

INVARIANTS AND AUTOMORPHISMS FOR SLICE REGULAR FUNCTIONS

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ABSTRACT. Let A be one of the following Clifford algebras : $\mathbb{R}_2 \cong \mathbb{H}$ or \mathbb{R}_3 . For the algebra A , the automorphism group $Aut(A)$ and its invariants are well known. In this paper we will describe the invariants of the automorphism group of the algebra of slice regular functions over A .

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1. INTRODUCTION

The theory of slice regular functions was introduced by G. Gentili and D. Struppa in two seminal papers in 2006 [GS06] and in [GS07] : they used the fact that $\forall I \in \mathbb{S}_{\mathbb{H}} = \{J \in \mathbb{H} \mid J^2 = -1\}$ the real subalgebra \mathbb{C}_I generated by 1 and I is isomorphic to \mathbb{C} and they decomposed the algebra \mathbb{H} into a "book-structure" via these complex "slices" :

$$\mathbb{H} = \cup_{I \in \mathbb{S}_{\mathbb{H}}} \mathbb{C}_I$$

On an open set $\Omega \subset \mathbb{H}$, they defined a differentiable function $f: \Omega \rightarrow \mathbb{H}$ to be (Cullen or) slice regular if, for each $I \in \mathbb{S}$, the restriction of f to $\Omega_I = \Omega \cap \mathbb{C}_I$ is a holomorphic function from Ω_I to \mathbb{H} , both endowed with the complex structure defined by left multiplication with I . This definition covers all functions given by convergent power series of the form :

$$\sum_{n \in \mathbb{N}_0} q^n a_n$$

with $\{a_n\}_{n \in \mathbb{N}_0} \subset \mathbb{H}$.

Later on, the approach introduced by R. Ghiloni and A. Perotti in 2011, [GP11], for an alternative $*$ -algebra A over \mathbb{R} makes use of the complexified algebra $A \otimes_{\mathbb{R}} \mathbb{C}$, denoted by $A_{\mathbb{C}}$.

Let us denote its elements as $a + \iota b$ where $a, b \in A$ and ι is to be considered as the imaginary unit of \mathbb{C} .

For any slice regular function, and for any $I \in \mathbb{S}_{\mathbb{H}}$, the restriction $f: \mathbb{C}_I \rightarrow \mathbb{H}$ can be lifted through the map $\phi_I: \mathbb{H}_{\mathbb{C}} \rightarrow \mathbb{H}$, $\phi_I(a + \iota b) :=$

$a + Ib$ and it turns out that the lift does not depend on I . In other words, there exists a holomorphic function $F: \mathbb{C} \cong \mathbb{R}_{\mathbb{C}} \rightarrow \mathbb{H}_{\mathbb{C}}$ which makes the following diagram commutative for all $I \in \mathbb{S}_{\mathbb{H}}$.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F} & \mathbb{H}_{\mathbb{C}} \\ \phi_I \downarrow & & \downarrow \phi_I \\ \mathbb{H} & \xrightarrow{f} & \mathbb{H} \end{array}$$

After the first definitions were given, the theory of slice regular functions knew a big development : see, among the others, the following references [GSS22], [BW21b], [BDMW23], [BW21a], [BW20], [AB19a],[AB19b], [BS17], [BG18], [BDM22], [?].

Non-commutative associative division algebras admit many automorphisms, because $x \mapsto y^{-1}xy$ is an automorphism for every invertible element y .

The “essential” properties of a number or a function should not be changed by automorphisms.

In the case of the algebras under consideration here, there is an *antinvolution* $x \mapsto \bar{x}$ which commutes with all automorphisms. As a consequence, $N(x) = x\bar{x}$ and $\text{Tr}(x) = x + \bar{x}$ are *invariant* under automorphisms. In fact, for $A = \mathbb{H}$ we have the equivalence:

$$N(z) = N(w) \text{ and } \text{Tr}(z) = \text{Tr}(w) \iff \exists \phi \in \text{Aut}(A) : \phi(z) = w$$

(Here ϕ is an automorphism of A as an \mathbb{R} -algebra.)

This raises the question whether a similar correspondence holds not only for the elements in the algebra, but also for slice-regular functions of this algebra.

As it turns out, essentially this is true, but only via the associated stem functions and up to a condition on the multiplicity with which values in the center of $A_{\mathbb{C}}$ are assumed. To state the latter condition, in §6.2 we introduce the notion of a “*central divisor*” *cdiv*.

More precisely, the conjugation $x \mapsto \bar{x}$ on the algebra A defines a conjugation on the space of slice-regular functions which allows the definition of Tr and N as before.

This conjugation corresponds to a conjugation on the associated space of stem functions $F: \mathbb{C} \rightarrow A \otimes_{\mathbb{R}} \mathbb{C}$ defined as

$$F^c : z \mapsto \overline{(F(z))}$$

where $\overline{(q \otimes w)}$ is defined as $\bar{q} \otimes (w)$ (with $q \in A, w \in \mathbb{C}$). Again, conjugation induces Tr and N as

$$\text{Tr}(F) : z \mapsto (F + F^c)(z), \quad N(F) : z \mapsto (F(z))(F^c(z))$$

With these definitions, the correspondence between regular functions and stem functions is compatible with conjugation. As a consequence, if F is the stem function for f , then F^c is the stem function for f^c . Moreover $N(F)$, resp. $\text{Tr}(F)$, are the stem functions for $N(f)$, resp. $\text{Tr}(f)$.

As it turns out, essentially $N(F)$, $\text{Tr}(F)$ and $cdiv(F)$ (or, equivalently, $N(f)$, $\text{Tr}(f)$ and $cdiv(f)$) characterize F (equivalently, f) up to replacing F with $z \mapsto \phi(z) (F(z))$ for some holomorphic map ϕ from \mathbb{C} to the automorphism group of the complex algebra $A \otimes_{\mathbb{R}} \mathbb{C}$.

Here $cdiv$ is an additional invariant which we introduce in §6.2 for slice regular functions which are not slice preserving.

In this paper we describe the group of automorphisms of the algebra of slice regular functions with values in \mathbb{H} and $\mathbb{R}_3 \cong \mathbb{H} \oplus \mathbb{H}$.¹

Our Main Theorem is the following :

Theorem 1.1. *Let \mathbb{H} denote the algebra of quaternions, $\mathbb{H}_{\mathbb{C}} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$, $G = \text{Aut}(\mathbb{H}) \cong SO(3, \mathbb{R})$, $G_{\mathbb{C}} = \text{Aut}(\mathbb{H}_{\mathbb{C}}) \cong SO(3, \mathbb{C})$. Let $D \subset \mathbb{C}$ be a symmetric domain and let $\Omega_D \subset \mathbb{H}$ denote the corresponding axially symmetric domain.*

Let $f, h : \Omega_D \rightarrow \mathbb{H}$ be slice regular functions and let $F, H : D \rightarrow \mathbb{H}_{\mathbb{C}}$ denote the corresponding stem functions.

a) *Assume that neither f nor h are slice preserving.*

Then the following are equivalent:

- (i) *f and h have the same invariants $cdiv$, Tr , N .*
- (ii) *F and H have the same invariants $cdiv$, Tr , N .*
- (iii) *$cdiv(F) = cdiv(H)$ and for every $z \in D$ there exists an element $\alpha \in \text{Aut}(\mathbb{H}_{\mathbb{C}}) = G_{\mathbb{C}}$ such that $F(z) = \alpha(H(z))$.*
- (iv) *There is a holomorphic map $\phi : D \rightarrow G_{\mathbb{C}}$ such that $F(z) = \phi(z) (H(z)) \forall z \in D$.*
- (v) *There is a holomorphic map $\alpha : D \rightarrow \mathbb{H}_{\mathbb{C}}^*$ such that*

$$F(z) = \alpha(z)^{-1} \cdot H(z) \cdot \alpha(z)$$

b) *Assume that f is slice preserving. Then the following are equivalent:*

- (i) *$f = h$.*
- (ii) *$F = H$.*
- (iii) *For every $z \in D$ there exists an element $\alpha \in \text{Aut}(\mathbb{H}_{\mathbb{C}}) = G_{\mathbb{C}}$ such that $F(z) = \alpha(H(z))$.*
- (iv) *There is a holomorphic map $\phi : D \rightarrow G_{\mathbb{C}}$ such that $F(z) = \phi(z) (H(z)) \forall z \in D$.*

¹In a forthcoming paper we will investigate other algebras as well.

(v) There is a holomorphic map $\alpha : D \rightarrow \mathbb{H}_{\mathbb{C}}^*$ such that

$$F(z) = \alpha(z)^{-1} \cdot H(z) \cdot \alpha(z)$$

Remark. The notion of a “central divisor” is defined only if the function is not slice preserving. This is similar to the ordinary complex situation where the divisor of a holomorphic function is defined only if it is not constantly zero.

Theorem 1.1 is proved in §18.

We also derive a corresponding result for the Clifford algebra \mathbb{R}_3 (which as \mathbb{R} -algebra is isomorphic to $\mathbb{H} \oplus \mathbb{H}$) (Theorem 8.1).

Quaternions can be used to describe orthogonal complex structures (OCS) on 4-dimensional Euclidean space, since their imaginary units parametrize automorphisms of $\mathbb{R}^4 \cong \mathbb{C}^2$. An injective slice regular function on a symmetric slice domain of \mathbb{H} minus the reals, defines a new OCS via the push-forward of the standard one. It would be very interesting in the authors’ opinion to understand if slice regular functions that are in the same orbit, by the action of automorphisms as explained in this paper, induce the same OCS or isomorphic OCS, [GSS14].

1.1. Related work. In [ADF20b], Altavilla and de Fabritiis investigated equivalence relations for *semi-regular* functions.

These semi-regular functions are locally $*$ -quotient of slice regular functions and correspond to meromorphic functions in complex analysis.

Semi-regular functions have the advantage that they are always invertible unless they are zero-divisors. This eases the use of linear algebra.

In [ADF20b] it is proved that for any two semi-regular functions f and g the following properties are equivalent:

- $\text{Tr}(f) = \text{Tr}(g)$ and $\text{N}(f) = \text{N}(g)$.
- There is a semi-regular function h with $h * f * h^{-*} = g$.
- The “Sylvester-operator” $S_{f,-g} : x \mapsto f * x - x * g$ is not invertible.

In comparison, we obtain a stronger conclusion (namely, conjugation by a *regular* function instead by a function which is only semi-regular), but for this we need an additional assumption, namely equality not only of trace Tr and norm N , but also of the “central divisor” *cdiv* which we introduce in §6.2. See § 4 for an instructive example.

2. PREPARATIONS

Here we collect basic facts and notions needed for our main result. First we discuss conjugation, norm and trace, then types of domains, then slice regular functions and stem functions, followed by investigating conjugation, norm and trace for function algebras.

2.1. Conjugation, norm and trace. Let A be an alternative \mathbb{R} -algebra with 1 and let $x \mapsto \bar{x}$ be an *antiinvolution*, i.e., an \mathbb{R} -linear map such that $\overline{\bar{y}} = y$ and $\overline{xy} = \bar{y} \cdot \bar{x}$ for all $x, y \in A$. (An \mathbb{R} -algebra with an antiinvolution is often called $*$ -algebra.)²

Definition 2.1. *Given an \mathbb{R} -algebra A with antiinvolution $x \mapsto \bar{x}$, we define:*

$$\text{Trace: } \text{Tr}(x) = x + \bar{x}$$

$$\text{Norm: } \text{N}(x) = x\bar{x}$$

Consider

$$C = \{x \in A : x = \bar{x}\}.$$

We assume that C is *central* and associates with all other elements, i.e.,

$$\forall c \in C, x, y \in A : cx = xc \text{ and } c(xy) = (cx)y$$

It is easy to verify that C is a subalgebra (under these assumptions, i.e., if C is assumed to be central).

Lemma 2.2. *Under the above assumptions the following properties hold:*

- (i) $\forall x \in \mathbb{R} : x = \bar{x}$.
- (ii) $\forall x \in A : \text{N}(x), \text{Tr}(x) \in C = \{y \in A : y = \bar{y}\}$
- (iii) $\forall x \in A : x\bar{x} = \bar{x}x$
- (iv) $\forall x \in A : \text{N}(x) = \text{N}(\bar{x})$.
- (v) $\forall x, y \in A : \text{N}(xy) = \text{N}(x)\text{N}(y)$.

Proof. (i)

$$\bar{1} = 1 \cdot \bar{1} \implies 1 = \overline{\bar{1}} = 1 \cdot \bar{1} = \bar{1}$$

By \mathbb{R} -linearity of the antiinvolution this yields $\forall x \in \mathbb{R} : x = \bar{x}$.

²In this article we are mainly concerned with the algebra of quaternions and $\mathbb{R}_3 \cong \mathbb{H} \oplus \mathbb{H}$, but we would also like to prepare for a future article on other \mathbb{R} -algebras.

