A GRADIENT MODEL FOR THE BERNSTEIN POLYNOMIAL BASIS

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ABSTRACT. We introduce and study a symmetric, gradient exclusion process, in the class of non-cooperative kinetically constrained lattice gases, modelling a non-linear diffusivity where mass transport is constrained by the local density not being too small or too large. Maintaining the gradient property is the main technical challenge. The resulting model enjoys of properties in common with the Bernstein polynomial basis, and is associated with the diffusion coefficient $D_{n,k}(\rho) = \binom{n+k}{k} \rho^n (1-\rho)^k$, for n, k arbitrary natural numbers. The dynamics generalizes the Porous Media Model, and we show, via the entropy method, the hydrodynamic limit for the empirical measure associated with a perturbed, irreducible version of the process. The hydrodynamic equation is proved to be a Generalized Porous Media Equation.

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1. INTRODUCTION

1.1. Main result and strategy. This paper is part of an ongoing effort to rigorously understand macroscopic behaviour and thermodynamic properties in non-equilibrium statistical mechanics. We adopt the use of Markovian interacting particle systems to model the time evolution of physical systems on a microscopic scale. Concretely, we introduce and study a

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collection of nearest-neighbour interacting models, where the diffusivity is constrained by the local density of the media. Next, we describe the macroscopic evolution of the local density of particles as the weak solution of a generalized porous media equation. The main technical novelties in this work are the definition of a so-called *gradient* model satisfying target properties, and the rigorous derivation, at the macroscopic level, of a novel PDE from the theory of particle systems. Further applications of the collection of models here introduced are discussed in Subsection 1.2.

The "gradient property" means precisely that the microscopic current can be expressed as the (discrete) gradient of some function, $\mathbf{j} = -\nabla \mathbf{H}$. One can see this relation as the current being proportional to a *static* electric field, identifying \mathbf{H} as a potential. Indeed, in the scaling-limit for a large variety of models, it is known that the bulk diffusion coefficient can be formally expressed by a variational problem [15, Part II Subsection 2.2], and it is known that the gradient property is satisfied if [15, Part II Subsection 2.4] and only if [14] a dynamical part of the aforementioned variational formula is zero, providing this property a physical meaning. This property facilitates substantially the study of *hydrodynamic limits*, as we shall see in this work, while for non-gradient models the analysis is much more demanding - see [7] and references therein. This is the main feature of our model, and including it was the largest technical obstacle in this work. To the best of our knowledge, only the very brief discussion in [15, Page 184] alludes to a strategy for the construction of gradient models, very similar to our own.

Our discrete set-up is the one-dimensional torus, $\mathbb{T}_N := \mathbb{T}/N\mathbb{Z}$, where $N \in \mathbb{N}$ is a very large fixed number, and our stochastic setting is that of symmetric exclusion processes in homogeneous media, that we explain next. A *configuration* of particles (denoted by the Greek letters η , ξ) is a binary string with state-space $\Omega_N = \{0,1\}^{\mathbb{T}_N}$. The value 1 is associated with the presence of a *particle*, while 0 with its absence, or the presence of a "hole". For any site $x \in \mathbb{T}_N$, we denote by $\eta(x) \in \{0,1\}$ the occupation value of η at the site x. We will also denote by $\overline{\eta}$ the configuration with the sites and holes flipped, $\overline{\eta}(x) = 1 - \eta(x)$ for each $x \in \mathbb{T}_N$.

