SOME ISOPERIMETRIC INEQUALITIES FOR HOMOGENEOUS NORMS ON STRATIFIED LIE GROUPS

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ABSTRACT. We continue the program initiated in [IKZ11; Qiu24] and study L^1 -type functional inequalities for some probability measures defined in terms of homogeneous norms on a stratified Lie group, our goal being to obtain isoperimetric inequalities beyond the nondegenerate setting of probability measures defined in terms of the Carnot-Carathéodory distance. We then provide some examples on and beyond stratified Lie groups.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we continue the program initiated in [IKZ11; Qiu24] and study $L^1\Phi$ -entropy type inequalities for probability measures defined on stratified Lie groups using an adaptation of the methods presented in the latter, our goal being to obtain isoperimetric inequalities for some probability measures defined in terms of homogeneous norms. Consider the probability measure

(1.1)
$$d\mu_p = Z^{-1} e^{-\psi^p} d\xi$$

where ψ is a homogeneous norm (in the sense of [BLU07, §5]) on a stratified Lie group \mathbb{G} with Lebesgue measure $d\xi$, p > 0, and Z is a normalisation constant (here and in the sequel). It was shown in [HZ10, Corollary 4.1] that if ψ is the Carnot-Carathéodory distance d on the Heisenberg group $\mathbb{G} = \mathbb{H}^n$ and $p \ge 2$ then μ_p satisfies a q-logarithmic Sobolev inequality

(1.2)
$$\int |f|^q \log\left(\frac{|f|^q}{\int |f|^q d\mu_p}\right) d\mu_p \lesssim \int |\nabla_{\mathbb{G}} f|^q d\mu_p$$

for $q \in (1, 2]$ Hölder conjugate to p and $f : \mathbb{H}^n \to \mathbb{R}$ smooth. Here and in the sequel \leq means the inequality holds for some constant C > 0 independent of f appearing on the right hand side. Note for p > 2 this inequality improves on and implies the standard 2-logarithmic Sobolev inequality since it implies, by Hölder's inequality, a defective 2-logarithmic Sobolev inequality (meaning with an additional L^2 -norm appearing on the right hand side) which can be tightened by the 2-Poincaré inequality of [BZ05, Corollary 2.5] and [BGL+14, Proposition 5.1.3]. Despite some technical differences between the Carnot-Carathéodory distance and the euclidean norm (for instance, smoothness), the measures they define in the sense of (1.1) are analogous at the level of functional inequalities; both satisfy (1.2), and according to [IKZ11, Theorem 5.6], both share the same isoperimetric profile up to constants.

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YAOZHONG QIU

It was later proven in [Qiu24, Theorem 2] that if ψ is the Kaplan norm $N = N_{\mathbb{G}}$ on (again for simplicity) the Heisenberg group $\mathbb{G} = \mathbb{H}^n$ and p > 2, then μ_p satisfies a *q*-super-Poincaré inequality

(1.3)
$$\int |f|^q d\mu_p \le \varepsilon \int |\nabla_{\mathbb{G}} f|^q d\mu_p + \beta_q(\varepsilon) \left(\int |f|^{q/2} d\mu_p \right)^2$$

for $q \in (1,2)$ Hölder conjugate to p, all $\varepsilon > 0$ and growth

 $\beta_q(\varepsilon) = C \exp(C' \varepsilon^{-2(p-1)/(p-2)}), \quad C, C' > 0.$

[Ing12, Lemma 4.11] showed this q-variant is stronger than the 2-super-Poincaré inequality, the latter introduced in [Wan00] to study the essential spectrum of the operator whose Dirichlet form is $\mathcal{E}(f, f) = \int |\nabla_{\mathbb{G}} f|^2 d\mu$. One of the observations made in [Qiu24] was that the growth β_q in (1.3) for the Kaplan norm measure $d\mu_p = Z^{-1}e^{-N^p}d\xi$ implies a 2super-Poincaré inequality with a growth β_2 which coincided with the growth of the "model" measure $d\nu_r = Z^{-1}e^{-|x|^r}dx$ for $r = \frac{2p}{p+2} \in (1, 2)$ on \mathbb{R}^n . Thus the measure $d\mu_p$, despite having supergaussian decay of tails, can only support a subgaussian 2-super-Poincaré inequality.

In this paper, we would like to answer the question of whether such a subgaussian qsuper-Poincaré inequality implies subgaussian isoperimetric content. That is, does $d\mu_p = Z^{-1}e^{-N^p}d\xi$, p > 2, share the same isoperimetric profile as $d\nu_r = Z^{-1}e^{-|x|^r}dx$, $r = \frac{2p}{p+2}$, up
to constants? An issue however with the q-super-Poincaré inequality for q > 1 is that the
method of extracting isoperimetric content through the consequent F-Sobolev inequality

(1.4)
$$\int f^2 F(f^2) d\mu_p \le c_1 \int |\nabla_{\mathbb{G}} f|^2 d\mu_p + c_2, \quad c_1, c_2 > 0 \text{ and } \int f^2 d\mu_p = 1.$$

as done in [Wan00, Theorem 3.4], fails in the subelliptic setting of stratified Lie groups due to the absence of curvature lower bounds. In particular the consequences of [Qiu24] are apparently partial in the sense they cannot recover the expected isoperimetric content.

Although in light of the previous discussion the choice of q being Hölder conjugate to p is somehow the "natural" choice of setting, it is not known to us how to extract isoperimetric content from q-super-Poincaré inequalities (more precisely, the q-variant of the F-Sobolev inequality they imply) in the subelliptic setting due to the dependence on curvature lower bounds. In contrast, the methods developed in [IKZ11] for isoperimetric inequalities require no such assumptions. Motivated by their work, we will complete the picture by first proving a 1-super-Poincaré inequality for μ_p , show this passes to an L^1 -analogue of the F-Sobolev inequality following the arguments of [Wan00, Theorem 3.2], and finally show this implies via [IKZ11, Theorem 4.5] the expected isoperimetric inequality. We will conclude by providing an argument which shows these ideas can be greatly simplified but are included nonetheless for the sake of completeness and possible applications in future work. We will not be able to affirmatively answer the question of whether a q-super-Poincaré inequality, q > 1, directly implies isoperimetric content but we will see it "predicts" a 1-super-Poincaré inequality (at least in the cases of interest) which does. We now introduce the main objects of interest. The methods allow for generalisation to other spaces with a similar structure, but to provide a clearer exposition of the main ideas, the underlying space of interest (for the moment) is a stratified Lie group \mathbb{G} . This means that $\mathbb{G} = (\mathbb{R}^n, \circ)$ is a Lie group, which we will identify with \mathbb{R}^n equipped with a composition law $\circ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, whose Lie algebra $(\mathfrak{g}, [-, -])$ of left invariant vector fields admits the decomposition (or stratification)

$$\mathfrak{g} = \bigoplus_{i=0}^{r-1} \mathfrak{g}_i$$
 such that $\mathfrak{g}_i = [\mathfrak{g}_i, \mathfrak{g}_{i+1}], \quad 1 \le i \le r-1.$

In other words, \mathfrak{g}_0 generates the Lie algebra \mathfrak{g} through successive Lie bracketing. The prototypical example is the Heisenberg group $\mathbb{H}^1 = (\mathbb{R}^3, \circ)$ whose Lie algebra \mathfrak{h}^1 is generated by $X_1 = \partial_x - \frac{y}{2}\partial_z$ and $X_2 = \partial_y + \frac{x}{2}\partial_z$. There is a canonical basis $\{X_1, \dots, X_\ell\}$ for \mathfrak{g}_0 and ℓ the topological dimension of \mathfrak{g}_0 whose components form the subgradient $\nabla_{\mathbb{G}} := (X_1, \dots, X_\ell)$ and sublaplacian $\Delta_{\mathbb{G}} = \nabla_{\mathbb{G}} \cdot \nabla_{\mathbb{G}} = \sum_{i=1}^{\ell} X_i^2$ on \mathbb{G} replacing their euclidean counterparts $\nabla = (\partial_1, \dots, \partial_n)$ and $\Delta = \sum_{i=1}^n \partial_i^2$.

