(TM)$ and $\eta, \beta \in \Gamma^{\infty}(T^*M)$, and the proof is complete.

4 Integrability of some generalized almost plastic structures

Let M be a smooth manifold and let ∇ be an affine connection on M.

DEFINITION 4.1. A generalized structure \hat{J} on M is called ∇ -integrable if its Nijenhuis tensor field $N^{\nabla}(\hat{J})$ with respect to ∇ :

$$N^{\nabla}(\hat{J})(\sigma,\tau) := [\hat{J}\sigma, \hat{J}\tau]_{\nabla} - \hat{J}[\hat{J}\sigma, \tau]_{\nabla} - \hat{J}[\sigma, \hat{J}\tau]_{\nabla} + \hat{J}^2[\sigma, \tau]_{\nabla}$$

vanishes for all $\sigma, \tau \in \Gamma^{\infty}(TM \oplus T^*M)$.

We shall study the integrability of some generalized almost plastic structures, characterizing it also in terms of quasi-statistical structures. We recall the following definition.

DEFINITION 4.2. [5, 7] Let g be a non-degenerate (0, 2)-tensor field on M and let ∇ be an affine connection on M with torsion operator T^{∇} . Then, (g, ∇) is called a *quasi-statistical structure* on M if

$$(\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) + g(T^{\nabla}(X, Y), Z) = 0$$

for all $X, Y, Z \in \Gamma^{\infty}(TM)$, where T^{∇} is the torsion of the given connection ∇ . If (g, ∇) is a quasi-statistical structure on M, then, (M, g, ∇) is called a *quasi-statistical manifold*.

For generalized quasi-statistical structures, see [1, 2, 3].

4.1 Integrability of generalized almost plastic structures induced by two almost plastic structures on M

PROPOSITION 4.3. Let M be a smooth manifold, let ∇ be an affine connection on M, and let J_1, J_2 be two almost plastic structures on M. Then, the generalized almost plastic structure defined by

$$\hat{J} := \begin{pmatrix} J_1 & 0\\ 0 & J_2^* \end{pmatrix},$$

is ∇ -integrable if and only if the following conditions are satisfied:

(7)
$$\begin{cases} N(J_1) = 0\\ \nabla_{J_1X} J_2 = J_2(\nabla_X J_2) \end{cases}$$

for all $X \in \Gamma^{\infty}(TM)$.

PROOF. A direct computation gives:

$$N^{\nabla}(\hat{J})(X+\eta,Y+\beta) = N(J_1)(X,Y) + ((\nabla_{J_1X}J_2^*) - J_2^*(\nabla_XJ_2^*))(\beta) - ((\nabla_{J_1Y}J_2^*) - J_2^*(\nabla_YJ_2^*))(\eta)$$

for all $X, Y \in \Gamma^{\infty}(TM)$ and $\eta, \beta \in \Gamma^{\infty}(T^*M)$. Moreover, for any $X, Y \in \Gamma^{\infty}(TM)$ and for any $\eta \in \Gamma^{\infty}(T^*M)$:

$$((\nabla_{J_1X} J_2^*)(\eta))(Y) = \eta((\nabla_{J_1X} J_2)Y)$$

and

$$((J_2^*(\nabla_X J_2^*))(\eta))(Y) = \eta(J_2(\nabla_X J_2)Y),$$

and the proof is complete.

REMARK 4.4. We remark that if $J_1 = J_2 = J$, then (7) becomes:

$$\begin{cases} T^{\nabla}(JX, JY) - JT^{\nabla}(JX, Y) - JT^{\nabla}(X, JY) + J^2 T^{\nabla}(X, Y) = 0\\ \nabla_{JX}J = J(\nabla_X J) \end{cases}$$

for all $X, Y \in \Gamma^{\infty}(TM)$. In particular, for a torsion-free affine connection, the integrability condition becomes:

$$\nabla_{JX}J = J(\nabla_X J)$$

for all $X \in \Gamma^{\infty}(TM)$.