A configuration evolves generically in the following way. Independent, exponential clocks are associated with each bond $\{x, x+1\}$, for each $x \in \mathbb{T}_N$. They ring with rate $\mathbf{c}_{x,x+1}(\eta)\mathbf{a}_{x,x+1} + \mathbf{c}_{x+1,x}(\eta)\mathbf{a}_{x+1,x}$, where $\mathbf{a}_{x,x+1}(\eta) = \eta(x)\overline{\eta}(x+1)$, and $\mathbf{c}_{x,x+1}, \mathbf{c}_{x,x+1} : \Omega_N \to [0, +\infty)$ are maps "restricting" or "reinforcing" a jump from x to x+1 and x+1 to x, respectively. We say that a model is symmetric if $\mathbf{c}_{x,x+1} = \mathbf{c}_{x+1,x}$, and we are going to restrict ourselves to this class of models. The function $\mathbf{c}_{x,x+1} \ge 0$ can be restrictive, even if $\eta(x) + \eta(x+1) = 1$, in the sense that it can be that $\mathbf{c}_{x,x+1}(\eta) = 0$, and in this case the model is said to be kinetically constrained. In this way, if $\mathbf{c}_{x,x+1}(\eta) > 0$ and there is exactly one particle in $\{x, x+1\}$, then the occupations of the sites x and x+1 are exchanged; otherwise, the exchange is suppressed. The homogeneity of the media is modelled by translation-invariant constraints meaning precisely that, letting $\tau : \eta \mapsto \tau \eta$ be the shift operator (such that $\tau \eta(z) = \eta(z+1)$ for any $z \in \mathbb{T}_N$), and short-writing $\tau^i = \circ_{j=1}^i \tau$ for its *i*-th composition, one can express $\mathbf{c}_{x,x+1} = \tau^x \mathbf{c}(\eta)$ for every $x \in \mathbb{T}_N$, and where $\mathbf{c} \equiv \mathbf{c}_{0,1} \ge 0$. We refer to a generic function \mathbf{c} as just described by a constraint.

This describes a Markov process, generically being characterized by its infinitesimal generator. Throughout this text, Markovian infinitesimal generators shall be denoted by variants of \mathcal{L} . It is important to note that, because the dynamics is symmetric (in the sense just described) and of exclusion type, evolving in a lattice with periodic boundary conditions, the process induced by **c** is reversible with respect to the Bernoulli product measure, ν_{α}^{N} , parametrized by any constant $\alpha \in (0, 1)$.

In the setting just described, we say that a constraint **c** induces a model with generator \mathcal{L} . In this context, we see that $\mathcal{L}\eta(x) = -\nabla^+ \mathbf{j}_{x,x+1}$, where $\mathbf{j}_{x,x+1}(\eta) = -\mathbf{c}(\tau^x \eta)\nabla^+ \eta(x)$ is the current associated with the node $\{x, x+1\}$, and $\nabla^+ := \tau - \mathbf{1}$ is the forward difference operator and $\mathbf{1}$ the identity in Ω_N . The gradient property corresponds then to the existence of some $\mathbf{h}: \Omega_N \to \mathbb{R}$ such that

$$\mathbf{c}(\eta)\nabla^+\eta(0) = \nabla^+\mathbf{h}(\eta).$$

In order to explain precisely our contribution, we recall the Porous Media Model (PMM), introduced in [9]. Fixed $n \in \mathbb{N}$, we shorten to PMM(n) the (porous media) model induced by the constraint

$$\mathbf{c}^{n}(\eta) := \frac{1}{n+1} \sum_{i=0}^{n} \prod_{\substack{i=-n-1+j\\i\neq 0,1}}^{j} \eta(i),$$
(1.1)

The corresponding dynamics is illustrated in Figure 1, and can be explained as a particle jumping to its nearest-neighbour (free) site with rate given by the proportion of groups of n aligned particles around the node where the jump may occur. The PMM belongs to the class of *non-cooperative kinetically constrained* models. The "non-cooperative" aspect corresponds to the presence of *mobile clusters*: representing a particle by \bullet , a hole by \circ , and letting \blacksquare be a cluster of particles (that is, some fixed finite box in \mathbb{T}_N composed by particles and vacant sites), we say that \blacksquare constitutes a *mobile* cluster if

- the transitions $\blacksquare \bullet \leftrightarrow \bullet \blacksquare$ and $\blacksquare \circ \leftrightarrow \circ \blacksquare$ are possible with a finite number of jumps, and independent of the rest of the configuration;
- and it is always possible for a jump to occur in a node, if there exists a cluster in the vicinity of the respective node, that is, $\blacksquare \circ \bullet \leftrightarrow \blacksquare \bullet \circ$ and $\circ \bullet \blacksquare \leftrightarrow \bullet \circ \blacksquare$.