The identification between \mathfrak{g} and \mathbb{G} through the exponential map equips \mathbb{G} with the coordinates $\mathbb{G} \cong \bigotimes_{i=0}^{r-1} \mathbb{R}_i$ where dim $(\mathbb{R}_i) = \dim(\mathfrak{g}_i)$. The first strata \mathbb{R}_0 , called the horizontal strata, is distinguished in the sense it represents the "euclidean" part of \mathbb{G} ; for a function f on \mathbb{G} depending only on coordinates in \mathbb{R}_0 , the action of $\nabla_{\mathbb{G}}$ and $\Delta_{\mathbb{G}}$ is simply the euclidean one. For a point $\xi = (x_0, \dots, x_{r-1}) \in \bigotimes_{i=0}^{r-1} \mathbb{R}_i \cong \mathbb{G}$, we write $x = x(\xi) = x_0$ for the horizontal part of ξ , and |x| for its euclidean norm, called the horizontal norm. The horizontal norm will appear in the estimates on the homogeneous norm ψ .

Every stratified Lie group is equipped with a family of automorphisms

$$\delta_{\lambda}(\xi) = \delta_{\lambda}(x_0, \cdots, x_{r-1}) = (\lambda x_0, \cdots, \lambda^r x_{r-1}), \quad \lambda > 0$$

called dilations. We will be interested in studying measures defined in terms of homogeneous norms, meaning smooth almost everywhere functions ψ satisfying

- (1) $\psi(\xi) = 0$ if and only if $\xi = e$,
- (2) $\psi(\xi^{-1}) = \psi(\xi)$ for each $\xi \in \mathbb{G}$, and
- (3) $\psi(\delta_{\lambda}(\xi)) = \lambda \psi(\xi)$ for each $\xi \in \mathbb{G}$.

Two homogeneous norms are particularly distinguished. The first is the Carnot-Carathéodory distance mentioned in the introduction is the metric on G, defined as

$$d(\xi_1, \xi_2) = \sup_{|\nabla_{\mathbb{G}} f|^2 \le 1} f(\xi_1) - f(\xi_2)$$

for all $\xi_1, \xi_2 \in \mathbb{G}$ and where the supremum is taken over all f smooth. Abusing notation, we will write $d(\xi) \equiv d(\xi, 0)$ for the homogeneous norm d. It can also be defined in terms of geodesics which are integral curves of X_1, \dots, X_ℓ , that is paths whose tangent vectors belong to $\text{Span}\{X_1, \dots, X_\ell\}$ at each point. That the space is actually connected by such paths is because the vector fields X_1, \dots, X_ℓ satisfy Hörmander's rank condition and therefore the Carathéodory-Chow theorem applies, see for instance [BLU07, Chapter 19].

YAOZHONG QIU

The second is the Kaplan norm N which, on a Heisenberg group, is defined as the fundamental solution to $\Delta_{\mathbb{G}}$, namely if $Q(\mathbb{G}) = \sum_{i=0}^{r-1} (i+1) \dim(\mathfrak{g}_i)$ is the homogeneous dimension of \mathbb{G} , then

$$\Delta_{\mathbb{G}} N^{2-Q(\mathbb{G})} = \delta_0$$

in the sense of distributions. On more general stratified Lie groups, the Kaplan norm is typically called the Korányi-Folland gauge. Both d and N are homogeneous norms in the sense of [BLU07, §5], and are equivalent up to constants.

Interestingly, the Carnot-Carathéodory distance satisfies like the euclidean norm on \mathbb{R}^n the eikonal equation $|\nabla_{\mathbb{G}}d| = 1$ in the distributional sense [MC01, Theorem 3.1], but is not smooth on $\mathbb{G} \setminus \{0\}$. On the other hand, the Kaplan norm is smooth on all of $\mathbb{G} \setminus \{0\}$ but need not satisfy the eikonal equation and instead $|\nabla_{\mathbb{G}}N|$ may vanish in certain directions, for instance along the zero locus $\{|x| = 0\}$ of the horizontal norm when \mathbb{G} is a Heisenberg group. These differences manifest in the functional inequalities their respective measures satisfy; see, for instance, [HZ10, Theorem 6.3] wherein it was shown on a *H*-type group that μ_p can never satisfy a logarithmic Sobolev inequality when ψ is any homogeneous norm smooth away from the origin (in particular when ψ is the Kaplan norm). For more details on stratified Lie groups, we refer the reader to [BLU07].

The functional inequalities of interest are q-super-Poincaré, F-Sobolev, and isoperimetric inequalities. If (X, \mathcal{A}, ν) is a probability space and L is a selfadjoint operator on $L^2(\nu)$ generating a Markov semigroup $(P_t)_{t\geq 0}$ with Dirichlet form $\mathcal{E}(f, f)$, then the 2-super-Poincaré inequality as introduced by [Wan00] is the family of inequalities

(1.5)
$$\int f^2 d\nu \le \varepsilon \mathcal{E}(f, f) + \beta_2(\varepsilon) \left(\int |f| \, d\nu \right)^2$$

valid for real-valued $f: X \to \mathbb{R}$ in the domain $\mathcal{D}(\mathcal{E})$ of the Dirichlet form and $\varepsilon > \varepsilon_0 \ge 0$. As before, we call $\beta_2: (\varepsilon_0, \infty) \to \mathbb{R}$ the growth function. One of the remarkable properties of the 2-super-Poincaré inequality is that it not only implies [Wan00, Theorem 2.1] the essential spectrum $\sigma_{\text{ess}}(-L)$ of -L is contained in $[\varepsilon_0^{-1}, \infty)$, but also implies [Wan00, Theorems 3.1 and 3.2] via *F*-Sobolev inequalities (defined below) a number of other functional inequalities, including in particular the Poincaré and logarithmic Sobolev. For more details on 2-super-Poincaré inequalities, we refer the reader to [Wan00; Wan02], and also to [Wan06; Bak04; BGL+14] for more general discussion of functional inequalities.

In the sequel, we will always work in the diffusion setting where $X = \mathbb{G}$ is a stratified Lie group, $d\nu = Z^{-1}e^{-U}d\xi$, and $L = \Delta_{\mathbb{G}} - \nabla_{\mathbb{G}}U \cdot \nabla_{\mathbb{G}}$ with $U \in W^{2,1}_{\text{loc}}(\mathbb{G})$ satisfying $Z = \int e^{-U}d\xi < \infty$ where $d\xi$ is the Lebesgue measure on \mathbb{G} . Then L is essentially selfadjoint in $L^2(\nu)$, its unique selfadjoint extension generates a Markov semigroup $(P_t)_{t\geq 0}$ in $L^2(\nu)$, and its Dirichlet form $\mathcal{E}(f, f) = (-Lf, f) = \int |\nabla_{\mathbb{G}}f|^2 d\mu_p$ has domain $\mathcal{D}(\mathcal{E}) = W^{1,2}(\nu)$. The 1-super-Poincaré inequality in this diffusion setting is the family of inequalities

(1.6)
$$\int |f| \, d\nu \le \varepsilon \int |\nabla_{\mathbb{G}} f| \, d\nu + \beta_1(\varepsilon) \left(\int |f|^{1/2} \, d\nu \right)^2, \quad \varepsilon \ge \varepsilon_0 > 0.$$

It was shown for $\varepsilon_0 = 0$ in [Ing12, Lemma 4.11] to imply (1.5) with $\beta_2(\varepsilon) = C\beta_1(C'\varepsilon^{1/2})$ for some C, C' > 0. The implication is still true for general $\varepsilon_0 > 0$, but the ε_0 in (1.6) need not be the same ε_0 in (1.5).