(ii)

$$\overline{(\operatorname{Tr}(x))} = \overline{x + \bar{x}} = \bar{x} + x = \operatorname{Tr}(x)$$

and

$$\overline{(\operatorname{N}(x))} = \overline{(x\bar{x})} = \overline{(\bar{x})x} = x\bar{x} = \operatorname{N}(x)$$

(iii) $x + \bar{x}$ is central, hence $x(x + \bar{x}) = (x + \bar{x})x$ which implies $x^2 + x\bar{x} = x^2 + \bar{x}x$ and consequently $x\bar{x} = \bar{x}x$.(iv) $\operatorname{N}(\bar{x}) = (\bar{x})\overline{(\bar{x})} = (\bar{x})x = x(\bar{x}) = \operatorname{N}(x)$.(v) If A is associative we argue as follows:

$$\begin{aligned} \operatorname{N}(xy) &= (xy)\overline{(xy)} = xy(\bar{y}\bar{x}) \\ &= x(y\bar{y})\bar{x} = x\bar{x}(y\bar{y}) \\ &= \operatorname{N}(x)\operatorname{N}(y) \end{aligned}$$

For the general case we observe that x, \bar{x}, y, \bar{y} are all contained in the C -algebra A_0 generated by x and y (note that $x + \bar{x}, y + \bar{y} \in C$). Artin's theorem ([?], Theorem 3.1) implies that A_0 is associative. Thus all the calculations in the above sequence of equations take place *within* an associative algebra, namely A_0 and the proof is therefore still valid, even if A itself is not associative, but only alternative. □

2.2. Relation with notions in linear algebra and number theory. The norm and trace as considered here for the algebra of quaternions are closely related to notions in linear algebra and number theory.

The algebra \mathbb{H} is a *central simple* \mathbb{R} -algebra with $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \operatorname{Mat}(2 \times 2, \mathbb{C})$.

In the theory of central simple algebras (see e.g. [Lor08], Chapter 29), one considers *reduced traces* $\operatorname{Tr}d$ and *reduced norms* Nrd defined as $\operatorname{Tr}(\phi(x))$ resp. $\det(\phi(x))$ for $x \in \mathbb{H}$, where ϕ denotes the embedding of \mathbb{H} into $\operatorname{Mat}(2 \times 2, \mathbb{C})$ via the natural embedding $\mathbb{H} \subset \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ composed with an isomorphism $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \operatorname{Mat}(2 \times 2, \mathbb{C})$.

There is also a connection with notions of norm and trace in algebraic number theory:

If $A = \mathbb{C}$ and $x \mapsto x^c$ is complex conjugation, then Tr and N defined as here agree with the number-theoretical notions Tr and N for the Galois field extension \mathbb{C}/\mathbb{R} .

2.3. Stem functions. A *stem function* is a (usually holomorphic) function F defined on a *symmetric domain*³ D in \mathbb{C} with values in

³ $D \subset \mathbb{C}$ is called *symmetric* iff $z \in D \iff \bar{z} \in D$.

$A_{\mathbb{C}} = A \otimes_{\mathbb{R}} \mathbb{C}$ satisfying $\overline{F(z)} = F(\bar{z})$ where we use the *complex conjugation* on $A \otimes_{\mathbb{R}} \mathbb{C}$.

For a stem function F we define $(F^c)(z) = (F(z))^c$, i.e., we apply quaternionic conjugation pointwise. Thus we obtain a *conjugation* on the algebra of stem functions defined on a domain $D \subset \mathbb{C}$. As before, we define norm and trace and obtain:

$$N(F)(z) = (FF^c)(z) = (F(z))(F^c(z)) = (F(z))(F(z))^c = N(F(z))$$

and

$$(\text{Tr } F)(z) = (F + F^c)(z) = F(z) + (F(z))^c = \text{Tr}(F(z)).$$

Globally defined slice regular functions are given as globally convergent power series:

$$f(q) = \sum_{k=0}^{+\infty} q^k a_k$$

In this case

$$(f^c)(q) = \sum_{k=0}^{+\infty} q^k a_k^c$$

The space of slice regular functions on an axially symmetric domain forms an associative \mathbb{R} -algebra with the $*$ -product as multiplication.

Hence $(\text{Tr } f) = f + f^c$ and $N(f) = f * f^c$.

Immediately from the construction we obtain:

Proposition 2.3. *Let f be a slice function and F its associated stem function.*

Then $N(F)$, $\text{Tr}(F)$ and F^c are the stem functions associated to $N(f)$, $\text{Tr}(f)$ and f^c .

Remark. In general we have $f^c(q) \neq \overline{f(q)}$. Only for real points $q \in \mathbb{R}$ we have $f^c(q) = \overline{f(q)}$ and consequently $(Nf)(q) = N(f(q))$ and $(\text{Tr } f)(q) = \text{Tr}(f(q))$.

2.4. Compatibility. Recall that conjugation for slice regular functions is *not* just pointwise conjugation of the function values.

Therefore in general

$$(\text{Tr } f)(q) \neq \text{Tr}(f(q)), (Nf)(q) \neq N(f(q)).$$

Let B be a \mathbb{R} -sub algebra of an \mathbb{R} algebra A equipped with an anti-involution which preserves B .

Then for $x \in B$ the notions $\text{Tr}(x)$ and $N(x)$ defined with respect to this antiinvolution are the same regardless whether we regard x as an element of B or as an element of A .

As a consequence

- For $x \in \mathbb{H}$ the notions $N(x)$, $\text{Tr}(x)$ agree independent of whether we consider x in \mathbb{H} or in $\mathbb{H}_{\mathbb{C}}$.
- For an element $x \in \mathbb{H}$ the notions $N(x)$, $\text{Tr}(x)$ agree whether we regard x in \mathbb{H} or as a constant slice regular function with value x .

2.5. **Other notions.** For $x \in \mathbb{H}$ the term $\frac{1}{2} \text{Tr}(x)$ is often called *real part* of x , sometimes denoted as x_0 .

$N(x)$ is also called the *symmetrization* of x and denoted as x^s .

3. AN EXAMPLE

Warning. *The conditions of the Main Theorem 1.1 do not imply that for any $q \in \mathbb{H}$ we can find an element $\alpha \in \text{Aut}(\mathbb{H})$ such that $f(q) = \alpha(g(q))$.*

Example 3.1. *Let $f(q) = I$ and let $g(q) = \cos(q)I + \sin(q)J$, i.e.,*

$$g(q) = \sum_{k=0}^{+\infty} \left(q^{2k} \frac{(-1)^k}{(2k)!} I + q^{2k+1} \frac{(-1)^k}{(2k+1)!} J \right).$$

where I, J are any two orthogonal imaginary units in \mathbb{H} with $K = IJ$. We recall the classical identities

$$\sin(it) = i \sinh(t), \quad \cos(it) = \cosh(t) \quad \forall t \in \mathbb{R}.$$

For $t \in \mathbb{R}$ we deduce

$$\begin{aligned} g(tJ) &= \cos(tJ)I + \sin(tJ)J \\ &= \cosh(t)I + \sinh(t) \underbrace{J^2}_{=-1} \\ &= \cosh(t)I - \sinh(t). \end{aligned}$$

In particular, the “real part” $\frac{1}{2} \text{Tr}(g(tJ))$ is non-zero if $\sinh(t) \neq 0$, i.e., if $t \neq 0$. Hence for $t \in \mathbb{R}^*$, we have

$$\text{Tr}(g(tJ)) \neq 0 = \text{Tr}(I) = \text{Tr}(f(tJ))$$

Since $\text{Tr}(\phi(q)) = \text{Tr}(q)$ for every $q \in \mathbb{H}$ and every automorphism ϕ of \mathbb{H} , it follows that there is no automorphism of \mathbb{H} mapping $g(tJ)$ to $f(tJ) = I$.

On the other hand

$$\begin{aligned}
g^c(q) &= -\cos(q)I - \sin(q)J \\
&\implies \\
(g * g^c)(q) &= -(\cos(q)I + \sin(q)J) * (\cos(q)I + \sin(q)J) \\
&= -\left(\cos^2(q) \underbrace{I^2}_{=-1} + \sin^2(q) \underbrace{J^2}_{=-1} + \sin(q)\cos(q) \underbrace{(IJ + JI)}_{=0} \right) \\
&= \left(\underbrace{\cos^2(q) + \sin^2(q)}_{=1} \right) = 1 = f * f^c
\end{aligned}$$

and

$$\begin{aligned}
g(q) + g^c(q) &= \underbrace{\cos(q)I + \sin(q)J}_{g(q)} + \underbrace{-\cos(q)I - \sin(q)J}_{g^c(q)} = 0 \\
&= I + (-I) = f(q) + f^c(q), \quad \forall q \in \mathbb{H}.
\end{aligned}$$

Thus $\text{Tr}(f) = \text{Tr}(g) = 0$ and $\text{N}(f) = \text{N}(g) = 1$. The stem functions $F, G : \mathbb{C} \rightarrow \mathbb{H} \otimes \mathbb{C}$ associated to f and g are:

$$F(z) = 1 \otimes I, \quad G(z) = \cos(z) \otimes I + \sin(z) \otimes J$$

Since $\sin^2(z) + \cos^2(z) = 1 \quad \forall z \in \mathbb{C}$, both F and G avoid the center $\mathbb{R} \otimes \mathbb{C}$ of $\mathbb{H} \otimes \mathbb{C}$. Hence the central divisors $\text{cdiv}(F)$ and $\text{cdiv}(G)$ (as defined in §6.2) are both empty. Therefore we have verified that in this example

$$\text{Tr}(f) = \text{Tr}(g), \quad \text{N}(f) = \text{N}(g), \quad \text{cdiv}(f) = \text{cdiv}(g),$$

but for some $q \in \mathbb{H}$ there is no $\phi \in \text{Aut}(\mathbb{H})$ with

$$f(q) = \phi(g(q)).$$

On the other hand, our main result implies that there exists a holomorphic map $\phi : \mathbb{C} \rightarrow \text{Aut}(\mathbb{H}_{\mathbb{C}})$ such that the corresponding stem functions F and G satisfy $F(z) = \phi(z)(G(z))$.

[ADF20b] and [ADF20a] pointed out that there exists a slice regular function h such that $f * h = h * g$. This implies that

$$h^{-*} * f * h = g$$

where h is slice regular, but h^{-*} is possibly only semi-regular.

In contrast, our result implies the existence of such a slice regular function h which is invertible in the sense that h^{-*} is likewise slice regular, and not only semi-regular.

To give an explicit example, let

$$H(z) = \cos(z/2) - K \sin(z/2)$$

Then

$$\begin{aligned} & H(z)^{-1} \cdot F(z) \cdot H(z) \\ &= \left(\cos\left(\frac{z}{2}\right) + K \sin\left(\frac{z}{2}\right) \right) \cdot I \cdot \left(\cos\left(\frac{z}{2}\right) - K \sin\left(\frac{z}{2}\right) \right) \\ &= I \cos z + J \sin z = G(z) \end{aligned}$$

4. ANOTHER EXAMPLE

Consider the stem functions

$$\begin{aligned} F(z) &= I + zJ + \frac{1}{2}z^2K \\ G(z) &= \left(1 + \frac{1}{2}z^2 \right) I \end{aligned}$$

We have

$$\begin{aligned} \text{Tr}(F) &= \text{Tr}(G) = 0 \\ \text{N}(F) &= \text{N}(G) = 1 + z^2 + \frac{1}{4}z^4 \\ \text{cdiv}(F) &= \{\}, \quad \text{cdiv}(G) = 1\{\sqrt{2}i\} + 1\{-\sqrt{2}i\} \end{aligned}$$

F and G are not equivalent in our sense because $\text{cdiv}(F) \neq \text{cdiv}(G)$. Therefore there does not exist a stem function $H : \mathbb{C} \rightarrow \mathbb{H}_{\mathbb{C}}^*$ such that both H and H^{-1} are holomorphic and

$$F = H^{-1} \cdot G \cdot H$$

In contrast, Altavilla and de Fabritiis do not need a condition on cdiv for their results in [ADF20b] and [ADF20a]. Hence their results imply that there exists a *meromorphic* stem function H with

$$F = H^{-1} \cdot G \cdot H$$

Indeed, an explicit calculation shows that

$$H(z) = I \left(2 + \frac{1}{2}z^2 \right) + Jz + \frac{1}{2}z^2K, \quad H^{-1}(z) = -\frac{I \left(2 + \frac{1}{2}z^2 \right) + Jz + \frac{1}{2}z^2K}{\frac{1}{2}z^4 + 3z^2 + 4}$$

is such a function.

5. MORE PREPARATIONS

5.1. The Clifford algebra \mathbb{R}_3 . The *Clifford algebra* \mathbb{R}_3 may be realized as the associative \mathbb{R} -algebra generated by e_1, e_2, e_3 with the relations

$$e_{jk} + e_{kj} = -2\delta_{jk} \quad j, k \in \{1, 2, 3\}$$

where δ_{jk} is the Kronecker symbol and $e_{jk} = e_j \cdot e_k$, i.e.,

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

It contains the idempotents ω_+, ω_- satisfying $\omega_+\omega_- = 0$, namely:

$$\omega_+ = \frac{1}{2}(e_1e_2e_3 + 1), \quad \omega_- = \frac{1}{2}(e_1e_2e_3 - 1)$$

This yields a direct sum decomposition of \mathbb{R}_3 :

$$\mathbb{R}_3 = \omega_+\mathbb{H} \oplus \omega_-\mathbb{H}$$

where $\mathbb{H} \cong \mathbb{R}_2$ is embedded in \mathbb{R}_3 as the subalgebra generated by e_1, e_2 .