The PMM allows for the first derivation of the hydrodynamic limit for a non-cooperative kinetically constrained model, and a substantial amount of research has been done proceeding its introduction, also in long-range and open boundary contexts [2, 4, 11, 6], giving us a good picture at the hydrodynamical level, yet with important open problems due to its non-irreducibility. The PMM(n) is associated, in the *continuum*/macroscopic scale, with the Porous Media Equation, $\partial_t \rho = \partial_u (\rho^n \partial_u \rho)$ via the hydrodynamic limit of the empirical measure in the diffusive time-scaling N^2 . This will be further explored and explained later on. In particular, one can see \mathbf{c}^n as a microscopic analogous of the diffusivity $D(\rho) = \rho^n \in [0, 1]$.

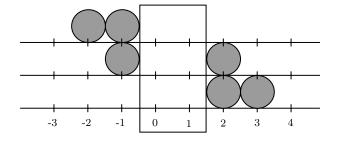


FIGURE 1. PMM(2) constraints for a jump in $\{0, 1\}$.

The PMM allows diffusion of mass when a minimal threshold value for the local density it attained. The "combinatorial" aspect in its definition, as we shall see, guarantees the gradient property. Our question is: can we define a dynamics modelling zero diffusivity at both high and low densities? This can represent, for instance, a system where at low densities the particles are too far away for any interaction, while at a high density they are too packed for any movement. Our goal is then to provide a gradient toy-model generalizing the collection of PMM in the previously described sense. More precisely, we are going to derive the diffusivity $D_{n,k}(\rho) = \binom{n+k}{k} \rho^n (1-\rho)^k$, for any $n, k \in \mathbb{N}_+$, obtaining that the local average of particles is governed, macroscopically, by the the hydrodynamic equation

$$\partial_t \rho = \binom{n+k}{k} \partial_u^2 \Phi^{n,k}(\rho), \qquad \text{on} \quad \mathbb{T} \times [0,T], \tag{1.2}$$

where T > 0 is some finite and fixed time-horizon, and $\Phi^{n,k}$ is identified with the Incomplete Beta function,

$$\Phi^{n,k}(x) = \int_0^x u^n (1-u)^k du = \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \frac{x^{n+\ell+1}}{n+\ell+1}, \quad \text{for } x \in [0,1].$$
(1.3)

The differential equation in (1.2) is a degenerated parabolic equation, belonging to the class of *generalized porous media equations* [16]. One can see in Figure 2 that, depending on the values of n and k, the diffusivity is either constant, linearly or non-linearly increasing, or non-linear non-monotonic.

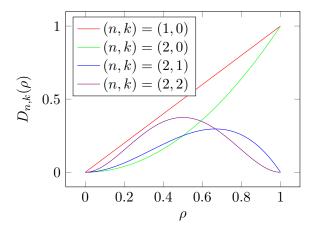


FIGURE 2. Plot of $D_{n,k}(\rho)$ for different values of $n, k \in \mathbb{N}_+$.

It is important to note that optimal diffusivity it attained with $\max_{\rho \in [0,1]} D_{n,k}(\rho) = D_{n,k}(\rho^*)$ for $\rho^* = \frac{n}{n+k}$. For any positive integer L, the collection $\{D_{L,L-n}\}_{n=0,\dots,L}$ is also known as the *Bernstein basis of degree L*.

Remarkably, our indirect approach led to the very simple model in Definition 2.1 that we describe next. For any $n, k \in \mathbb{N}$ fixed, we set $\mathcal{L}_N^{n,k}$ as the Markov generator of the model induced by the constraint

$$\mathbf{b}^{n,k}(\eta) = \frac{1}{n+k+1} \sum_{j=0}^{n+k} \mathbf{1}\left\{\langle \eta \rangle_{W_j} = \rho^\star\right\},\tag{1.4}$$

where $\langle \eta \rangle_{W_j} = \frac{1}{|W_j|} \sum_{z \in W_j} \eta(z)$ and $W_j := [-j, -j+n+k+1] \setminus \{0, 1\}$, for $0 \le j \le n+k$. The model induced by this constraint will be referred to as B(n, k), and we represent its dynamics in Figure 3. Concretely, the occupation value of two neighbouring sites are exchanged only if there is at least one window W_j of length n + k around them with density $\rho^* = n/(n+k)$ (that is, with exactly *n* particles), and the precise rate is given by the proportion of windows with the prescribed density of particles.