The goal of this paper is to study the 1-super-Poincaré inequality (1.6) for a class of homogeneous norms ψ on a stratified Lie group \mathbb{G} defining the probability measure $d\nu =$ $d\mu_p = Z^{-1}e^{-\psi^p}d\xi$ where Z is again the normalisation constant, with its dependence on p suppressed for convenience. The main ingredients will be estimates for $\nabla_{\mathbb{G}}\psi$, $\Delta_{\mathbb{G}}\psi$, and $\nabla_{\mathbb{G}}|x| \cdot \nabla_{\mathbb{G}}\psi$. The proofs will also pass through the L^1 -analogue of a U-bound

(1.7)
$$\int \eta f^2 d\nu \lesssim \mathcal{E}(f,f) + \int f^2 d\nu$$

which we continue to call a U-bound (since we will not consider inequalities of any other L^{p} -type)

(1.8)
$$\int \eta |f| d\mu_p \lesssim \int |\nabla_{\mathbb{G}} f| d\mu_p + \int |f| d\mu_p$$

for some $\eta : \mathbb{G} \to \mathbb{R}$. They were introduced in [HZ10] and shown under generic conditions (going beyond the case of stratified Lie groups) to provide Poincaré and logarithmic Sobolev inequalities. One reason they might be interesting to study is that there are some tractable ways to prove them, for instance the original integration by parts method of [HZ10]. They are also stable under (certain types of) perturbations beyond the typical functions of bounded oscillation. In the L^2 -case of (1.7), they are essentially quadratic form lower bounds $\eta \leq$ -L + 1. For more discussion on U-bounds, we refer the reader to [HZ10; IKZ11; Qiu24] and the references therein.

After proving the 1-super-Poincaré inequality, we will prove an L^1 -analogue of the F-Sobolev inequality

(1.9)
$$\int f^2 F(f^2) d\nu \le c_1 \mathcal{E}(f, f) + c_2, \quad \int f^2 d\nu = 1 \text{ and } c_1, c_2 > 0$$

which we continue to call a F-Sobolev inequality

(1.10)
$$\int |f| F_1(|f|) d\mu_p \le c_1 \int |\nabla_{\mathbb{G}} f| d\mu_p + c_2, \quad \int |f| d\mu_p = 1 \text{ and } c_1, c_2 > 0$$

and in particular establish a correspondence between 1-super-Poincaré inequalities and F-Sobolev inequalities analogous to [Wan00, Theorems 3.1 and 3.2]. As mentioned earlier, such a F-Sobolev inequality is an $L^1\Phi$ -entropy inequality in the language of [IKZ11] and, following their methods, we will conclude by deducing the titular isoperimetric inequality which we present in greater generality than necessary (as far as the examples we present are concerned) for possible applications in future work.

Theorem 1. Let \mathbb{G} be a stratified Lie group with Lebesgue measure $d\xi$, N a homogeneous norm satisfying the estimates

(1.11)
$$\frac{|x|^{\alpha}}{N^{\alpha}} \lesssim |\nabla_{\mathbb{G}}N| \lesssim \frac{|x|^{\beta}}{N^{\beta}}, \quad |\Delta_{\mathbb{G}}N| \lesssim \frac{|x|^{\gamma}}{N^{\gamma+1}}, \quad |\nabla_{\mathbb{G}}|x| \cdot \nabla_{\mathbb{G}}N| \lesssim \frac{|x|^{\delta}}{N^{\delta}}$$

for some $\alpha, \beta, \gamma, \delta \geq 0$. Let

$$\sigma = \max(-2\alpha, -\beta, -\gamma, -\delta + 1).$$

If $\delta \geq \sigma + 2\alpha$ and $p > 1 + \sigma + 2\alpha$ then the measure $d\mu_p = Z^{-1}e^{-N^p}d\xi$ satisfies the 1-super-Poincaré inequality (1.6) with $\varepsilon_0 = 0$ and

$$\beta_1(\varepsilon) = C \exp\left(C'\varepsilon^{-p(1+\sigma+2\alpha)/(p-1-\sigma-2\alpha)}\right), \quad C, C' > 0.$$

Proposition 1. The 1-super-Poincaré inequality (1.6) with growth

$$\beta_1(\varepsilon) = C \exp(C' \varepsilon^{-1/\delta}), \quad C, C' > 0$$

implies the F-Sobolev inequality (1.10) with

$$F_1(x) = \log(1+x)^{\delta}$$

Theorem 2. Assume as in Theorem 1. Let d be the Carnot-Carathéodory distance and for a Borel set A define

$$A_{\varepsilon} = \{x \in \mathbb{G} \mid d(x, A) < \varepsilon\} \text{ and } \mu_p^+(A) = \liminf_{\varepsilon \to 0^+} \frac{\mu_p(A_{\varepsilon} \setminus A)}{\varepsilon}$$

the ε -neighbourhood of A and the surface measure with respect to μ_p of A respectively. Then the isoperimetric profile

$$\mathcal{I}_{\mu_p}(t) = \inf\{\mu_p^+(A) \mid A \text{ is Borel and } \mu_p(A) = t\}$$

of μ_p satisfies

$$\mathcal{U}_r(t) \lesssim \mu_p^+(A), \quad r = \frac{p(1+\sigma+2\alpha)}{p-(1+\sigma+2\alpha)}$$

that is $I_{\mu_p} \gtrsim \mathcal{U}_r$, for all A such that $\mu_p(A) = t$ and where $\mathcal{U}_r = \mathcal{I}_{\nu_{r^*}}$ is the isoperimetric profile of the measure

$$d\nu_{r^*} = Z^{-1} e^{-|x|^{r^*}} dx$$

where

$$r^* = \frac{(1+\sigma+2\alpha)p}{(\sigma+2\alpha)p+(1+\sigma+2\alpha)}$$

is Hölder conjugate to r.

In other words, there exist measures μ_p on stratified Lie groups with (up to constants) the concentration, but not isoperimetric, properties of ν_p .

Remark 1. The isoperimetric profile \mathcal{U}_r of ν_{r^*} itself satisfies

$$\mathcal{U}_r(t) \gtrsim \min(t, 1-t) \log\left(\frac{1}{\min(t, 1-t)}\right)^{1-1/r^*}, \quad t \in (0, 1).$$

2. Proof of main results

The main idea of the proof follows the generalisation provided in [Qiu24, §2.3] to general step two stratified Lie groups; the estimates (1.11) provide a U-bound (1.8) of the form $\eta = |x|^s N^t$ for some s, t > 0, and then a suitable Hardy inequality allows passage to the q-super-Poincaré inequality. Unlike in [Qiu24], we will actually need the Hardy inequality as opposed to the associated uncertainty principle. We therefore start by proving a U-bound for μ_p assuming (1.11). In what follows, all integration happens over \mathbb{G} and $f \in W^{1,2}(\mu_p)$. Moreover, we write $I_1(f) \leq I_2(f)$ whenever $I_1(f) \leq CI_2(f)$ for some constant C > 0 which does not depend on f.