See e.g. [Gar11],[Lam05], Chapter V for more details on Clifford algebras.

5.2. Invariants for \mathbb{R}_3 . For more clarity in this paragraph we use N_A resp. Tr_A in order to denote the norm and trace for a given algebra A .

The Clifford algebra \mathbb{R}_3 is isomorphic (as \mathbb{R} -algebra with an anti-involution which we call conjugation) to $\mathbb{H} \oplus \mathbb{H}$.

As a consequence we have

$$\begin{aligned} N_{\mathbb{R}_3}(q_1, q_2) &= N_{\mathbb{H} \oplus \mathbb{H}}(q_1, q_2) &&= (N_{\mathbb{H}}(q_1), N_{\mathbb{H}}(q_2)) \\ \text{Tr}_{\mathbb{R}_3}(q_1, q_2) &= \text{Tr}_{\mathbb{H} \oplus \mathbb{H}}(q_1, q_2) &&= (\text{Tr}_{\mathbb{H}}(q_1), \text{Tr}_{\mathbb{H}}(q_2)) \end{aligned}$$

for $q_1, q_2 \in \mathbb{H} \oplus \mathbb{H} \cong \mathbb{R}_3$.

Similar for the complexified algebra $\mathbb{R}_3 \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{H}_{\mathbb{C}} \oplus \mathbb{H}_{\mathbb{C}}$ and the corresponding function algebras.

Caveat: Since $(z, w) \mapsto (w, z)$ is an automorphism of $\mathbb{H} \oplus \mathbb{H}$ (see §11), these functions N, Tr are not invariants. They are only invariants up to changing the order of the two components, i.e., they are invariants for the connected component $\text{Aut}^0(\mathbb{H} \oplus \mathbb{H})$ which equals $\text{Aut}(\mathbb{H}) \times \text{Aut}(\mathbb{H})$ acting component-wise on $A = \mathbb{H} \oplus \mathbb{H}$.

5.3. Slice preserving functions.

Proposition 5.1. *Let $A = \mathbb{H}$ and let $f : A \rightarrow A$ be a slice regular function with stem function $F : \mathbb{C} \rightarrow A_{\mathbb{C}}$.*

Then the following are equivalent:

- (i) $f = f^c$,
- (ii) $F = F^c$,
- (iii) $F(\mathbb{C}) \subset \mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} \subset A \otimes_{\mathbb{R}} \mathbb{C} = A_{\mathbb{C}}$.
- (iv) $f(\mathbb{C}_I) \subset \mathbb{C}_I$ for all $I \in \mathbb{S} = \{q \in A : q^2 = -1\}$ (with $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$).

Definition 5.2. *If one (hence all) of these properties are fulfilled, f is called “slice preserving”.*

Proof. These equivalences are well known. (iv) \iff (iii) follows from representation formula. (i) \iff (ii) \iff (iii) by construction of $(\)^c$. \square

5.4. Slice regular functions for $A = \mathbb{R}_3$. In [GP11] *slice regular functions* are considered for real alternative algebras, e.g., Clifford algebras. For generalities on Clifford algebras see e.g. [Lam05], Ch.V.

A *quadratic cone* $Q_A \subset A$ is defined such that every $q \in Q_A$ is contained in the image of some \mathbb{R} -algebra homomorphism from \mathbb{C} to A . While $Q_A = A$ for $A = \mathbb{H}$, we have $Q_A \neq A$ for the Clifford algebra $A = \mathbb{R}_3 \cong \mathbb{H} \oplus \mathbb{H}$. More precisely (see [GP11]):

$$\begin{aligned} Q_A &= \mathbb{R} \cup \{x \in A : \text{Tr}(x), \text{N}(x) \in \mathbb{R}, 4\text{N}(x) > \text{Tr}(x)^2\} \\ &= \{x \in A : \text{Tr}(x), \text{N}(x) \in \mathbb{R}\} \\ &\cong \{(q_1, q_2) \in \mathbb{H} \oplus \mathbb{H} : \text{Tr}_{\mathbb{H}}(q_1) = \text{Tr}_{\mathbb{H}}(q_2), \text{N}_{\mathbb{H}}(q_1) = \text{N}_{\mathbb{H}}(q_2)\} \end{aligned}$$

(For $A = \mathbb{R}_3$ the condition $4\text{N}(x) > \text{Tr}(x)^2$ is automatically satisfied for every element $x \in A \setminus \mathbb{R}$ with $\text{N}(x), \text{Tr}(x) \in \mathbb{R}$.)

Given a symmetric domain $D \subset \mathbb{C}$, there is the “*circularization*”

$$\Omega_D = \{(x + yH_1, x + yH_2) : x, y \in \mathbb{R}, H_1, H_2 \in \mathbb{S}_{\mathbb{H}}, x + yi \in D\}.$$

(with $\mathbb{S}_{\mathbb{H}} = \{J \in \mathbb{H} : J^2 = -1\}$.) Evidently $\Omega_D \subset Q_A$.

But due to the special direct sum nature of \mathbb{R}_3 we may regard a larger set, namely :

$$(5.1) \quad W_D = \{(x_1 + y_1H_1, x_2 + y_2H_2) : H_j \in \mathbb{S}_{\mathbb{H}}, x_j, y_j \in \mathbb{R}, x_j + y_j i \in D \forall j \in \{1, 2\}\}$$

Note that $\Omega_D = Q_A \cap W_D$. Thus Ω_D relates to W_D like the quadratic cone Q_A to the whole algebra $A = \mathbb{R}_3$.

Slice regular functions as defined in [GP11] are certain functions defined on the cone Q_A . It is shown in [GP11] that these *slice regular*

functions correspond to *holomorphic stem functions*, i.e., holomorphic functions $F : D \rightarrow A_{\mathbb{C}}$ such that $\overline{F(z)} = F(\bar{z}) \forall z \in \mathbb{C}$.

We remark that there is a natural way to extend a function $f : \Omega_D \rightarrow \mathbb{R}_3$ to a function $\tilde{f} : W_D \rightarrow \mathbb{R}_3$.

Namely, we use the decomposition $\mathbb{R}_3 \cong \mathbb{H} \oplus \mathbb{H}$ to represent f as $f = (f_1, f_2)$ with $f_i : \Omega_D \rightarrow \mathbb{H}$ and define

$$\begin{aligned} & \tilde{f}(x_1 + y_1 H_1, x_2 + y_2 H_2) \\ &= (f_1(x_1 + y_1 H_1, x_1 + y_1 H_2), f_2(x_2 + y_2 H_1, x_2 + y_2 H_2)) \end{aligned}$$

In this way the algebra of slice regular functions on Ω_D can be identified with a certain algebra of functions on W_D .

Let us now discuss the special case of globally defined functions (i.e. $D = \mathbb{C}$, $\Omega_D = Q_A$ and $W_D = A = \mathbb{R}_3$.)

In this case stem functions are holomorphic functions with values in $A_{\mathbb{C}}$ which are defined on the whole of \mathbb{C} . Such a holomorphic function may be defined by a convergent power series $\sum_{k=0}^{+\infty} z^k a_k$ with $a_k \in A$. (A priori, $\overline{a_k} \in A_{\mathbb{C}}$ for a holomorphic function $F : \mathbb{C} \rightarrow A_{\mathbb{C}}$, but the condition $\overline{F(z)} = F(\bar{z})$ ensures that $a_k \in A$.)

Such a power series may also be regarded as power series in a variable in A ; in this case it defines a function $f : A \rightarrow A$. (It is easily shown that it is again globally convergent.)

Thus there are bijective correspondances between the following three classes of functions:

- *Slice regular functions* $f : Q_A \rightarrow A$ as defined in [GP11].
- *Entire functions*, i.e., functions $f : A \rightarrow A$ which are defined by globally convergent power series $\sum_{k=0}^{+\infty} q^k a_k$ ($a_k \in A$).
- *Stem functions*, i.e., holomorphic functions $F : \mathbb{C} \rightarrow A_{\mathbb{C}}$ with $\overline{F(z)} = F(\bar{z})$.

In particular, slice regular functions $f : Q_A \rightarrow A$ extend naturally to functions $f : A \rightarrow A$ defined by globally convergent power series.

Thus for the special case of the Clifford algebra \mathbb{R}_3 there is no need to restrict the domain of definition to (subdomains of) the quadratic cone Q_A .

Let D, Ω_D, W_D be as in the preceding subsection.

We described above how to associate a function $\tilde{f} : W_D \rightarrow A \cong \mathbb{H} \oplus \mathbb{H}$ to a given function $f : \Omega_D \rightarrow A$.

6. CENTRAL DIVISORS

6.1. Divisors for vector-valued functions. Normally, *divisors* are defined for holomorphic functions with values in \mathbb{C} . Here we extend this notion to holomorphic maps from Riemann surfaces to higher-dimensional complex vector spaces.

Definition 6.1. *Let $f : X \rightarrow V = \mathbb{C}^n$ be a holomorphic map from a Riemann surface X to a complex vector space $V = \mathbb{C}^n$. Assume that f is not identically zero.*

The divisor of f is the divisor corresponding to the pull back of the ideal sheaf of the origin, i.e., for $f = (f_1, \dots, f_n)$, $f_i : X \rightarrow \mathbb{C}$ we have $\text{div}(f) = \sum_{p \in X} m_p \{p\}$ where m_p denotes the minimum of the multiplicities $\text{mult}_p(f_i)$.

6.2. Central divisor for stem functions. In [BW21a], Definition 3.1, we introduced the notion of a *slice divisor*. Here we will need a different notion of divisor.

Namely, we need a notion of divisor which measures how far the function is from being *slice-preserving*. This we call “*central divisor*”. Here we define it for the quaternion case.

Definition 6.2. *Let $Z \cong \mathbb{C}$ be the center of $\mathbb{H}_{\mathbb{C}}$. $D \subset \mathbb{C}$ be a symmetric domain, $F : D \rightarrow A_{\mathbb{C}}$ a holomorphic map. Assume $F(D) \not\subset Z$.*

The central divisor $\text{cdiv}(F)$ is defined as the divisor (in the sense of Definition 6.1) of the map from D to $\mathbb{H}_{\mathbb{C}}/Z$ induced by F .

Let $(1, i, j, k)$ be the standard basis of \mathbb{H} and let $W \otimes_{\mathbb{R}} \mathbb{C}$ denote the complex vector subspace of $\mathbb{H}_{\mathbb{C}}$ generated by $i \otimes 1, j \otimes 1, k \otimes 1$.

Then $\mathbb{H}_{\mathbb{C}}$ may be represented as the vector space direct sum $\mathbb{H}_{\mathbb{C}} = Z \oplus W \otimes_{\mathbb{R}} \mathbb{C}$ where $Z \cong \mathbb{C}$ denotes the center. and we can decompose $F : D \rightarrow \mathbb{H}_{\mathbb{C}}$ as $F = (F', F'') : D \rightarrow Z \times (W \otimes_{\mathbb{R}} \mathbb{C})$ and the central divisor $\text{cdiv}(F)$ equals $\sum_{p \in D} n_p \{p\}$ where n_p denotes the vanishing order of F'' at p .

Let $F = (F', F'')$ be a stem function for a slice function f . Then f is *slice-preserving* if $F'' \equiv 0$ (Definition 5.2).

Hence the assumption that f is not slice preserving implies that F'' does not vanish identically and we therefore may define $\text{cdiv}(F)$ as above.

Example 6.3. *Consider $F : \mathbb{C} \rightarrow \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ defined as*

$$F(z) = 1 \otimes z + i \otimes z^2(z - 1) + j \otimes z^3(z - 1)^2$$

Then

$$\text{cdiv}(F) = 2 \cdot \{0\} + 1 \cdot \{1\}$$

Caveat: These central divisors do *not* satisfy the usual functoriality:

Example 6.4. *Let*

$$F(z) = 1 + iz, \quad G(z) = 1 + j(1 + z)$$

Then $\text{cdiv}(F) = 1 \cdot \{0\}$ and $\text{cdiv}(G) = 1 \cdot \{-1\}$, but $\text{cdiv}(FG) = \text{cdiv}(1 + iz + j(z + 1) + k(z + 1)z)$ is empty. Thus

$$\text{cdiv}(FG) \neq \text{cdiv}(F) + \text{cdiv}(G).$$

6.3. Central divisor for slice functions. We simply define the *central divisor* for a given slice regular function as the central divisor of the corresponding stem function (as defined in Definition 6.2).

6.4. Central divisors for \mathbb{R}_3 . The *central divisor* cdiv of a slice regular function $f : \mathbb{R}_3 \rightarrow \mathbb{R}_3$ is defined via the stem function F . Given a stem function $F : D \rightarrow A_{\mathbb{C}} \cong \mathbb{H}_{\mathbb{C}} \oplus \mathbb{H}_{\mathbb{C}}$, we define

$$\text{cdiv}(F) = (\text{cdiv } F_1, \text{cdiv } F_2)$$

where F_1, F_2 are the components of the stem function F with respect to the decomposition of \mathbb{R}_3 into a direct sum of two copies of \mathbb{H} .