4.2 Integrability of generalized almost plastic structures induced by a polynomial structure J on M such that $J^3 - J + I = 0$

Taking inspiration from (3), we consider the generalized almost plastic structure defined by:

(8)
$$\hat{J} := \begin{pmatrix} J & (I-J^2)g^{-1} \\ g & 0 \end{pmatrix},$$

where J is a polynomial structure on M such that $J^3 - J + I = 0$.

PROPOSITION 4.5. Let g be a pseudo-Riemannian metric and let ∇ be an affine connection on M. If J is integrable, $\nabla J = 0$, and (M, g, ∇) is a quasi-statistical manifold, then \hat{J} given by (8) is ∇ -integrable.

PROOF. A direct computation gives:

$$N^{\nabla}(\hat{J})(X,Y) = N(J)(X,Y) + (I - J^2)g^{-1}((\nabla_Y g)X - (\nabla_X g)Y + g(T^{\nabla}(Y,X))) + (\nabla_{JX}g)Y - (\nabla_Y g)JX + g(T^{\nabla}(JX,Y)) + (\nabla_X g)JY - (\nabla_{JY}g)X + g(T^{\nabla}(X,JY)) + (\nabla_Y g)X - (\nabla_X g)Y + g(T^{\nabla}(Y,X)) + (\nabla_Y J^*)g(X) - (\nabla_X J^*)g(Y) + J^*((\nabla_Y g)X - (\nabla_X g)Y + g(T^{\nabla}(Y,X)))$$

for all $X, Y \in \Gamma^{\infty}(TM)$;

$$\begin{split} N^{\nabla}(\hat{J})(X,g(Z)) &= -(\nabla_{JX}J^2)Z - (\nabla_{(I-J^2)Z}J)X + J(\nabla_XJ^2)Z \\ &\quad -T^{\nabla}(JX,(I-J^2)Z) + JT^{\nabla}(X,(I-J^2)Z) \\ &\quad -(I-J^2)g^{-1}((\nabla_{JX}g)Z) + J(I-J^2)(g^{-1}(\nabla_Xg)Z) \\ &\quad +(\nabla_Xg)Z - (\nabla_Zg)X + g(T^{\nabla}(X,Z)) \\ &\quad -(J^*)^2((\nabla_Xg)Z) + (\nabla_{J^2Z}g)X - g(T^{\nabla}(X,J^2Z)) \\ &= -g^{-1}((\nabla_JXg)(I-J^2)Z)) - (\nabla_{(I-J^2)Z}g)JX + g(T^{\nabla}(JX,(I-J^2)Z))) \\ &\quad + Jg^{-1}((\nabla_Xg)(I-J^2)Z) - (\nabla_{(I-J^2)Z}g)X + g(T^{\nabla}(X,(I-J^2)Z))) \\ &\quad +(\nabla_Xg)Z - (\nabla_Zg)X + g(T^{\nabla}(X,Z)) \\ &\quad -(\nabla_Xg)J^2Z + (\nabla_{J^2Z}g)X - g(T^{\nabla}(X,J^2Z)) \end{split}$$

for all $X, Z \in \Gamma^{\infty}(TM)$;

$$N^{\nabla}(\hat{J})(g(Z), g(W)) = g^{-1}((\nabla_W g)Z - (\nabla_Z g)W + g(T^{\nabla}(W, Z))) + g^{-1}((\nabla_{J^2W} g)J^2Z - (\nabla_{J^2Z} g)J^2W + g(T^{\nabla}(J^2W, J^2Z))) + g^{-1}((\nabla_Z g)J^2W - (\nabla_{J^2W} g)Z + g(T^{\nabla}(Z, J^2W))) + g^{-1}((\nabla_{J^2Z} g)W - (\nabla_W g)J^2W + g(T^{\nabla}(J^2Z, W)))$$

for all $Z, W \in \Gamma^{\infty}(TM)$, and we get the conclusion.

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