Lemma 2. Given (1.11), it holds

$$\int |x|^{\sigma+2\alpha} N^{p-1-\sigma-2\alpha} |f| d\mu_p \lesssim \int |\nabla_{\mathbb{G}} f| d\mu_p + \int |f| d\mu_p$$

where

$$\sigma = \max(-2\alpha, -\beta, -\gamma, -\delta + 1)$$

Proof. The proof follows the integration by parts strategy of [HZ10]. Assume without loss of generality $f \geq 0$. We can develop the integral with respect to $d\xi$ of $\nabla_{\mathbb{G}}(fe^{-U}) \cdot |x|^{\varepsilon} N^{\zeta} \nabla_{\mathbb{G}} N$ with $\varepsilon, \zeta \in \mathbb{R}$ in two ways. Let $V = N^p$. We may either expand the subgradient and obtain

$$\int \nabla_{\mathbb{G}} (fe^{-V}) \cdot \nabla_{\mathbb{G}} |x|^{\varepsilon} N^{\zeta} \nabla_{\mathbb{G}} N d\xi = \int \nabla_{\mathbb{G}} f |x|^{\varepsilon} N^{\zeta} \nabla_{\mathbb{G}} N d\mu_p - \int f |x|^{\varepsilon} N^{\zeta} \nabla_{\mathbb{G}} V \cdot \nabla_{\mathbb{G}} N d\mu_p$$

or we may integrate by parts and obtain

$$\int \nabla_{\mathbb{G}} (fe^{-V}) \cdot |x|^{\varepsilon} N^{\zeta} \nabla_{\mathbb{G}} N d\xi = -\int f \nabla_{\mathbb{G}} \cdot (|x|^{\varepsilon} N^{\zeta} \nabla_{\mathbb{G}} N) d\mu_{p}.$$

Equating, we find

(2.1)
$$\int f |x|^{\varepsilon} N^{\zeta} \nabla_{\mathbb{G}} V \cdot \nabla_{\mathbb{G}} N d\mu_{p} = \int \nabla_{\mathbb{G}} f |x|^{\varepsilon} N^{\zeta} \nabla_{\mathbb{G}} N d\mu_{p} + \int f \nabla_{\mathbb{G}} \cdot (|x|^{\varepsilon} N^{\zeta} \nabla_{\mathbb{G}} N) d\mu_{p}.$$

On the left hand side of (2.1) we have $\nabla_{\mathbb{G}} V \cdot \nabla_{\mathbb{G}} N = \nabla_{\mathbb{G}} N^p \cdot \nabla_{\mathbb{G}} N = p N^{p-1} |\nabla_{\mathbb{G}} N|^2 \gtrsim N^{p-1} |x|^{2\alpha} N^{-2\alpha}$ and therefore

(2.2)
$$\int f |x|^{\varepsilon} N^{\zeta} \nabla_{\mathbb{G}} V \cdot \nabla_{\mathbb{G}} N d\mu_p \gtrsim \int f |x|^{\varepsilon + 2\alpha} N^{\zeta + p - 1 - 2\alpha} d\mu_p$$

On the right hand side of (2.1) and using the estimates (1.11), the first addend is controlled just by taking absolute values inside the integrand

(2.3)
$$\int \nabla_{\mathbb{G}} f |x|^{\varepsilon} N^{\zeta} \nabla_{\mathbb{G}} N d\mu_{p} \lesssim \int |\nabla_{\mathbb{G}} f| |x|^{\varepsilon+\beta} N^{\zeta-\beta} d\mu_{p}$$

while the second addend is expanded to find

(2.4)
$$\int f \nabla_{\mathbb{G}} \cdot (|x|^{\varepsilon} N^{\zeta} \nabla_{\mathbb{G}} N) d\mu_{p} \lesssim \int f |x|^{\varepsilon + \delta - 1} N^{\zeta - \delta} d\mu_{p} + \int f |x|^{\varepsilon + 2\alpha} N^{\zeta - 1 - 2\alpha} d\mu_{p} + \int f |x|^{\varepsilon + \gamma} N^{\zeta - 1 - \gamma} d\mu_{p}.$$

The weight $|x|^{\varepsilon+\beta} N^{\zeta-\beta}$ in (2.3) must satisfy $\varepsilon + \beta \geq 0$ and be equal to $\mathcal{O}(1/N^{\theta})$ for some $\theta \geq 0$. Clearly if $\theta = 0$ it is automatic the right hand side of (2.3) is controlled by $\int |\nabla_{\mathbb{G}} f| d\mu_p$, while if $\theta > 0$ we can glue the U-bound restricted to a function supported outside the unit ball B_1 with respect to N where $1/N^{\theta} \leq 1$, with the trivial U-bound $\int_{B_1} f\eta d\mu_p \leq \int f d\mu_p$ inside B_1 , by compactness of B_1 and continuity of $\eta = |x|^{\varepsilon+2\alpha} N^{\zeta+p-1-2\alpha}$. This implies that $\varepsilon \geq -\beta$ and $\zeta \leq -\varepsilon$. The formality of the decomposition $\mathbb{G} = B_1 \sqcup B_1^c$ can be resolved modulo constants by writing $f = f\chi + f(1-\chi)$ where $\chi = \min(1, \max(2-N, 0))$ satisfies $|\nabla_{\mathbb{G}}\chi| \leq 1$, see also [Wan00; Wan02; Cat+09; HZ10].

The remaining weights can be handled similarly. $|x|^{\varepsilon+\delta-1} N^{\zeta-\delta}$ for instance must satisfy $\varepsilon + \delta - 1 \ge 0$ and be again equal to $\mathcal{O}(1/N^{\theta})$ for some $\theta \ge 0$ which implies by our previous argument $\varepsilon \ge -\delta + 1$ and $\zeta \le -\varepsilon + 1$. In short, we conclude

$$\varepsilon \ge \max(-\beta, -\delta + 1, -2\alpha, -\gamma) \text{ and } \zeta \le \min(-\varepsilon, -\varepsilon + 1, -\varepsilon + 1, -\varepsilon + 1) = -\varepsilon.$$

To make the U-bound as large as possible, we choose ε and ζ as small and large respectively as possible. Thus we arrive at the U-bound

$$\int f |x|^{\sigma+2\alpha} N^{p-1-\sigma-2\alpha} d\mu_p \lesssim \int |\nabla_{\mathbb{G}} f| d\mu_p + \int f d\mu_p$$
$$= \max(-\beta, -\delta + 1, -2\alpha, -\gamma) \text{ and } \zeta = -\varepsilon.$$

Remark 2. There is perhaps some scope for improvement in these U-bounds in special cases where the estimated can be improved. For instance, if either c or C is possible, then the first

where the estimates can be improved. For instance, if either ε or ζ is negative, then the first two terms in (2.4) come with negative signs. In particular the second term is automatically controlled since it is nonnegative, while the first term can also be controlled in cases such as a Heisenberg group where $\nabla_{\mathbb{G}} |x| \cdot \nabla_{\mathbb{G}} N = |x|^3 N^{-3}$ is also nonnegative.