What we need, is to treat \mathbb{R}_3 consequently as product, i.e., we regard slice regular functions from (a domain in) \mathbb{R}_3 to \mathbb{R}_3 as a pair of the quaternionic slice regular functions and *define* the “central divisor” of a slice regular function $f = (f_1; f_2)$ as

$$\text{cdiv}(f) = \text{cdiv}(f_1; f_2) = (\text{cdiv}(f_1), \text{cdiv}(f_2))$$

7. STRATEGY

Let A be a \mathbb{R} -algebra, $A_{\mathbb{C}} = A \otimes_{\mathbb{R}} \mathbb{C}$ its complexification and $G_{\mathbb{C}}$ the automorphism group of the complex algebra $A_{\mathbb{C}}$.

For certain⁴ holomorphic maps $F, H : \mathbb{C} \rightarrow A_{\mathbb{C}}$ we want to show the following statement:

If for every $x \in \mathbb{C}$ there exists an element $g \in G_{\mathbb{C}}$ such that $F(x) = g(H(x))$, then there exists a holomorphic map $\phi : \mathbb{C} \rightarrow G_{\mathbb{C}}$ with $F(x) = \phi(x)(H(x)) \quad \forall x$.

This amounts to find a section for a certain projection map, namely $\pi : V \rightarrow \mathbb{C}$ with

$$V = \{(x, g) \in \mathbb{C} \times G_{\mathbb{C}} : F(x) = g(H(x))\}$$

and $\pi(x, g) = x$.

First we discuss the locus where F and H assume zero as value.

Outside this zero locus D , i.e., restricted to $V_0 = \pi^{-1}(\mathbb{C} \setminus D)$, the map π has some nice properties if $A \cong \mathbb{H}$:

⁴“Stem functions”

- (i) all π -fibers are of the same dimension, homogeneous and moreover biholomorphic to a Lie subgroup of $G_{\mathbb{C}}$.
- (ii) There are everywhere local holomorphic sections.

Moreover, for $A = \mathbb{H}$ there is a Zariski open subset Ω in \mathbb{C} on which π is a locally holomorphically trivial fiber bundle with \mathbb{C}^* as fiber.

We show that there exists a global holomorphic section on Ω .

We will use strongly that the generic isotropy group (i.e. \mathbb{C}^*) is both commutative and one-dimensional.

8. MAIN RESULT FOR \mathbb{R}_3

Theorem 8.1. *Let $D \subset \mathbb{C}$ be a symmetric domain, $\Omega_D \subset \mathbb{R}_3$ as defined above (equation(5.1)), $f, h : \Omega_D \rightarrow \mathbb{R}_3$ slice regular functions as defined above.*

Let $F, H : D \rightarrow A_{\mathbb{C}}$ be the associated stem functions.

Let $G_{\mathbb{C}}$ denote the connected component of the neutral element of the group of \mathbb{C} -algebra automorphisms of $A_{\mathbb{C}} = \mathbb{R}_3 \otimes_{\mathbb{R}} \mathbb{C}$, i.e., $G_{\mathbb{C}} = \text{Aut}(\mathbb{H}_{\mathbb{C}}) \times \text{Aut}(\mathbb{H}_{\mathbb{C}})$.

Let N, Tr and cdiv be defined as in subsection 5.2.

Then the following are equivalent:

- (1) *There exists a holomorphic map $\phi : D \rightarrow G_{\mathbb{C}}$ with*

$$F(z) = \phi(z) (H(z)) \quad \forall z \in D$$

- (2) *There exists a holomorphic map $\alpha : D \rightarrow A_{\mathbb{C}}^* \cong \mathbb{H}_{\mathbb{C}}^* \times \mathbb{H}_{\mathbb{C}}^*$ with*

$$F(z) = \alpha(z) (H(z)) (\alpha(z))^{-1} \quad \forall z \in D$$

- (3) *$\text{cdiv}(F) = \text{cdiv}(H)$, $\text{Tr}(F) = \text{Tr}(H)$ and $N(F) = N(H)$.*

Proof. Using $A \cong \mathbb{H} \oplus \mathbb{H}$ and $G_{\mathbb{C}}^0 \cong \text{Aut}(\mathbb{H}_{\mathbb{C}}) \times \text{Aut}(\mathbb{H}_{\mathbb{C}})$, the equivalence of the existence of such a map $\phi : D \rightarrow G_{\mathbb{C}}$ with the condition

$$\text{cdiv}(F) = \text{cdiv}(H), \text{Tr}(F) = \text{Tr}(H) \text{ and } N(F) = N(H)$$

follows from the respective result for quaternions (Theorem 1.1, proved in §18). □

9. LOCAL EQUIVALENCE

Assumptions 9.1. *Let G be a connected complex Lie group acting holomorphically on a complex manifold X such that all the orbits have the same dimension d .*

Let

$$F, H : \Delta = \{z \in \mathbb{C} : |z| < 1\} \rightarrow X$$

be holomorphic maps such that for every $z \in \Delta$ there exists an element $g \in G$ (depending on z , not necessarily unique) with $F(z) = g \cdot H(z)$.

Let

$$V = \{(z, g) \in \Delta \times G : F(z) = g \cdot H(z)\}$$

Lemma 9.2. *Under the above assumptions (9.1) for every $t \in \Delta$ there is a biholomorphic map between the fiber*

$$V_t = \{(z, g) \in V : z = t\}$$

and the isotropy group

$$G_{F(t)} = \{g \in G : g \cdot F(t) = F(t)\}$$

Proof. Fix $t \in \Delta$ and choose a point $(t, g_0) \in V_t$. Note that

$$F(t) = g_0 \cdot H(t)$$

because $(t, g_0) \in V_t \subset V$.

We define a map

$$\zeta : G_{F(t)} \xrightarrow{\sim} V_t$$

as

$$\zeta(g) = (t, g \cdot g_0)$$

We claim that this is a biholomorphic map from $G_{F(t)}$ to V_t . First we verify that $\zeta(g) \in V_t$. Indeed, $F(t) = g_0 \cdot H(t)$ (as seen above) and $g \cdot F(t) = F(t)$, because $g \in G_{F(t)}$.

Combined these facts imply

$$F(t) = g \cdot \underbrace{g_0 \cdot H(t)}_{F(t)} \implies \zeta(g) = (t, gg_0) \in V$$

ζ is obviously injective. Let us check surjectivity. Let $(t, p) \in V_t$. Then $F(t) = p \cdot H(t)$. Recall that $F(t) = g_0 \cdot H(t)$. It follows that

$$(p \cdot g_0^{-1})F(t) = (p \cdot g_0^{-1})g_0 \cdot H(t) = pH(t) = F(t)$$

Hence

$$p \cdot g_0^{-1} \in G_{F(t)}.$$

We claim that $\zeta(p \cdot g_0^{-1}) = (t, p)$. Indeed

$$\zeta(p \cdot g_0^{-1}) = (t, (p \cdot g_0^{-1}) \cdot g_0) = (t, p).$$

Therefore

$$\zeta : G_{F(t)} \xrightarrow{\sim} V_t$$

is biholomorphic. □

Lemma 9.3. *Under the above assumptions (9.1) V is smooth.*

Proof. Consider

$$Z := \{(z, x, x) : z \in \Delta, x \in X\}.$$

Note that Z is a submanifold of $\Delta \times X \times X$.

Define

$$\phi : \Delta \times G \rightarrow \Delta \times X \times X, \quad \phi(z, g) = (z, F(z), g \cdot H(z))$$

and observe that $V = \phi^{-1}(Z)$.

In order to verify the smoothness of V it suffices to show that $D\phi$ has everywhere the same rank.

Let $(x, g) \in \Delta \times G$ and consider

$$D\phi_{(x,g)} : T_{(x,g)}(\Delta \times G) \rightarrow T_{\phi(x,g)}(\Delta \times X \times X).$$

We observe that

$$T_{(x,g)}(\Delta \times G) \cong (T_x\Delta) \times \text{Lie}(G).$$

From $\phi(z, g) = (z, F(z), g(H(z)))$ we infer that

$$\begin{aligned} (v, w) \in \ker D\phi_{(x,g)} &\subset (T_x\Delta) \times \text{Lie}(G) \\ \iff v = 0 \text{ and } w \in \ker D\zeta &\text{ for } \zeta : g \mapsto g(H(z)) \end{aligned}$$

Here $\zeta : G \rightarrow X$ is the orbit map $g \mapsto g(H(z))$. Standard theory of transformation groups implies that $w \in \ker D\zeta$ iff $w \in T_gG \cong \text{Lie}(G)$ is contained in the Lie subalgebra of the isotropy group of the G -action at $g(H(z))$.

By assumption all the G -orbits in X have the same dimension d . Hence every isotropy group is of dimension $\dim(G) - d$. It follows that

$$\dim \ker D\phi_{z,g} = \dim(G) - d \quad \forall z, g$$

Thus ϕ is a map of constant rank and $V = \phi^{-1}(Z)$ is smooth. \square

Proposition 9.4. *Under the above assumptions, there exists $0 < r < 1$ and a holomorphic map*

$$\phi : \Delta_r = \{z : |z| < r\} \rightarrow G$$

such that

$$F(z) = \phi(z)(H(z)) \quad \forall z \in \Delta_r.$$

Proof. Let $g_0 \in G$ such that $F(0) = g_0(H(0))$. We may replace $H(z)$ by the function $\tilde{H}(z) = g_0(H(z))$. In this way we see: There is no loss of generality in assuming $F(0) = H(0)$.

We may replace G by its universal covering and therefore assume that G is simply-connected. Then we can use the fact that every simply-connected complex Lie group is a Stein manifold (see [MM60]). Hence we may assume that G is a Stein manifold.

Recall that

$$V = \{(z, g) \in \Delta \times G : F(z) = g(H(z))\}.$$

Let $\pi : V \rightarrow \Delta$ be the natural projection and $V_t = \pi^{-1}(t)$ (for $t \in \Delta$). Due to Lemma 9.2 there is a bijective map from V_t to the isotropy group of the G -action at $F(t)$. Since all the G -orbits are assumed to have the same dimension d , each isotropy group has the dimension $\dim(G) - d$. Therefore all the fibers of $\pi : V \rightarrow \Delta$ have the same dimension (namely $\dim(G) - d$).

Due to Lemma 9.3 the complex space V is smooth.

We consider the relative symmetric product $S_\Delta^m V$, i.e., $S_\Delta^m V$ is the quotient of

$$\{(z; v_1, \dots, v_m) \in \Delta \times V^m : \pi(v_i) = z, \forall i\}$$

under the natural action of the symmetric group S_m permuting the components of V^m .

V may be embedded into $S_\Delta^m V$ diagonally as

$$V \ni v = (z, g) \xrightarrow{\delta} [(z; v, \dots, v)] \in S_\Delta^m V.$$

Note that V is Stein as a closed analytic subspace of the Stein manifold $\Delta \times G$. It follows that

$$\{(z; v_1, \dots, v_m) \in \Delta \times V^m : \pi(v_i) = z, \forall i\}$$

is likewise Stein. Thus $S_\Delta^m V$ is the quotient of a Stein space by a finite group (namely S_m). Hence $S_\Delta^m V$ is Stein and therefore admits an embedding $j : S_\Delta^m V \hookrightarrow \mathbb{C}^N$ for some $N \in \mathbb{N}$. Recall that $S_\Delta^m V$ is a quotient of a subspace of $\Delta \times V^m$ by a S_m -action which is trivial on the first factor Δ . The natural projection from $\Delta \times V^m$ onto its first factor thus yields a natural map $p : S_\Delta^m V \rightarrow \Delta$.

Now we define an embedding $\xi : S_\Delta^m V \rightarrow \Delta \times \mathbb{C}^N$ as

$$\xi : w \mapsto (p(w), j(w))$$

We obtain a commutative diagram

$$\begin{array}{ccccc}
V & \xrightarrow{\delta} & S_{\Delta}^m V & \xrightarrow{\xi} & \Delta \times \mathbb{C}^N \\
\pi \downarrow & & \downarrow p & & \downarrow pr_1 \\
\Delta & \xlongequal{\quad} & \Delta & \xlongequal{\quad} & \Delta
\end{array}$$

Here $pr_1 : \Delta \times \mathbb{C}^N \rightarrow \Delta$ is the projection to the first factor.

Because V is smooth (see Lemma 9.3), the “tubular neighbourhood theorem” (see [For17], Theorem 3.3.3.) implies the existence of a holomorphic retraction of an open neighborhood W of $Y = \xi(\delta(V))$ in $\Delta \times \mathbb{C}^N$ onto Y .

This can be done in a relative way, over Δ , cf. [For04], Lemma 3.3; [For17], Theorem 3.3.4.

Thus there is a holomorphic map $\rho : W \rightarrow Y$ with $\rho|_Y = id_Y$ and $pr_1(w) = pr_1(\rho(w))$ for all $w \in W$.

$$\begin{array}{ccccc}
\xi(\delta(V)) & \xlongequal{\quad} & Y & \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\rho} \end{array} & W & \begin{array}{c} \xrightarrow{\text{open}} \\ \xleftarrow{\quad} \end{array} & \Delta \times \mathbb{C}^N \\
& & \searrow & & \downarrow & & \swarrow \\
& & & & \Delta & &
\end{array}$$

Let C be a (local) smooth complex curve in V through the point $(0, e)$ which is not contained in the fiber $V_0 = \pi^{-1}(0)$. Now $C \cap V_0$ is a discrete subset of C . Hence by shrinking C we may assume that with

$$C \cap V_0 = \{(0, e)\}$$

Now $\pi|_C : C \rightarrow \Delta$ is a non-constant holomorphic map between one-dimensional complex manifolds.

By one-dimensional complex analysis, after appropriately shrinking Δ and C , the projection map from C onto Δ is a finite ramified covering of some degree $m \in \mathbb{N}$.

Thus we obtain a local multisection, i.e., there is an open neighborhood Δ' of 0 in Δ and a closed analytic subset $C' \subset \pi^{-1}(\Delta')$ such that $\pi|_{C'} : C' \rightarrow \Delta'$ is a finite ramified covering. Let m denote the degree of $\pi|_{C'}$.

To each point $t \in \Delta'$ we associate the finite space $\pi^{-1}(t) \cap C'$. This is a finite subspace of $\pi^{-1}(t) \subset V$ of degree m which is mapped by π into one single point of Δ . Such a finite subset corresponds to a point in the relative symmetric product $S_{\Delta}^m V$. Therefore our multisection yields a section $s : \Delta' \rightarrow S_{\Delta}^m V$.

We recall that we assumed $C \cap V_0 = \{(0, e)\}$. Hence

$$s(0) = \delta(0, e) \implies \xi(s(0)) \in Y \subset W$$

Now W is open, and $(\xi \circ s)(0) \in W$. Let $\Delta'' = (\xi \circ s)^{-1}(W)$.

We compose $\xi \circ s : \Delta'' \rightarrow W$ with the holomorphic retraction

$$\rho : W \rightarrow Y = \xi(\delta(V)).$$

Since $\xi \circ \delta : V \rightarrow Y$ is an isomorphism, there is a unique holomorphic map $\sigma : \Delta'' \rightarrow V$ satisfying

$$\xi \circ \delta \circ \sigma = \rho \circ \xi \circ s,$$

namely $\sigma = (\xi \circ \delta)^{-1} \circ \rho \circ \xi \circ s$.

By construction, this map σ is a section of $\pi : V \rightarrow \Delta$ on Δ'' , i.e., a holomorphic map $\sigma : \Delta'' \rightarrow V$ with $\pi \circ \sigma(z) = z$, $\forall z \in \Delta''$.

Recall that V is defined as

$$V = \{(z, g) \in \Delta \times G : F(z) = g(H(z))\}.$$

Therefore a section σ is given as

$$\sigma : z \mapsto (z, \phi(z))$$

where $\phi : \Delta'' \rightarrow G$ is a map which fulfills

$$F(z) = \phi(z)(H(z)) \quad \forall z \in \Delta''.$$

This yields the statement of the proposition if we choose $r \in (0, 1)$ such that $\Delta_r \subset \Delta''$. \square

Remark. *It is essential to assume that all the orbits have the same dimension. For example, let $G = \mathbb{C}^*$ act on $X = \mathbb{C}$ as usual, let $F(z) = z^2$ and $H(z) = z^3$. Then $\forall z, \exists \lambda \in G : \lambda z^3 = z^2$, but there is no holomorphic function $\phi : \Delta \rightarrow G$ with $\phi(z)z^3 = z^2$.*

10. AUTOMORPHISMS OF QUATERNIONS

10.1. Automorphisms of \mathbb{H} . We will need the following classical result (see e.g. [CS03], §6.8). It is based on the fact that \mathbb{H} is a central simple \mathbb{R} -algebra and the Skolem-Noether theorem (see e.g. [Lor08], §29) which states that every automorphism of a central simple algebra is inner.

Proposition 10.1. *Every ring automorphism of the \mathbb{R} -algebra of quaternions \mathbb{H} is already a \mathbb{R} -algebra automorphism (thus \mathbb{R} -linear and continuous) and preserves the scalar product on \mathbb{H} defined as*

$$\langle x, y \rangle = \frac{1}{2}(x\bar{y} + y\bar{x})$$

Let G be the group of ring automorphisms of \mathbb{H} .

Then G is isomorphic to $SO(3, \mathbb{R})$, acting trivially on the center \mathbb{R} and by the standard action of $SO(3, \mathbb{R})$ on \mathbb{R}^3 , the orthogonal complement W of \mathbb{R} , if we identify W with \mathbb{R}^3 using the standard basis i, j, k of W .

For $q \in \mathbb{H}$ let G_q be its isotropy group, i.e., $G_q = \{g \in G = \text{Aut}(\mathbb{H}) : g(q) = q\}$.

Then $G = G_q$, if q is in the center \mathbb{R} of \mathbb{H} and $G_q \cong S^1 \cong SO(2, \mathbb{R})$ for $q \notin \mathbb{R}$.

Remark. We have $G_{\mathbb{C}} \cong PSL_2(\mathbb{C}) \cong SO(3, \mathbb{C})$ acting on a three-dimensional vector space preserving a non-degenerate bilinear form. This representation must be isomorphic to the adjoint representation of $PSL_2(\mathbb{C})$ on its Lie algebra $\text{Lie}(PSL_2(\mathbb{C}))$ which preserves the Killing form $\text{Kill}(x, y) = \text{Trace}(\text{ad}(x)\text{ad}(y))$. Let $v \in \text{Lie}(PSL_2(\mathbb{C})) \setminus \{0\}$. Then $\text{Kill}(v, v) = 0$ if the one-parameter subgroup $\exp(v\mathbb{C})$ is unipotent, while $\exp(v\mathbb{C}) \cong \mathbb{C}^*$ if $\text{Kill}(v, v) \neq 0$. The isotropy group of $G_{\mathbb{C}}$ at $v \in \text{Lie}(G_{\mathbb{C}})$ is the centralizer of $\exp(v\mathbb{C})$. For $G_{\mathbb{C}} = PSL_2(\mathbb{C})$ and $v \neq 0$ this is always $\exp(v\mathbb{C})$ itself. Hence the isotropy group of $G_{\mathbb{C}}$ acting on $\text{Lie}(G_{\mathbb{C}})$ at an element $v \neq 0$ is isomorphic to \mathbb{C}^* if

$$\text{Kill}(v, v) \neq 0$$

and isomorphic to \mathbb{C} if $\text{Kill}(v, v) = 0$. Similarly for $G_{\mathbb{C}}$ acting on $W \otimes_{\mathbb{R}} \mathbb{C}$, replacing $\text{Kill}(\cdot, \cdot)$ by $B(\cdot, \cdot)$.

As a consequence one obtains:

Corollary 10.2. Let $p, q \in \mathbb{H} \setminus \{0\}$.

Then the following conditions are equivalent:

- $N(p) = N(q)$ and $\text{Tr}(p) = \text{Tr}(q)$.
- There is an \mathbb{R} -algebra automorphism ϕ of \mathbb{H} such that $\phi(p) = q$.

11. AUTOMORPHISMS OF $\mathbb{R}_3 = \mathbb{H} \oplus \mathbb{H}$

\mathbb{R}_3 denotes the Clifford algebra associated to \mathbb{R}^3 endowed with a positive definite quadratic form. As an \mathbb{R} -algebra, $\mathbb{R}_3 \cong \mathbb{H} \oplus \mathbb{H}$. This direct sum structure allows an easy determination of the automorphism group of \mathbb{R}_3 .

11.1. Algebraic Preparation.

Lemma 11.1. Let R be a (possibly non-commutative) ring with 1 and without zero-divisors. Then every ring automorphism of $A = R \oplus R$ is either of the form

$$(x, y) \mapsto (\phi(x), \psi(y)) \quad \phi, \psi \in \text{Aut}(R)$$

or of the form

$$(x, y) \mapsto (\psi(y), \phi(x)) \quad \phi, \psi \in \text{Aut}(R)$$

Proof. Since R has no zero-divisors, the only idempotents are 0 and 1, i.e.,

$$\{x \in R : x^2 = x\} = \{0, 1\}$$

As a consequence,

$$I = \{z = (x, y) \in R \times R : z^2 = z\} = \{0, 1\} \times \{0, 1\}$$

Every ring automorphism α of A must stabilize I and fix $0 = (0, 0)$ and $1 = (1, 1)$. Hence either α fixes $(0, 1)$ and $(1, 0)$, or

$$(0, 1) \xrightarrow{\alpha} (1, 0) \xrightarrow{\alpha} (0, 1)$$

In the first case, α stabilizes both $(0, 1)A = \{0\} \times R$ and $(1, 0)A = R \times \{0\}$, which implies that $\alpha(x, y) = (\phi(x), \psi(y))$ for some $\phi, \psi \in \text{Aut}(R)$.

Similarly in the second case

$$(x, y) \mapsto (\psi(y), \phi(x))$$

for some $\phi, \psi \in \text{Aut}(R)$. □

11.2. Description of $\text{Aut}(\mathbb{R}_3)$. As a consequence of Lemma 11.1 and the description of the automorphisms of \mathbb{H} , we obtain:

Proposition 11.2. *The automorphism group of the Clifford algebra $\mathbb{R}_3 \cong \mathbb{H} \oplus \mathbb{H}$ is generated by $(z, w) \mapsto (w, z)$ and $SO(3, \mathbb{R}) \times SO(3, \mathbb{R})$ where $SO(3, \mathbb{R})$ acts as the group of all orientation preserving orthogonal linear transformations of the imaginary parts of the factors of the product $\mathbb{H} \times \mathbb{H}$.*

The automorphism group of the complex algebra $\mathbb{R}_3 \otimes \mathbb{C}$ is generated by $(z, w) \mapsto (w, z)$ and $SO(3, \mathbb{C}) \times SO(3, \mathbb{C})$.

11.3. Isotropy groups. Using the above description of the full automorphism group and the product structure $\mathbb{R}_3 \cong \mathbb{H} \oplus \mathbb{H}$ it is easy to determine the isotropy groups of an element (q_1, q_2) :

- The isotropy group contains all automorphisms of the form $\phi : (x_1, x_2) \mapsto (\phi_1(x_1), \phi_2(x_2))$ with $\phi_i \in \text{Aut}(\mathbb{H})$ and $\phi_i(q_i) = q_i$ ($i \in \{1, 2\}$).
- If there is an automorphism $\alpha \in \text{Aut}(\mathbb{H})$ with $\alpha(q_1) = q_2$, then the isotropy group contains in addition all automorphisms of the form $\phi : (x_1, x_2) \mapsto (\phi_1(x_2), \phi_2(x_1))$ with $\phi_i \in \text{Aut}(\mathbb{H})$ and

$$(q_1, q_2) = (\phi_1(q_2), \phi_2(q_1))$$

In view of $\alpha(q_1) = q_2$ the above equation is equivalent to

$$\begin{aligned} (\alpha \circ \phi_1)(q_2) &= \alpha(q_1) &= q_2 \\ (\alpha^{-1} \circ \phi_2)(q_1) &= \alpha^{-1}(q_2) &= q_1 \end{aligned}$$

Therefore $\alpha \circ \phi_1$ must be in the isotropy group of q_2 and $\alpha^{-1} \circ \phi_2$ in the isotropy group of q_1 .

12. ORBITS IN THE COMPLEXIFIED ALGEBRA

The proposition below is principally applied to the situation, where $A \cong \mathbb{H}$ and $A = \mathbb{R} \oplus V$ as vector space, V being the subspace of totally imaginary elements.

In fact, we have seen that in this case $\text{Aut}(A)$ acts trivially on \mathbb{R} and by orthogonal transformations on V such that every sphere centered at the origin in V is one orbit.

Proposition 12.1. *Let $V = \mathbb{R}^n$, and let G be a connected real Lie group acting by orthogonal linear transformations on V such that the unit sphere $S = \{v \in \mathbb{R}^n : \|v\| = 1\}$ is a G -orbit.*

Let $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. Let B denote the \mathbb{C} -bilinear form on $V_{\mathbb{C}}$ extending the standard euclidean scalar product on $V = \mathbb{R}^n$.

Let $G_{\mathbb{C}}$ be the smallest complex Lie subgroup of $GL(V_{\mathbb{C}})$ containing G . Then the $G_{\mathbb{C}}$ -orbits in $V_{\mathbb{C}}$ are the following:

- $H_{\lambda} = \{v \in V_{\mathbb{C}} : B(v, v) = \lambda\}$ for $\lambda \in \mathbb{C}^*$.
- $H_0 = \{v \in V_{\mathbb{C}} : B(v, v) = 0\} \setminus \{0\}$
- $\{0\}$.

Proof. First we observe that $B(\cdot, \cdot)$ is invariant under the $G_{\mathbb{C}}$ -action, because G acts by orthogonal transformations.