Lemma 3. Given (1.11) and if $\delta \geq \sigma + 2\alpha$, it holds for all r > 0 and sufficiently small $\varepsilon > 0$ that

$$\int |f| \, d\mu_p \lesssim \varepsilon \int |\nabla_{\mathbb{G}} f| \, d\mu_p + \varepsilon^{-r} \int |x|^r \, |f| \, d\mu_p.$$

Proof. We follow the proof of [Qiu24, Lemma 4] and start by proving an L^1 -Hardy inequality for μ with respect to the horizontal norm |x|. Consider

$$\int |f| \nabla_{\mathbb{G}} \cdot h d\mu_p \leq \int |\nabla_{\mathbb{G}} f| |h| d\mu_p + \int |f| \nabla_{\mathbb{G}} V \cdot h d\mu_p$$

by taking $\varepsilon = \sigma$

for h a vector field of the same length as $\nabla_{\mathbb{G}}$. With $h = \frac{x}{|x|} = \nabla_{\mathbb{G}} |x|$ we find

$$(d-1)\int \frac{|f|}{|x|}d\mu_p \lesssim \int |\nabla_{\mathbb{G}}f| \, d\mu_p + \int |f| \, \nabla_{\mathbb{G}}V \cdot \nabla_{\mathbb{G}} |x| \, d\mu_p$$
$$\lesssim \int |\nabla_{\mathbb{G}}f| \, d\mu_p + \int |f| \, N^{p-1}\nabla_{\mathbb{G}}N \cdot \nabla_{\mathbb{G}} |x| \, d\mu_p$$
$$\lesssim \int |\nabla_{\mathbb{G}}f| \, d\mu_p + \int |x|^{\delta} \, N^{p-1-\delta} \, |f| \, d\mu_p$$

where $d = \dim(\mathfrak{g}_0) = \dim(\mathbb{R}_0)$ is the horizontal dimension. We would like to control the weight $|x|^{\delta} N^{p-1-\delta}$ in the second addend by the U-bound $\eta = |x|^{\sigma+2\alpha} N^{p-1-\sigma-2\alpha}$, possibly up to a constant plus an arbitrarily small factor of $|x|^{-1}$ which can be carried over to the left hand side. If $\delta \geq \sigma + 2\alpha$, this is automatic since we have the estimate $|x| \leq N$, since |x| can be controlled by a euclidean-like norm (see [BLU07, Example 5.1.2]) and all homogeneous norms are equivalent (see [BLU07, §5]), while if $\delta < \sigma + 2\alpha$, this is impossible since $N^{p-1-\delta}$ grows faster than $N^{p-1-\sigma-2\alpha}$ at infinity and |x| can be fixed away from zero independently of $N \to \infty$. It follows we require $\delta \geq \sigma + 2\alpha$, in which case

$$(d-1)\int \frac{|f|}{|x|}d\mu_p \lesssim \int |\nabla_{\mathbb{G}}f|\,d\mu_p + \int |f|\,d\mu_p$$

The apparent issue at d = 1 can be sidestepped by the fact a nontrivial stratified Lie group \mathbb{G} can never have horizontal dimension d = 1 as otherwise the Lie algebra \mathfrak{g} of \mathbb{G} is generated by a single left invariant vector field X which is impossible.

By the scalar inequality

$$|x|^r + \frac{1}{|x|} \gtrsim 1 \implies \varepsilon^{-r} |x|^r + \frac{\varepsilon}{|x|} \gtrsim 1,$$

we obtain

$$\int |f| d\mu_p \lesssim \varepsilon \int \frac{|f|}{|x|} d\mu_p + \varepsilon^{-r} \int |x|^r |f| d\mu_p \lesssim \varepsilon \int |\nabla_{\mathbb{G}} f| d\mu + \varepsilon^{-r} \int |x|^r |f| d\mu_p$$

y $\varepsilon > 0.$

for any $\varepsilon > 0$.

We now conclude with the proof of the main results.

Proof of Theorem 1. We follow again the arguments of [Qiu24]. It is possible by the L^1 -Sobolev inequality [VSC91, Theorem IV.7.1]

(2.5)
$$\left(\int |f|^{Q/(Q-1)} d\xi\right)^{(Q-1)/Q} \lesssim \int |\nabla_{\mathbb{G}} f| d\xi, \quad f \in C_c^{\infty}(\mathbb{G}),$$

where Q is the homogeneous dimension of \mathbb{G} , together with Hölder interpolation to prove a 1-super-Poincaré inequality with respect to Lebesgue measure

(2.6)
$$\int |f| d\xi \lesssim \varepsilon \int |\nabla_{\mathbb{G}} f| d\xi + \widetilde{\beta}_1(\varepsilon) \left(\int |f|^{1/2} d\xi \right)^2$$

where $\beta_1(\varepsilon) = C(1 + \varepsilon^{-Q})$ for some C > 0. Assume $f \ge 0$ without loss of generality. Let B_R be the N-ball of radius R. On B_R , by compactness, we apply (2.6) to fe^{-N^p} to find

$$(2.7) \qquad \int_{B_R} |f| \, d\mu_p \lesssim \delta \int_{B_R} |\nabla_{\mathbb{G}} f| \, d\mu_p + \int_{B_R} |\nabla_{\mathbb{G}} N^p| \, |f| \, d\mu_p + \widetilde{\beta}_1(\delta) \left(\int_{B_R} |f|^{1/2} \, e^{-N^p/2} d\xi \right)^2$$
$$(2.7) \qquad \qquad \lesssim \delta \int |\nabla_{\mathbb{G}} f| \, d\mu_p + \delta R^{p-1} \int |f| \, d\mu_p + \widetilde{\beta}_1(\delta) \sup_{B_R} e^{N^p} \left(\int |f|^{1/2} \, d\mu_p \right)^2$$
$$\lesssim \delta \int |\nabla_{\mathbb{G}} f| \, d\mu_p + \delta R^{p-1} \int |f| \, d\mu_p + \widetilde{\beta}_1(\delta) e^{R^p} \left(\int |f|^{1/2} \, d\mu_p \right)^2.$$

since $|\nabla_{\mathbb{G}}N| \leq 1$ under (1.11). We will choose R to be a suitable negative power of $\varepsilon > 0$ sufficiently small, and then choose δ to be a positive power of ε such that δ and δR^{p-1} are both comparable to ε . As a result, the dependence on δ will not play any role in the leading asymptotics which are controlled by e^{R^p} .

To determine the appropriate scale of R, on B_R^c we apply the Hardy inequality with $r = \sigma + 2\alpha$ to obtain

$$(2.8) \qquad \int_{B_R^c} |f| \, d\mu_p \lesssim \varepsilon \int_{B_R^c} |\nabla_{\mathbb{G}} f| \, d\mu_p + \varepsilon^{-(\sigma+2\alpha)} \int_{B_R^c} |x|^{\sigma+2\alpha} \, |f| \, d\mu_p$$
$$\lesssim \varepsilon \int |\nabla_{\mathbb{G}} f| \, d\mu_p + \varepsilon^{-(\sigma+2\alpha)} R^{-(p-1-\sigma-2\alpha)} \int |x|^{\sigma+2\alpha} \, N^{p-1-\sigma-2\alpha} \, |f| \, d\mu_p$$
$$\lesssim \varepsilon \int |\nabla_{\mathbb{G}} f| \, d\mu_p + \varepsilon^{-(\sigma+2\alpha)} R^{-(p-1-\sigma-2\alpha)} \left(\int |\nabla_{\mathbb{G}} f| \, d\mu_p + \int |f| \, d\mu_p \right)$$

by the U-bound of Lemma 2. It suffices to choose R such that

$$\varepsilon^{-(\sigma+2\alpha)}R^{-(p-1-\sigma-2\alpha)} = \varepsilon,$$

that is

(2.9)
$$R = \varepsilon^{-\tau}, \quad \tau = \frac{1 + \sigma + 2\alpha}{p - 1 - \sigma - 2\alpha}$$

We conclude by adding (2.7) and (2.8) which yields the expected 1-super-Poincaré inequality with growth

$$\beta_1(\varepsilon) \lesssim \exp\left(C\varepsilon^{-p(1+\sigma+2\alpha)/(p-1-\sigma-2\alpha)}\right), \quad C > 0.$$

The formality of the decomposition $\mathbb{G} = B_R \sqcup B_R^c$ is addressed using the same argument appearing in the proof of Lemma 2.