Since G acts transitively on S , and G acts linearly on V , the G -orbits in V are precisely $\{0\}$ and the spheres $S_r = \{v \in V : \|v\| = r\}$ ($r > 0$).

Let $v = u + \iota w \in V_{\mathbb{C}}$, $u, w \in V$. If $v \neq 0$, then $(u, w) \neq (0, 0)$. If $u \neq 0$, then the projection onto the real part yields a map from the G -orbit through v onto a sphere in \mathbb{R}^n , which implies that the real tangent space of Gv contains at least $n - 1$ \mathbb{C} -linearly independent tangent vectors. It follows that the $G_{\mathbb{C}}$ -orbit through v is of complex dimension $\geq n - 1$.

On the other hand the complex hypersurfaces

$$C_{\lambda} = \{v \in V_{\mathbb{C}} : B(v, v) = \lambda\}$$

are irreducible, complex $(n - 1)$ -dimensional and $G_{\mathbb{C}}$ -invariant. This implies the assertion. \square

Corollary 12.2. *Let A be a finite-dimensional \mathbb{R} -algebra with \mathbb{R} as center. Let $A = \mathbb{R} \oplus V$ as vector space and let G be a real Lie group acting trivially on \mathbb{R} and by orthogonal linear transformations on V . Assume that G acts transitively on the unit sphere of V .*

Let $G_{\mathbb{C}}$ be the smallest complex Lie subgroup of $GL(A \otimes_{\mathbb{R}} \mathbb{C})$ containing G .

Then all the $G_{\mathbb{C}}$ -orbits in $(V \otimes_{\mathbb{R}} \mathbb{C}) \setminus \{0\}$ are complex hypersurfaces. In particular, they all have the same dimension.

Corollary 12.3. *Under the assumptions of Corollary 12.2 for every point $q \in (V \otimes_{\mathbb{R}} \mathbb{C}) \setminus \{0\}$ the isotropy group I of the $G_{\mathbb{C}}$ -action at q satisfies*

$$\dim_{\mathbb{C}} I + \dim_{\mathbb{R}} V - 1 = \dim_{\mathbb{C}}(G_{\mathbb{C}})$$

In particular, we have $\dim_{\mathbb{C}}(I) = 1$ for $A \cong \mathbb{H}$.

Proof. The $G_{\mathbb{C}}$ -orbit through q is a complex hypersurface (Corollary 12.2). Hence its complex dimension equals $\dim_{\mathbb{C}}(V \otimes_{\mathbb{R}} \mathbb{C}) - 1$ and therefore

$$\dim_{\mathbb{C}}(G_{\mathbb{C}}) = \dim_{\mathbb{C}} I + \dim_{\mathbb{C}}(V \otimes_{\mathbb{R}} \mathbb{C}) - 1$$

Now $\dim_{\mathbb{R}}(V) = \dim_{\mathbb{C}}(V \otimes_{\mathbb{R}} \mathbb{C})$. This yields the assertion. \square

Corollary 12.4. *Under the same assumptions, there is a Zariski open subset, namely $\Omega \subset A \otimes_{\mathbb{R}} \mathbb{C} \overset{\sim}{\simeq} \mathbb{C} \oplus (V \otimes_{\mathbb{R}} \mathbb{C})$ defined as $\Omega = \{q : \zeta(q) = (x, v), B(v, v) \neq 0\}$ such that all the isotropy groups of $G_{\mathbb{C}}$ at points in Ω are conjugate.*

Proof. By assumption, $A = \mathbb{R} \oplus V$ as vector space and correspondingly

$$A \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \underbrace{(V \otimes_{\mathbb{R}} \mathbb{C})}_{V_{\mathbb{C}}}$$

With respect to this vector space sum decomposition we define

$$\Omega = \{r + v : r \in \mathbb{C}, v \in V_{\mathbb{C}}, B(v, v) \neq 0\}$$

Due to Proposition 12.1, the $G_{\mathbb{C}}$ -orbit through a given point $x = r + v \in \Omega$ is

$$\{r + w, w \in V_{\mathbb{C}}, B(w, w) = B(v, v)\}.$$

The isotropy groups in the same orbit are clearly conjugate. Let $x = r + v, y = s + w \in \Omega$ (with $r, s \in \mathbb{C}, v, w \in V_{\mathbb{C}}$) be in different orbits. Choose $\lambda \in \mathbb{C}^*$ such that $B(w, w) = B(\lambda v, \lambda v)$. Now the isotropy group at $r + v$ agrees with the isotropy group at $\lambda(r + v) = \lambda r + \lambda v$, because the group action is linear. Since $G_{\mathbb{C}}$ acts trivially on the center \mathbb{C} of $A \otimes_{\mathbb{R}} \mathbb{C}$, the isotropy groups at $\lambda r + \lambda v$ and $s + \lambda v$ coincide. Finally, due to $B(w, w) = B(\lambda v, \lambda v)$ the elements $s + \lambda v$ and $s + w$ are in the same $G_{\mathbb{C}}$ -orbit. Consequently, their isotropy groups are conjugate.

Combined, these facts imply that the isotropy group at $r + v$ and $s + w$ are conjugate. \square

Corollary 12.5. *Let $A = \mathbb{H}$ and $A_{\mathbb{C}} = A \otimes_{\mathbb{R}} \mathbb{C}$.*

Let $p, q \in A_{\mathbb{C}} \setminus \{0\}$. Then the following properties are equivalent:

- (i) $\text{Tr}(p) = \text{Tr}(q)$ and $\text{N}(p) = \text{N}(q)$.
- (ii) *There is an automorphism $\phi \in \text{Aut}(A_{\mathbb{C}})$ such that $\phi(p) = q$.*

Proof. By construction, we have $\text{N}(x) = x(x^c) = B(x, x)$ for all $x \in A_{\mathbb{C}}$ and $x \mapsto \frac{1}{2}\text{tr}(x)$ equals the projection of x to \mathbb{C} with respect to the direct sum decomposition $A_{\mathbb{C}} = \mathbb{C} \oplus W$.

Let $p = p' + p''$, $q = q' + q''$ with $p', q' \in \mathbb{C}$, $p'', q'' \in W$. Since $\text{Aut}(A_{\mathbb{C}})$ acts trivially on \mathbb{C} and by linear, B -preserving transformations on W , (ii) implies that $p' = q'$ and $B(p'', p'') = B(q'', q'')$ which in turn implies $\text{Tr}(p) = \text{Tr}(q)$, $\text{N}(p) = \text{N}(q)$.

Conversely,

$$(\text{Tr}(p) = \text{Tr}(q)) \wedge (\text{N}(p) = \text{N}(q)) \implies (p' = q') \wedge (B(p'', p'') = B(q'', q''))$$

and the latter condition implies $\exists \phi \in \text{Aut}(A_{\mathbb{C}}) : \phi(p) = q$ due to Proposition 12.1. \square

13. CONSTRUCTING AN AUXILIARY VECTOR FIELD

Proposition 13.1. *Let G be a complex Lie group acting holomorphically on a complex manifold X . Assume that all isotropy groups are connected and one-dimensional.*

Let Z be a non-compact Riemann surface.

Let $C, F : Z \rightarrow X$ be holomorphic maps and let

$$V = \{(g, t) \in G \times Z : g \cdot C(t) = F(t)\}$$

Let $\pi : V \rightarrow Z$ be the natural projection map: $\pi(g, t) = t$.

Then there is an integrable holomorphic vector field \overline{X} on V such that for every $t \in Z$ the associated flow (one-parameter-group) stabilizes $V_t = \pi^{-1}(\{t\})$ and coincides with the action of the isotropy group

$$G_{F(t)} = \{g \in G : g \cdot F(t) = F(t)\}$$

on V_t (with $g : (h, t) \mapsto (gh, t)$).

Proof. We define a line bundle on Z by associating to each point $p \in Z$ the Lie algebra of

$$\{g \in G : g \cdot C(p) = C(p)\}$$

On a non-compact Riemann surface every holomorphic line bundle is trivial. Thus we obtain \overline{X} as any nowhere vanishing section of this trivial holomorphic line bundle. \square

14. REDUCTION TO THE CASE $\text{Tr} = 0$

Lemma 14.1. *Let $A_{\mathbb{C}}$ be a \mathbb{C} -algebra with an antiinvolution $(\)^c$ such that $Z = \{x \in A_{\mathbb{C}} : x = x^c\}$ is central in $A_{\mathbb{C}}$. Let $G_{\mathbb{C}}$ be a complex Lie group acting by \mathbb{C} -algebra homomorphisms on $A_{\mathbb{C}}$ such that the Z is fixed pointwise. Let D be a domain in \mathbb{C} . Let $F, H : D \rightarrow A_{\mathbb{C}}$ be holomorphic maps. Assume that $\text{N}(F) = \text{N}(H)$ and $\text{Tr}(F) = \text{Tr}(H)$ (with $\text{Tr}(F) = F + F^c$ and $\text{N}(F) = FF^c$).*

Define $\hat{F} = \frac{1}{2}(F - F^c)$ and $\hat{H} = \frac{1}{2}(H - H^c)$.

Then:

- (i) $\text{Tr}(\hat{F}) = 0 = \text{Tr}(\hat{H})$.
- (ii) $\text{N}(\hat{F}) = \text{N}(\hat{H})$.
- (iii) *There exists a holomorphic map $\phi : D \rightarrow G_{\mathbb{C}}$ with $F = \phi(H)$ if and only there exists a holomorphic map $\phi : D \rightarrow G_{\mathbb{C}}$ with $\hat{F} = \phi(\hat{H})$.*

Proof. (i)

$$\text{Tr}(\hat{F}) = \frac{1}{2} \text{Tr}(F - F^c) = \frac{1}{2} (F + F^c - (F^c + F)) = 0$$

and similarly $\text{Tr}(\hat{H}) = 0$.

- (ii) Let $F_0 = \frac{1}{2} \text{Tr}(F)$ and $H_0 = \frac{1}{2} \text{Tr}(H)$. Then $F_0^c = F_0$ and $H_0^c = H_0$. Observe that F_0 and H_0 are central in $A_{\mathbb{C}}$.

By construction

$$F = F_0 + \hat{F}, \quad H = H_0 + \hat{H}, \quad \hat{F}^c = -\hat{F}, \quad \hat{H}^c = -\hat{H}$$

We obtain:

$$\begin{aligned} \text{N}(F) &= \text{N}(F_0 + \hat{F}) = (F_0 + \hat{F}) \cdot (F_0^c + (\hat{F})^c) \\ &= (F_0 + \hat{F}) \cdot (F_0 - \hat{F}) \\ &= F_0^2 - F_0 \hat{F} + \hat{F} F_0 - (\hat{F})^2 \\ &= F_0^2 - \hat{F} F_0 + \hat{F} F_0 - (\hat{F})^2 \text{ because } F_0 \text{ is central} \\ &= F_0^2 - (\hat{F})^2 \\ &= \frac{1}{4} (\text{Tr}(F))^2 + \text{N}(\hat{F}) \end{aligned}$$

Similarly we obtain

$$\text{N}(H) = \frac{1}{4} (\text{Tr}(H))^2 + \text{N}(\hat{H})$$

In combination with $N(F) = N(H)$ and $\text{Tr}(F) = \text{Tr}(H)$ this yields $N(\hat{F}) = N(\hat{H})$.

- (iii) Note that the image of $\text{Tr} : A_{\mathbb{C}} \rightarrow A_{\mathbb{C}}$ is contained in the center of $A_{\mathbb{C}}$ which is pointwise stabilized by the $G_{\mathbb{C}}$ -action. It follows that for every holomorphic map $\phi : D \rightarrow A_{\mathbb{C}}$ we have

$$\begin{aligned} F(z) &= \phi(z) (H(z)) \quad \forall z \in D \\ F_0(z) + \hat{F}(z) &= \underbrace{\phi(z) (H_0(z))}_{=H_0(z)=F_0(z)} + \phi(z) (\hat{H}(z)) \\ \hat{F}(z) &= \phi(z) (\hat{H}(z)) \end{aligned}$$

□

15. VANISHING ORDERS

Proposition 15.1. *Let X be a non-compact Riemann surface, V a vector space and let $f, g : X \rightarrow V$ be holomorphic maps which do not vanish identically.*

Assume that f, g have the same divisor (as defined in Definition 6.1).

Then there exist a holomorphic function $h : X \rightarrow \mathbb{C}$ and holomorphic maps $\tilde{f}, \tilde{g} : X \rightarrow V \setminus \{0\}$ such that $f = h\tilde{f}$, $g = h\tilde{g}$.

Proof. Recall that on a non-compact Riemann surface every divisor is a *principal* divisor, i.e., the divisor of a holomorphic function.

We choose a holomorphic function h on X with

$$\text{div}(h) = \text{div}(g) = \text{div}(f)$$

and define $\tilde{f} = f/h$, $\tilde{g} = g/h$. □

Lemma 15.2. *Let X be a Riemann surface, V a vector space and $f : X \rightarrow V$, $\phi : X \rightarrow GL(V)$ be holomorphic maps. Assume that f is not vanishing identically.*

Define $g(z) = \phi(z) (f(z))$.

Then f and g have the same divisor.

Proof. Let $\text{div}(f) = \sum_p m_p \{p\}$. Then for every $p \in X$ and $i \in \{1, \dots, n\}$ the germ of f_i at p is divisible by $z_p^{m_p}$ where z_p is a local coordinate with $z_p(p) = 0$. Since $\phi(p)$ is linear, the components g_i likewise have germs at p which are divisible by $z_p^{m_p}$. Hence $\text{div}(g) \geq \text{div}(f)$.

The same arguments show that also $\text{div}(f) \geq \text{div}(g)$, since

$$f(z) = \tilde{\phi}(z) (g(z))$$

for

$$\tilde{\phi}(z) = (\phi(z))^{-1}$$

Thus $\operatorname{div}(f) = \operatorname{div}(g)$. \square

16. SOME COHOMOLOGY

We need the lemma below as a preparation for the proof of Proposition 16.2.

Lemma 16.1. *Let X be a Riemann surface, $D \subset X$ a discrete subset, $\tau_0 : P \rightarrow X \setminus D$ an unramified $2 : 1$ -covering.*

Then τ_0 extends to a ramified covering $\tau : X' \rightarrow X$, i.e., we have a commutative diagram:

$$\begin{array}{ccc} P & \hookrightarrow & X' \\ \downarrow \tau_0 & & \downarrow \tau \\ X \setminus D & \hookrightarrow & X \end{array}$$

Proof. Let $q \in D$ and let $\psi : U \rightarrow \Delta$ be a coordinate chart with $\psi(q) = 0$ and $U \cap D = \{q\}$.

An unramified $2 : 1$ covering over $\Delta^* = \Delta \setminus \{0\}$ is either a product or given by a group homomorphism $\rho : \pi_1(\Delta^*) \rightarrow \mathbb{Z}/2\mathbb{Z}$. There is only one non-trivial group homomorphism from $\mathbb{Z} \cong \pi_1(\Delta^*)$ to $\mathbb{Z}/2\mathbb{Z}$.

Hence either τ_0 restricts on $U^* = U \setminus \{q\}$ to a direct product $U^* \times \mathbb{Z}/2\mathbb{Z}$ (in which case τ_0 trivially extends through q), or we can identify the restriction to U^* as the only non-trivial $2 : 1$ -covering, which can be realized as $z \mapsto z^2$ as a map from Δ^* to Δ^* . Thus we obtain the following commutative diagram:

$$\begin{array}{ccc} \tau^{-1}(U^*) & \xrightarrow{\sim} & \Delta^* \\ \downarrow \tau_0 & & \downarrow z \mapsto z^2 \\ U^* & \xrightarrow{\psi} & \Delta^* \end{array}$$

Now the covering map $z \mapsto z^2$ obviously extends from an unramified covering $\Delta^* \rightarrow \Delta^*$ to a ramified covering $\Delta \rightarrow \Delta$.

Performing this procedure around every point of D , we obtain the desired extension. \square

Proposition 16.2. *Let X be a non-compact Riemann surface, D a discrete subset, \mathcal{S} a sheaf which is locally isomorphic to $\underline{\mathbb{Z}}$ (the sheaf of locally constant \mathbb{Z} -valued functions) on $X \setminus D$. Let*

$$(16.1) \quad 0 \rightarrow \mathcal{S} \rightarrow \mathcal{O}_X \rightarrow \mathcal{A} \rightarrow 0$$

be a short exact sequence of \mathcal{O}_X module sheaves.

Then $H^1(X, \mathcal{A}) = \{0\}$ and $H^2(X, \mathcal{S}) = \{0\}$.

Proof. X is Stein, hence $H^k(X, \mathcal{O}) = 0$ for $k > 0$. Therefore the long exact cohomology sequence associated to the above sequence of sheaves (16.1) implies

$$H^1(X, \mathcal{A}) \cong H^2(X, \mathcal{S}).$$

Let \mathcal{S}_0 be the sheaf on X defined by

$$\mathcal{S}_0(U) = \begin{cases} \{0\} & \text{if } U \cap D \neq \emptyset \\ \mathcal{S}(U) & \text{if } U \cap D = \emptyset \end{cases}$$

Then we have a short exact sequence

$$(16.2) \quad 0 \rightarrow \mathcal{S}_0 \rightarrow \mathcal{S} \rightarrow \mathcal{F} \rightarrow 0$$

where \mathcal{F} is a skyscraper sheaf supported on D . Thus $H^k(X, \mathcal{F}) = 0$ for $k > 0$. Consequently the long exact cohomology sequence associated to (16.2) implies

$$H^2(X, \mathcal{S}) \cong H^2(X, \mathcal{S}_0).$$

By construction, \mathcal{S}_0 is a sheaf on $X \setminus D$ locally isomorphic to $\underline{\mathbb{Z}}$. $(\mathbb{Z}, +)$ admits only one non-trivial automorphism, namely $n \mapsto -n$. Hence we may regard \mathcal{S}_0 as the sheaf of sections in a locally trivial fiber bundle $B \rightarrow X \setminus D$ with structure group $\mathbb{Z}/2\mathbb{Z}$ and typical fiber \mathbb{Z} . We consider the sub-bundle P with typical fiber $\{+1, -1\}$. Such a bundle is an unramified 2:1-covering $\tau_0 : P \rightarrow X \setminus D$. It extends to a ramified 2:1-covering $\tau : X' \rightarrow X$ by adding one point above each point of D .

We consider the natural sheaf homomorphism $\underline{\mathbb{Z}}_X \rightarrow \tau_* \underline{\mathbb{Z}}_{X'}$ given by $f \mapsto f \circ \tau$. We define a sheaf homomorphism α from $\tau_* \underline{\mathbb{Z}}_{X'}$ to \mathcal{S}_0 as follows: Given a \mathbb{Z} -valued function f on $\{-1, +1\}$, let

$$d = |f(1) - f(-1)|,$$

and associate

$$\alpha(f) \stackrel{\text{def}}{=} \begin{cases} d & \text{if } f(1) \geq f(-1) \\ -d & \text{if } f(1) < f(-1) \end{cases}$$

This yields a short exact sequence of sheaves

$$0 \rightarrow \underline{\mathbb{Z}}_X \rightarrow \tau_* \underline{\mathbb{Z}}_{X'} \xrightarrow{\alpha} \mathcal{S}_0 \rightarrow 0$$

We note that outside D the covering map τ is locally biholomorphic, while for a point $p \in D$, the preimage of a small neighborhood of p in X will be a small neighborhood of $\tau^{-1}(p)$ in X' and will be contractible, since X' is a complex space.

It follows that the higher direct image sheaves $R^q\tau_*\underline{\mathbb{Z}}$ ($q > 0$) are trivial. We consider the Leray spectral sequence for the sheaf $\underline{\mathbb{Z}}$ on X' and the map $\tau : X' \rightarrow X$:

$$H^{p+q}(X', \underline{\mathbb{Z}}) = E^{p+q} \leftarrow E_2^{p+q} = H^p(X, R^q\tau_*\underline{\mathbb{Z}})$$

Since $R^q\tau_*\underline{\mathbb{Z}} = 0 \forall q > 0$, this spectral sequence degenerates. Therefore

$$H^k(X, \tau_*\underline{\mathbb{Z}}) \cong H^k(X', \underline{\mathbb{Z}}) \cong H^k(X', \mathbb{Z})$$

Since X' and X are non-compact Riemann surfaces, we have $H^k(X', \mathbb{Z}) = H^k(X, \mathbb{Z}) = 0$ for $k \geq 2$ and therefore

$$\dots \rightarrow \underbrace{H^2(X', \mathbb{Z})}_{=0} \rightarrow H^2(X, \mathcal{S}_0) \rightarrow \underbrace{H^3(X, \mathbb{Z})}_{=0} \rightarrow \dots$$

Therefore

$$H^1(X, \mathcal{A}) \cong H^2(X, \mathcal{S}) \cong H^2(X, \mathcal{S}_0) = \{0\}.$$

□

17. AUTOMORPHISMS AND CONJUGACY

For \mathbb{H} and $\mathbb{H}_{\mathbb{C}}$ every \mathbb{R} - (resp. \mathbb{C} -) algebra automorphism ϕ is *inner*, i.e., given by conjugation with an element: $\exists q : \phi : x \mapsto qxq^{-1}$. Conversely for every $q \in \mathbb{H}$ (resp. $q \in \mathbb{H}_{\mathbb{C}}$) conjugation by q defines an automorphism. This automorphism is trivial if and only if q is in the center.

Therefore we have a short exact sequence of Lie groups.

$$(17.1) \quad 1 \rightarrow \mathbb{C}^* \rightarrow \mathbb{H}_{\mathbb{C}}^* \xrightarrow{\zeta} G_{\mathbb{C}} \rightarrow 1$$

with $\zeta(g) : x \mapsto gxg^{-1}$.

In particular, $G_{\mathbb{C}} \cong \mathbb{H}_{\mathbb{C}}^*/\mathbb{C}^*$, which implies that $\mathbb{H}_{\mathbb{C}}^*$ is a \mathbb{C}^* -principal bundle over $G_{\mathbb{C}}$ (see e.g. [SM21], Prop. 13.25).

Proposition 17.1. *Let D be an open subset in \mathbb{C} and let $\phi : D \rightarrow G_{\mathbb{C}}$ be a holomorphic map.*

Then there exists a holomorphic map $\tilde{\phi} : D \rightarrow \mathbb{H}_{\mathbb{C}}^$ with $\phi = \zeta \circ \tilde{\phi}$.*

Proof. Due to (17.1) we have a \mathbb{C}^* -principal bundle on $G_{\mathbb{C}}$ which we may pull back via ϕ to obtain a \mathbb{C}^* -principal bundle on D . But D is a (not necessarily connected) Stein Riemann surface and therefore every \mathbb{C}^* -principal bundle on D is holomorphically trivial. Thus it admits a

section. Such a section corresponds to a lifting as below

$$\begin{array}{ccc} & & \mathbb{H}_{\mathbb{C}}^* \\ & \nearrow \tilde{\phi} & \downarrow \zeta \\ D & \xrightarrow{\phi} & G_{\mathbb{C}} \end{array}$$

□

18. PROOF OF THE MAIN THEOREM

We are now in a position to prove our Main Theorem 1.1.

First we prove the most difficult part of Theorem 1.1 which is the following.

Theorem 18.1. *Let $A = \mathbb{H}$ and $A_{\mathbb{C}} = A \otimes_{\mathbb{R}} \mathbb{C}$.*

Let $D \subset \mathbb{C}$ be a domain which is invariant under conjugation.

Let $F, H : D \rightarrow A_{\mathbb{C}} \setminus \{0\}$ be holomorphic maps such that $\overline{F(\bar{z})} = F(z)$, $\overline{H(\bar{z})} = H(z)$, $\text{Tr}(F) = \text{Tr}(H) = 0$, $N(F) = N(H)$.⁵

Then there exists a holomorphic map $\phi : D \rightarrow G_{\mathbb{C}}$ such that

$$\phi(z)(F(z)) = H(z) \quad \forall z \in D.$$

Proof. Throughout the proof we will use the fact that D is a non-compact Riemann surface.

Note that we assume $\text{Tr}(F) = \text{Tr}(H) = 0$. It follows that the images $F(D), H(D)$ are contained in $W \otimes_{\mathbb{R}} \mathbb{C}$.

Thus we may regard F and H as holomorphic maps from D to $(W \otimes_{\mathbb{R}} \mathbb{C}) \setminus \{0\} = (W \otimes_{\mathbb{R}} \mathbb{C}) \cap (A_{\mathbb{C}} \setminus \{0\})$.

From our assumption on F and H we deduce that for every $t \in D$ there is an element $g \in G_{\mathbb{C}}$ with $H(t) = g(F(t))$ (Corollary 12.5).

The case where both F and H are constant is trivial. Hence we may assume that at least one of the two maps is not constant. Wlog F is not constant.

We define

$$V = \{(\alpha, z) \in G_{\mathbb{C}} \times D : F(z) = \alpha H(z)\}$$

The isotropy groups for the $G_{\mathbb{C}}$ -action on $(W \otimes_{\mathbb{R}} \mathbb{C}) \setminus \{0\}$ have all the same dimension (namely 1 if $A = \mathbb{H}$) due to Corollary 12.3.

⁵We need no condition on *cdiv*, because the assumptions $\text{Tr}(F) = \text{Tr}(H) = 0$ and $F(D), H(D) \subset A_{\mathbb{C}} \setminus \{0\}$ imply that *cdiv*(F) and *cdiv*(H) are empty.

Therefore we may apply Proposition 9.4. It follows that for every $p \in D$ there is an open neighborhood U of p in D and a holomorphic map $\psi : U \rightarrow G_{\mathbb{C}}$ with

$$F(z) = \psi(z)H(z) \quad \forall z \in U.$$

Thanks to Proposition 13.1 we obtain a vector field which gives us a one-parameter group acting transitively on the fibers of $V \rightarrow D$, i.e., there is an action

$$\mu : (\mathbb{C}, +) \times V \rightarrow V.$$

We define \mathcal{A} as the sheaf on D of holomorphic maps ξ from D to $G_{\mathbb{C}}$ such that $\xi(t)$ is contained in the isotropy group

$$G_{F(t)} = \{g \in G : g(F(t)) = F(t)\}$$

for every t .