Proof of Proposition 1. The proof follows [Wan00, Theorem 3.2] except we replace

(1) $\mu(f^2) = 1$ with $\mu(f) = 1$, (2) $A_n = \{\delta^{n+1} > f^2 \ge \delta^n\}$ with $A_n = \{\delta^{n+1} > f \ge \delta^n\}$, and (3) $f_n = (f - \delta^{n/2}) \wedge (\delta^{(n+1)/2} - \delta^{n/2})$ with $f_n = (f - \delta^n) \wedge (\delta^{n+1} - \delta^n)$. The proof follows in exactly the same way up until the second lower bound for $\mu(|\nabla f|^2)$ where our analogue is

$$\mu(|\nabla_{\mathbb{G}}f|) \ge \sum_{n=0}^{\infty} \xi(\delta^n) \mu(f \ge \delta^{n+1}) (\delta^{n+1} - \delta^n)$$

meaning we obtain the same function F as in the case of the 2-super-Poincaré inequality but with different constants, namely c_1 is not $(\delta^{(n+1)/2} - \delta^{n/2})^2/(\delta^n - \delta^{n-1})$ but instead

$$c_1 = \frac{\delta^{n+1} - \delta^n}{\delta^n - \delta^{n-1}} = \delta.$$

It follows [Wan00, Corollary 3.3] holds with the same F, that is the 1-super-Poincaré inequality with growth $\beta_1(\varepsilon) \leq \exp(C\varepsilon^{-1/\tau})$ for some $C, \tau > 0$ implies the F-Sobolev inequality

$$\int |f| F_1(|f|) d\mu \le c_1 \int |\nabla_{\mathbb{G}} f| d\mu + c_2, \quad \int |f| d\mu = 1$$

for some $c_1, c_2 > 0$ and

$$F_1(x) = \log(1+x)^{\tau}.$$

Proof of Theorem 2. The *F*-Sobolev inequality implies by renormalisation a defective 1-logarithmic^{θ} Sobolev inequality of the form

$$\int |f| \log \left(1 + \frac{|f|}{\int |f| \, d\mu_p}\right)^{\theta} d\mu_p \lesssim \int |\nabla_{\mathbb{G}} f| \, d\mu_p + \int |f| \, d\mu_p$$

where

$$\theta = \frac{p - 1 - \sigma - 2\alpha}{p(1 + \sigma + 2\alpha)}$$

and which, in the language of [IKZ11], is a defective (meaning with an additional $\int |f| d\mu_p$ term on the right hand side) $L^1\Phi$ -entropy inequality for $\Phi(x) = x \log(1+x)^{\theta}$. To apply part (ii) of [IKZ11, Theorem 4.5], we also need a Cheeger inequality

$$\int \left| f - \int f d\mu_p \right| d\mu_p \lesssim \int \left| \nabla_{\mathbb{G}} f \right| d\mu_p.$$

By [IKZ11, Theorem 2.6] it suffices, given a local Cheeger inequality for Lebesgue measure $d\xi$, to exhibit a U-bound

(2.10)
$$\int N^{pr} |f| d\mu_p \lesssim \int |\nabla_{\mathbb{G}} f| d\mu_p + \int |f| d\mu_p$$

for any r > 0. But this is already provided by the U-bound and the L^1 -Hardy inequality together with the scalar inequality

$$cx^p + \frac{1}{x^q} \gtrsim c^{q/(q+p)}$$

for all c, x > 0, that is

(2.11)
$$\int N^{(p-1-\sigma-2\alpha)/(1+\sigma+2\alpha)} |f| d\mu_p \lesssim \int \frac{|f|}{|x|} d\mu_p + \int |x|^{\sigma+2\alpha} N^{p-1-\sigma-2\alpha} |f| d\mu_p$$
$$\lesssim \int |\nabla_{\mathbb{G}} f| d\mu_p + \int |f| d\mu_p.$$

The result follows from part (iii) of [IKZ11, Theorem 4.5] with $q = 1/\theta$, that is μ_p has the isoperimetric profile of $d\nu_{r^*}$ where

$$r^* = \frac{q}{q-1} = \frac{p(1+\sigma+2\alpha)}{p(1+\sigma+2\alpha) - (p-(1+\sigma+2\alpha))} = \frac{(1+\sigma+2\alpha)p}{(\sigma+2\alpha)p + (1+\sigma+2\alpha)}$$

is Hölder conjugate to q.

To complete the argument we note [IKZ11, Theorem 2.6] requires in addition to (2.10) also a local L^1 -Poincaré inequality for Lebesgue measure, which is provided by [Jer86, Theorem 2.1].

3. Examples

In the following examples, we will usually speak of μ_p as having the isoperimetric profile of ν_{r^*} , despite the fact the isoperimetric inequality we have established provides only a lower bound for the isoperimetric profile. That is, we have shown $\mathcal{I}_{\mu_p}(t) \gtrsim \mathcal{U}_r(t)$ but we do not know if $\mathcal{I}_{\mu_p}(t) \leq \mathcal{U}_r(t)$ as well.

3.1. Stratified Lie groups of step two. If $\mathbb{G} \cong \mathbb{R}^n_x \times \mathbb{R}^m_z$ is a step two stratified Lie group defined in [DZ21, §2] and

$$N(x,z) = (|x|^4 + a |z|^2)^{1/4}$$

is a homogeneous norm with parameter a > 0, then N satisfies (1.11) with $\alpha = \beta = 1$, $\gamma = 2$, and $\delta = 3$ according to [DZ21, Lemma 2]. When a = 16 and \mathbb{G} is a *H*-type group in the sense of [BLU07, Chapter 18], this is the Kaplan norm after [Kap80, Theorem 2], that is $N^{2-Q(\mathbb{G})}$ is the fundamental solution to the sublaplacian $\Delta_{\mathbb{G}}$. The conditions are readily checked: $\sigma = -1$ and $\delta = 3 \ge 1 = \sigma + 2\alpha$, so for $p > 1 + \sigma + 2\alpha = 2$ the measure

 $d\mu_p = Z^{-1} e^{-N^p} d\xi$

has the isoperimetric profile of $d\nu_{r^*} = Z^{-1} e^{-d^{r^*}} d\xi$ where

$$r^* = \frac{(1 + \sigma + 2\alpha)p}{(\sigma + 2\alpha)p + (1 + \sigma + 2\alpha)} = \frac{2p}{p+2}.$$

In other words these measures have supergaussian decay of tails but a subgaussian isoperimetric profile (which is only asymptotically gaussian as $p \to \infty$). This result covers Heisenberg groups, *H*-type groups, and Métivier groups.

Remark 3. This is what we mean in the introduction by the predictive power of the q-super-Poincaré inequality, q Hölder conjugate to p; not only does the growth $\beta_q(\varepsilon) \leq \exp(C\varepsilon^{-2(p-1)/(p-2)})$ imply a 2-super-Poincaré inequality for μ_p with, since $p/(p-2) = 2(p-1)/(p-2) \cdot q/2$, the same growth $\beta_2(\varepsilon) \leq \exp(C\varepsilon^{-p/(p-2)})$ as that which appears in the 2-super-Poincaré inequality for ν_{r^*} , but it also correctly predicted, since $2p/(p-2) = 2(p-1)/(p-2) \cdot q/1$, the growth $\beta_1(\varepsilon) \leq \exp(C\varepsilon^{-2p/(p-2)})$ of the 1-super-Poincaré inequality for μ_p .