Due to Proposition 13.1 we may identify \mathcal{A} with a sheaf of fiber-preserving automorphisms, namely for an open subset $U \subset D$ we define $\mathcal{A}(U)$ as the set of biholomorphic self maps of $\pi^{-1}(U)$ which may be written as

$$\eta_{\zeta} : x \mapsto \mu(\zeta(\pi(x)), x)$$

for some holomorphic function $\zeta \in \mathcal{O}(U)$.

The map associating η_{ζ} to ζ defines a morphism of sheaves on D . Let \mathcal{S} denote its kernel. Then we have a short exact sequence of sheaves

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{O} \xrightarrow{\zeta \mapsto \eta_{\zeta}} \mathcal{A} \rightarrow 0$$

\mathcal{S} may be regarded as the sheaf of those holomorphic functions ζ for which $\mu(\zeta(t))$ equals the identity map on the fiber $\pi^{-1}(t)$.

Generically, the isotropy group of $G_{\mathbb{C}}$ at a point in $A_{\mathbb{C}}$ is isomorphic to \mathbb{C}^* . Elsewhere, the isotropy group is \mathbb{C} . Hence the sheaf \mathcal{S} is locally isomorphic to $\underline{\mathbb{Z}}$ on an open subset Ω of D and trivial on the complement $D \setminus \Omega$.

Next we want to show that $H^1(D, \mathcal{A}) = \{0\}$.

If Ω is empty, then \mathcal{S} is trivial and consequently $\mathcal{A} \cong \mathcal{O}$. This implies $H^1(D, \mathcal{A}) = \{0\}$, because D is Stein.

Thus we may assume that Ω is not empty. Then Ω is Zariski open and dense and its complement $D \setminus \Omega$ is discrete. Then $H^1(D, \mathcal{A}) = 0$ follows from Proposition 16.2.

We choose an open cover $(U_i)_i$ of D with holomorphic maps $\psi_i : U_i \rightarrow G_{\mathbb{C}}$ such that

$$F(z) = \psi_i(z)H(z), \quad \forall z \in U_i$$

(This is possible thanks to Proposition 9.4.)

On each intersection $U_{ij} = U_i \cap U_j$ we have

$$F(z) = \psi_i(z)H(z) \text{ and } F(z) = \psi_j(z)H(z),$$

implying that

$$\psi_{ji}(z) \stackrel{\text{def}}{=} \psi_j(z) \circ (\psi_i(z))^{-1}$$

is contained in the isotropy group at $F(z)$.

Thus ψ_{ij} defines a 1-cocycle of \mathcal{A} . Since $H^1(D, \mathcal{A}) = 0$, this cocycle is a coboundary, i.e., there are $\phi_i \in \Gamma(U_i, \mathcal{A})$ with $\psi_{ij} = \phi_i(\phi_j)^{-1}$.

Therefore

$$\begin{aligned} \psi_i(z)(\psi_j(z))^{-1} &= \psi_{ij}(z) = \phi_i(z)(\phi_j(z))^{-1} \\ \implies \psi_i(z) &= \phi_i(z)(\phi_j(z))^{-1}(\psi_j(z)) \\ \implies (\phi_i(z))^{-1}\psi_i(z) &= (\phi_j(z))^{-1}(\psi_j(z)) \end{aligned}$$

It follows that

$$\phi(z) = (\phi_i(z))^{-1}\psi_i(z)$$

is well-defined. Since $F(z) = \psi_i(z)H(z)$ and $\phi_i(z)$ is in the isotropy group of $G_{\mathbb{C}}$ at $F(z)$, we infer

$$F(z) = \phi(z)H(z)$$

This completes the proof. □

Proof of the Theorem 1.1. First we deal with the case where neither f nor h is slice preserving.

We proceed as follows:

$$\begin{array}{ccccc} (i) & \iff & (ii) & \iff & (iii) \\ & & \Downarrow & \nearrow & \\ (v) & \iff & (iv) & & \end{array}$$

(ii) \implies (iv):

By assumption we have $\text{Tr}(F) = \text{Tr}(H)$. Define

$$\hat{F} = \frac{1}{2}(F - F^c), \quad \hat{H} = \frac{1}{2}(H - H^c)$$

Evidently $\text{Tr}(\hat{H}) = \text{Tr}(\hat{F}) = 0$. Moreover $N(F) = N(H)$ in combination with Lemma 14.1 implies that $N(\hat{F}) = N(\hat{H})$.

With respect to the decomposition $A_{\mathbb{C}} = Z \oplus W \otimes_{\mathbb{R}} \mathbb{C}$ the map \hat{F} resp. \hat{H} is just the second component of F resp. H . By the definition

of the central divisor (introduced in §6.2) this implies that $cdiv(F) = cdiv(\hat{F})$ and $cdiv(H) = cdiv(\hat{H})$.

Since $cdiv(F) = cdiv(H)$, it follows that there are holomorphic maps $\tilde{F}, \tilde{H} : D \rightarrow A_{\mathbb{C}} \setminus \{0\}$ and $\lambda : D \rightarrow \mathbb{C}$ such that

$$\hat{F} = \lambda \tilde{F}, \quad \hat{H} = \lambda \tilde{H}$$

(Proposition 15.1). Observe that

$$\begin{aligned} 0 &= \text{Tr}(\hat{F}) = \lambda \text{Tr}(\tilde{F}), \quad N(\hat{F}) = \lambda^2 N(\tilde{F}) \\ 0 &= \text{Tr}(\hat{H}) = \lambda \text{Tr}(\tilde{H}), \quad N(\hat{H}) = \lambda^2 N(\tilde{H}) \end{aligned}$$

and that λ does not vanish identically, because f and h are not slice preserving. Hence $\text{Tr}(\tilde{F}) = 0 = \text{Tr}(\tilde{H})$ and $N(\tilde{F}) = N(\tilde{H})$ and Theorem 18.1 implies that there is a holomorphic map $\phi : D \rightarrow G_{\mathbb{C}}$ such that

$$\phi(z) \left(\tilde{F}(z) \right) = \tilde{H}(z) \quad \forall z \in D.$$

which in turn implies

$$\phi(z) \left(\hat{F}(z) \right) = \hat{H}(z) \quad \forall z \in D,$$

because $G_{\mathbb{C}}$ acts linearly, $\hat{F} = h\tilde{F}$ and $\hat{H} = h\tilde{H}$. Finally

$$\phi(z) (F(z)) = H(z) \quad \forall z \in D$$

follows via Lemma 14.1.

(*iv*) \implies (*iii*): The implication (*iv*) $\implies cdiv(F) = cdiv(H)$ is due to Lemma 15.2, the other assertion is obvious.

For (*iii*) \iff (*ii*) see Corollary 12.5.

For (*i*) \iff (*ii*), see Proposition 2.3 and Section 6.2.

(*v*) \implies (*iv*). Every $\alpha \in \mathbb{H}_{\mathbb{C}}^*$ defines an automorphism of $\mathbb{H}_{\mathbb{C}}$ via $q \mapsto \alpha q \alpha^{-1}$. This defines a natural map from $\mathbb{H}_{\mathbb{C}}^*$ to $G_{\mathbb{C}}$. Composition with this map yields the desired implication.

(*iv*) \implies (*v*) follows from Proposition 17.1.

This finishes the proof for case *a*). Let us deal now with the case *b*), i.e. we assume that f is slice preserving.

(*i*) \iff (*ii*) is due to the correspondence between slice regular functions and stem functions.

(*ii*) \implies (*v*) \implies (*iv*) \implies (*iii*) is trivial.

(*iii*) \implies (*ii*): Because f is slice preserving, all the values of F are contained in the center $Z \cong \mathbb{C}$ of $\mathbb{H}_{\mathbb{C}}$. But $G_{\mathbb{C}}$ fixes the center pointwise. Hence the statement. \square

REFERENCES

- [AB19a] Amedeo Altavilla and Cinzia Bisi. Log-biharmonic and a Jensen formula in the space of quaternions. *Ann. Acad. Sci. Fenn. Math.*, 44(2):805–839, 2019.
- [AB19b] Daniele Angella and Cinzia Bisi. Slice-quaternionic Hopf surfaces. *J. Geom. Anal.*, 29(3):1837–1858, 2019.
- [ADF20a] Amedeo Altavilla and Chiara De Fabritiis. Applications of the Sylvester operator in the space of slice semi-regular functions. *Concr. Oper.*, 7:1–12, 2020.
- [ADF20b] Amedeo Altavilla and Chiara De Fabritiis. Equivalence of slice semi-regular functions via Sylvester operators. *Linear Algebra and its Applications*, 607:151–189, 2020.
- [Bae02] John C. Baez. The octonions. *Bull. Amer. Math. Soc. (N.S.)*, 39(2):145–205, 2002.
- [BDM22] Cinzia Bisi and Antonino De Martino. On Brolin’s theorem over the quaternions. *Indiana University Mathematics Journal*, 71(4):1675–1705, 2022.
- [BDMW23] Cinzia Bisi, Antonino De Martino, and Jörg Winkelmann. On a Runge theorem over \mathbb{R}_3 . *Ann. Mat. Pura Appl. (4)*, 202(4):1531–1556, 2023.
- [BG18] Cinzia Bisi and Graziano Gentili. On quaternionic tori and their moduli space. *J. Noncommut. Geom.*, 12(2):473–510, 2018.
- [BS17] Cinzia Bisi and Caterina Stoppato. Landau’s theorem for slice regular functions on the quaternionic unit ball. *Internat. J. Math.*, 28(3):1750017, 21, 2017.
- [BW20] Cinzia Bisi and Jörg Winkelmann. On a quaternionic Picard theorem. *Proc. Amer. Math. Soc. Ser. B*, 7:106–117, 2020.
- [BW21a] Cinzia Bisi and Jörg Winkelmann. The harmonicity of slice regular functions. *J. Geom. Anal.*, 31(8):7773–7811, 2021.
- [BW21b] Cinzia Bisi and Jörg Winkelmann. On Runge pairs and topology of axially symmetric domains. *J. Noncommut. Geom.*, 15(2):713–734, 2021.
- [CS03] John H. Conway and Derek A. Smith. *On quaternions and octonions: their geometry, arithmetic, and symmetry*. A K Peters, Ltd., Natick, MA, 2003.
- [FH91] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.
- [For04] Franc Forstnerič. Holomorphic submersions from Stein manifolds. *Ann. Inst. Fourier (Grenoble)*, 54(6):1913–1942 (2005), 2004.
- [For17] Franc Forstnerič. *Stein manifolds and holomorphic mappings*, volume 56 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Cham, second edition, 2017. The homotopy principle in complex analysis.
- [Gar11] D. J. H. Garling. *Clifford algebras: an introduction*, volume 78 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2011.

- [GP11] Riccardo Ghiloni and Alessandro Perotti. Slice regular functions on real alternative algebras. *Adv. Math.*, 226(2):1662–1691, 2011.
- [GPS21] Riccardo Ghiloni, Alessandro Perotti, and Caterina Stoppato. Slice regular functions and orthogonal complex structures over \mathbb{R}^8 . *Journal of Noncommutative Geometry*, 2021.
- [Gra58] Hans Grauert. Analytische Faserungen über holomorph-vollständigen Räumen. *Math. Ann.*, 135:263–273, 1958.
- [GS06] Graziano Gentili and Daniele C. Struppa. A new approach to Cullen-regular functions of a quaternionic variable. *C. R. Math. Acad. Sci. Paris*, 342(10):741–744, 2006.
- [GS07] Graziano Gentili and Daniele C. Struppa. A new theory of regular functions of a quaternionic variable. *Adv. Math.*, 216(1):279–301, 2007.
- [GSS14] Graziano Gentili, Simon Salamon, and Caterina Stoppato. Twistor transforms of quaternionic functions and orthogonal complex structures. *J. Eur. Math. Soc. (JEMS)*, 16(11):2323–2353, 2014.
- [GSS22] Graziano Gentili, Caterina Stoppato, and Daniele Struppa. *Regular Functions of a Quaternionic Variable*. Springer Monographs in Mathematics. Springer, second edition, 2022.
- [Lam05] T. Y. Lam. *Introduction to quadratic forms over fields*, volume 67 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2005.
- [Lor08] Falko Lorenz. *Algebra. Vol. II*. Universitext. Springer, New York, 2008. Fields with structure, algebras and advanced topics, Translated from the German by Silvio Levy, With the collaboration of Levy.
- [MM60] Yozô Matsushima and Akihiko Morimoto. Sur certains espaces fibrés holomorphes sur une variété de Stein. *Bull. Soc. Math. France*, 88:137–155, 1960.
- [SM21] Luiz A. B. San Martin. *Lie groups*. Latin American Mathematics Series. Springer, Cham, [2021] ©2021. Translated from the 2016 Portuguese original by José Emílio Maiorino and Carlos Augusto Bassani Varea with revisions by Simon Chiossi.
- [SV00] Tonny A. Springer and Ferdinand D. Veldkamp. *Octonions, Jordan algebras and exceptional groups*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2000.

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