3.2. Grushin and Heisenberg-Greiner sublaplacians. According to [Qiu24], the Kaplan norms in the Grushin and Heisenberg-Greiner settings, which respectively generalise the euclidean and Heisenberg settings, satisfy the estimates (1.11).

The Grushin subgradient (see for instance [DAm04]) is defined as the operator

$$\nabla_{\eta} = (\nabla_x, |x|^{\eta} \nabla_y), \quad \eta > 0,$$

acting on $\mathbb{R}^n_x \times \mathbb{R}^m_y$. The Grushin sublaplacian $\Delta_\eta = \nabla_\eta \cdot \nabla_\eta = \Delta_x + |x|^{2\eta} \Delta_y$ is not a sublaplacian on a stratified Lie group, but it shares some similarities. For instance, Δ_η generalises the euclidean laplacian at $\eta = 0$, is homogeneous of order 2 with respect to the anisotropic dilations $\delta_\lambda(x, y) = (\lambda x, \lambda^{1+\eta} y)$, and is hypoelliptic for $\eta > 0$, a result due to [Gru70, Theorem 1.2] and known already for $\eta \in 2\mathbb{Z}_{\geq 1}$ by Hörmander's condition. As before, for a function f depending only on x, the action of Δ_η is the euclidean one. It admits a Kaplan norm

$$N = (|x|^{2(1+\eta)} + (1+\eta)^2 |y|^2)^{1/(2+2\eta)}$$

which is the fundamental solution for Δ_{η} and satisfies (1.11) with $\alpha = \beta = \eta$, $\gamma = 2\eta$, and $\delta = 2\eta + 1$, and so the conditions are again checked with $\sigma = -\eta$ and so for $p > 1 + \eta$ the measure

$$d\mu_p = Z^{-1} e^{-N^p} d\xi$$

has the isoperimetric profile of $d\nu_{r^*} = Z^{-1} e^{-d^{r^*}} d\xi$ where

(3.1)
$$r^* = \frac{(1+\eta)p}{(1+\eta)p - (p - (1+\eta))} = \frac{(1+\eta)p}{\eta p + (\eta + 1)}$$

provided the "horizontal" dimension satisfies $n \ge 2$, so that the L^1 -Hardy inequality holds. Note the distance d here implicit in the definition of the isoperimetric profile is the Carnot-Carathéodory distance induced by the vector fields comprising ∇_{η} .

Similarly, the Heisenberg-Greiner operator (see for instance [DAm05, §2]) is defined as the operator

$$\nabla_{\zeta} = (X_{\zeta}^1, \cdots, X_{\zeta}^n, Y_{\zeta}^1, \cdots, Y_{\zeta}^n), \quad \zeta \ge 1,$$

acting on $\mathbb{R}^n_x \times \mathbb{R}^n_y \times \mathbb{R}^1_z$ and where

$$X_i^{\zeta} = \partial_{x_i} + 2\zeta y_i r^{2\zeta - 2} \partial_z, \quad Y_i^{\zeta} = \partial_{y_i} - 2\zeta x_i r^{2\zeta - 2} \partial_z, \quad r = |(x, y)|.$$

The Heisenberg-Greiner sublaplacian $\Delta_{\zeta} = \nabla_{\zeta} \cdot \nabla_{\zeta} = \sum_{i=1}^{n} X_i^2 + Y_i^2$ generalises the Heisenberg laplacian at $\zeta = 1$. It also admits a Kaplan norm

$$N(x, y, z) = ((|x|^2 + |y|^2)^{2\zeta} + z^2)^{1/(4\zeta)} = (r^{4\zeta} + z^2)^{1/(4\zeta)}$$

which is the fundamental solution for Δ_{ζ} and satisfies (1.11) with $\alpha = \beta = 2\zeta - 1$, $\gamma = 4\zeta - 2$, and $\delta = 4\zeta - 1$, and so for $\sigma = -(2\zeta - 1)$ and $p > 2\zeta$ the measure

$$d\mu_p = Z^{-1} e^{-N^p} d\xi$$

has the isoperimetric profile of $d\nu_{r^*} = Z^{-1} e^{-d^{r^*}} d\xi$ where

$$r^* = \frac{2\zeta p}{(2\zeta - 1)p + 2\zeta}$$

Note both of these cases recover the isoperimetric inequality in the step two stratified Lie group setting at $\eta = \zeta = 1$. Note also the case $\eta = 0$, which corresponds to the euclidean setting, recovers the expected isoperimetric inequality, that is $r^* = p$ itself.

To complete the argument, since the Grushin and Heisenberg-Greiner sublaplacians fall outside the scope of [VSC91], the existence of the L^1 -Sobolev inequality and the local L^1 -Poincaré inequality needed for [IKZ11, Theorem 2.6] are a priori suspect. However, these are valid for certain parameters, for instance at least when we have $\eta \in 2\mathbb{Z}_{\geq 1}$ and $\zeta \in \mathbb{Z}_{\geq 1}$, in which case the constituent vector fields satisfy Hörmander's condition; the L^1 -Sobolev inequality follows from [CDG94, Theorem 1.1] while the L^1 -Poincaré inequality again follows from [Jer86, Theorem 2.1], see also [FLW95, Theorem 1]. In fact in the Grushin case, [CDG94] proved the L^1 -Sobolev inequality for all $\eta > 0$ while [FGW94, Theorem 1] proved (a stronger version of) the L^1 -Poincaré inequality, so there is a full family of probability measures with arbitrarily fast (supergaussian) decay of tails but apparently with an isoperimetric profile which is arbitrarily close to linear (but still superexponential).

Remark 4. The case $\eta \in (0, 1)$ may also be of some interest. If $\eta \ge 1$ we see by (3.1) that $r^* = r^*(\eta, p) < 2$ for all $p > 1 + \eta$ and therefore μ_p never achieves gaussian isoperimetry. However, if $\eta < 1$ and $p \ge 2(1 + \eta)/(1 - \eta)$, then $r^*(\eta, p) \ge 2$ which then, according to the fourth and fifth parts of [IKZ11, Theorem 4.5], implies μ_p not only achieves (super)gaussian isoperimetry, but also satisfies the q-logarithmic Sobolev inequality (1.2) and the Bobkov-type functional isoperimetric inequalities of [Bob96; Bob97].

Remark 5. We return to the statement made in the introduction that the arguments can be greatly simplified. Recall in the proof of the Cheeger inequality we proved a *U*-bound of the form

$$\int N^{pr} |f| d\mu_p \lesssim \int |\nabla_{\mathbb{G}} f| d\mu_p + \int |f| d\mu_p$$

and observe that in all of the previous examples we had that the estimates provide the U-bound

$$\int |x|^{\alpha} N^{p-1-\alpha} |f| d\mu_p \lesssim \int |\nabla_{\mathbb{G}} N^p| |f| d\mu_p \lesssim \int |\nabla_{\mathbb{G}} f| d\mu_p + \int |f| d\mu_p.$$

These two inequalities automatically yield the expected isoperimetric content by [IKZ11, Theorems 2.1, 2.7, and 4.5] providing respectively the defective $L^1\Phi$ -entropy, Cheeger, and isoperimetric inequalities.

3.3. The case of horizontal dimension n = 1. In [CFZ23] the *q*-Poincaré inequality was proved for a probability measure of the form (1.1) on the Engel group $\mathcal{B}_4 \cong (\mathbb{R}^4, \circ)$ and more generally on a stratified Lie group $\mathbb{G} = \mathbb{G}_{n+1} \cong (\mathbb{R}^{n+1}, \circ)$ of step $n \geq 3$ with filiform Lie algebra \mathfrak{g}_{n+1} , meaning \mathfrak{g}_{n+1} is generated by two vector fields X_1, X_2 and higher strata are spanned by a single vector field. The proof made use of a U-bound which vanishes on $\{|x_1|=0\}$ together with ideas from [Ing12].

The main objects are

(1)

$$\|x\|^{n} = \sum_{j=2}^{n} \left(|x_{1}|^{(n+1)/2} + |x_{2}|^{(n+1)/2} + |x_{j}|^{(n+1)/(2(j-1))} \right)^{2n/(n+1)},$$
(2) $N(x) = (\|x\|^{n} + |x_{n+1}|)^{1/n},$
(3) $X_{1} = \partial_{x_{1}},$
(4) $X_{2} = \partial_{x_{2}} + x_{1}\partial_{x_{3}} + \frac{x_{1}^{2}}{2}\partial_{x_{4}} + \dots + \frac{x_{1}^{n-1}}{(n-1)!}\partial_{x_{n+1}}.$

What is interesting is that the proof of the U-bound does not involve taking the full subgradient but only the X_1 -derivative. Indeed, for $f \ge 0$,

$$\int X_1(fe^{-N^p}) \cdot X_1 |x_1| d\xi = \int X_1(f) X_1 |x_1| d\mu - \int f N^{p-1} X_1 N \cdot X_1 |x_1| d\mu$$
$$= -\int f X_1^2 |x_1| d\mu.$$

The integral in the second equality vanishes since $X_1^2 |x_1| = 0$ in the distributional sense, while in the first equality we have

$$|X_1(f)| \le |\nabla_{\mathbb{G}} f|, \quad |X_1| x_1|| \le 1, \quad X_1 N \cdot X_1 |x_1| \gtrsim |x_1|^{(n-1)/2} ||x||^{(n-1)/2} N^{-(n-1)/2}$$

by [CFZ23, Equation 4.4], giving

(3.2)
$$\int |x_1|^{n-1} N^{p-n} |f| d\mu_p \lesssim \int |x_1|^{(n-1)/2} ||x||^{(n-1)/2} N^{p-n} |f| d\mu_p \lesssim \int |\nabla_{\mathbb{G}} f| d\mu_p.$$

We do not know if the absence of the L^1 -Hardy inequality in dimension 1 is technical in the sense it can still be sidestepped and we can still expect formally there is also the U-bound

(3.3)
$$\frac{1}{|x_1|} + |x_1|^{n-1} N^{p-n} \gtrsim N^{(p-n)/n}$$

or if the U-bound, and therefore the subsequent isoperimetric inequality, should be different. With our current method, we have a partial answer which does not fall within the purview of the previous framework but is still based on the same types of estimates.

If ψ is a *smooth* homogeneous norm, then $X_1\psi = \mathcal{O}(1)$ and $X_1^2\psi = \mathcal{O}(1/\psi)$ away from zero. If $\delta \in (0, 1)$, we have

$$X_1 \frac{|x_1|^{1+\delta}}{\psi^{\delta}} = (1+\delta) \frac{|x_1|^{\delta}}{\psi^{\delta}} X_1 |x_1| - \delta \frac{|x_1|^{1+\delta}}{\psi^{1+\delta}} X_1 \psi \lesssim 1,$$

and

$$\begin{aligned} X_1^2 \frac{|x_1|^{1+\delta}}{\psi^{\delta}} &= \delta(1+\delta) \frac{|x_1|^{-1+\delta}}{\psi^{\delta}} + |x_1|^{1+\delta} X_1^2(\psi^{-\delta}) - 2\delta(1+\delta) \frac{|x_1|^{\delta}}{\psi^{1+\delta}} X_1 |x_1| \cdot X_1 \psi \\ &= \delta(1+\delta) \frac{1}{|x_1|^{1-\delta} \psi^{\delta}} + \mathcal{O}\left(\frac{1}{\psi}\right). \end{aligned}$$

Taking $\psi = N$, we arrive at

$$\int fX_1^2 \frac{|x_1|^{1+\delta}}{N^{\delta}} d\mu_p \gtrsim \delta(1+\delta) \int \frac{f}{|x_1|^{1-\delta} N^{\delta}} d\mu_p + \mathcal{O}\left(\int \frac{f}{N} d\mu_p\right)$$

$$\int X_1(fe^{-N^p}) \cdot X_1 \frac{|x_1|^{1+\delta}}{N^{\delta}} d\xi = \int X_1 f \cdot X_1 \frac{|x_1|^{1+\delta}}{N^{\delta}} d\mu_p - \int fN^{p-1} X_1 N \cdot X_1 \frac{|x_1|^{1+\delta}}{N^{\delta}} d\mu_p$$

$$\lesssim \int |\nabla_{\mathcal{B}_4} f| d\mu_p + \int fN^{p-1} |X_1 N| d\mu_p$$

$$\lesssim \int |\nabla_{\mathcal{B}_4} f| d\mu_p + \int f |x_1|^{(n-1)/2} ||x||^{(n-1)/2} N^{p-n} d\mu_p$$

$$\lesssim \int |\nabla_{\mathcal{B}_4} f| d\mu_p + \int f d\mu_p$$

It follows

$$\delta(1+\delta) \int \frac{f}{|x_1|^{1-\delta} N^{\delta}} d\mu_p \lesssim \int |\nabla_{\mathcal{B}_4} f| \, d\mu_p + \int |f| \, d\mu_p$$

from which we obtain via (3.2) the U-bound

$$\delta(1+\delta) \frac{1}{|x_1|^{1-\delta} N^{\delta}} + |x_1|^{n-1} N^{p-n} \gtrsim c(\delta) N^{(p-n)/n-q(\delta)}$$

where $c(\delta), q(\delta) \to 0$ as $\delta \to 0^+$. That is, we almost have an L^1 -Hardy inequality and (3.3). We can show by the previous arguments that μ_p satisfies the isoperimetric inequality

(3.4)
$$\mathcal{U}_{pn/(p-n)+q(\delta)}(\mu_p(A)) \lesssim \tilde{c}(\delta)\mu_p^+(A)$$

where $\tilde{c}(\delta) \to \infty$ as $\delta \to 0^+$, that is asymptotically up to a constant which blows up, μ_p has the isoperimetric profile of $d\nu_{r^*} = Z^{-1}e^{-d^{r^*}}d\xi$ for $r = \frac{pn}{p(n-1)+n}$. This can be seen by following the proof of [IKZ11, Theorem 4.5]; the constants on the right hand side of the isoperimetric inequality depend on the constant in the $L^1\Phi$ -entropy inequality linearly up to another constant depending on $pn/(p-n) + q(\delta)$ which can be uniformly bounded for $\delta \in (0, 1)$.

A similar argument can be applied to the homogeneous norms on stratified Lie groups satisfying the conditions of [CFZ22, Lemma 4.6], because the estimates are essentially the same (and the *U*-bound again vanishes on the zero set of a single horizontal coordinate). Indeed, there is a horizontal coordinate x_i and a horizontal vector field $X_i = \partial_{x_i}$ for which the homogeneous norm N satisfies

$$|X_1N| \lesssim \frac{|x_i|^{\eta-1}}{N^{\eta-1}} \lesssim X_i N \cdot X_i |x_i|$$

for some $\eta \geq 2$. The rest of the arguments go through with η replacing n, giving another isoperimetric inequality of the form (3.4).

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10	
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18

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