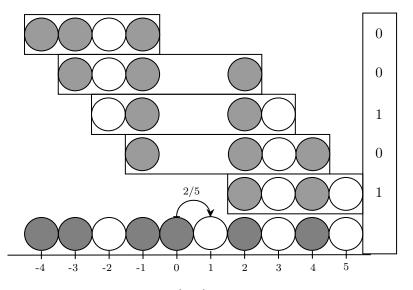


FIGURE 3. B(2,2) transition rates.

In Figure 3, the windows W_j , for $0 \le j \le 4$, are represented as the rectangles containing segments of the configuration of particles, in light gray, and the numbers 0, 1 on the right correspond to the value of $\mathbf{1}\{\langle \eta \rangle_{W_i} = \rho^*\}$, for each j.

We now summarize some relevant properties of the family of models $\{B(n,k)\}_{n,k\in\mathbb{N}_+}$.

Properties 1.1. The model induced by the constraint (1.4) satisfies the following:

(i) Gradient property: For each $n, k \in \mathbb{N}_+$ there exists a function $\mathbf{H}^{n,k} : \Omega_N \to \mathbb{R}$ such that, for every $\eta \in \Omega_N$, it holds that $\mathbf{b}^{n,k}(\eta)\nabla^+\eta(0) = \nabla^+\mathbf{H}^{n,k}(\eta)$, for some $\mathbf{H}^{n,k}$ which can be written as

$$\mathbf{H}^{n,k} = \mathbf{P}^{n,k} + \nabla^+ \mathbf{g}^{n,k} + \mathbf{f}^{n,k}, \tag{1.5}$$

where $\mathbf{f}^{n,k}$ has expectation zero, with respect to the invariant measure of the process; (ii) Range: For each $n, k \in \mathbb{N}_+$ and $\eta \in \Omega_N$ it holds that $\mathbf{b}^{n,k}(\eta) \in [0,1]$, and there is a

- local configuration η such that $\langle \eta \rangle_{W_j} = \frac{n}{n+k}$ for every $0 \le j \le n+k$.
- (iii) Interpolation: for each $n \in \mathbb{N}_+$ it holds that B(n,0)=PMM(n), that is, $\mathbf{b}^{n,0}=\mathbf{c}^n$;
- (iv) Symmetry: for any $\eta \in \Omega_N$ it holds that $\mathbf{b}^{n,k}(\eta) = \mathbf{b}^{k,n}(\overline{\eta})$;
- (v) Partition of the unity: for any integer 0 < L < N/2 it holds that

$$\mathcal{L}_{N}^{SSEP} = \sum_{n=0}^{L} \mathcal{L}_{N}^{n,L-n},$$

where $\mathcal{L}_N^{n,L-n}$ is the infinitesimal generator of the B(n,L-n), as in Definition 2.1, and \mathcal{L}_N^{SSEP} that of the Symmetric Simple Exclusion Process, identified with the B(0,0).

- (vi) Blocked configurations: If the number of particles in the system is either small or large enough, there are configurations which are blocked, in the sense that no jump can be performed. An example is the configuration where each particle is at a distance larger than n + k from each other, or each vacant site is at a distance larger than n + k from each other.
- (vii) Mobile Clusters: there exist mobile clusters.

The starting point for the definition of the model was to extend the combinatorial mechanism of the PMM(n + k), by adding an extra flip of k occupation variables on each window W_j . Since specific sites to flip in order to ensure the gradient condition are unknown, to different configurations on each window we associate a different weight, in this way defining a "probability distribution" for exactly k vacancies on each window (see Definition 2.6 and the discussion just after it). Forcing algebrically the gradient property in this prototype model, one obtains a linear system characterizing the (probability) weights. We then prove that, for general n and k, the uniform distribution yields a solution for this system. This is the content of Propositions 2.13 and 2.15, with the linear system as in (2.6). The constraint (1.4) corresponds to the uniform solution rescaled by a multiplicative factor $\binom{n+k}{k}$.

Regarding Properties 1.1, showing (i) with $\mathbf{H}^{n,k}$ as in (1.5) is a non-trivial problem, relying on a good understanding of some specific equivalence classes. The analysis is also combinatorial, and performed in Appendix A, resulting on the expressions as in Proposition 2.19. Properties (ii)-(iv) and (vi) are obtained directly from the definition of the model, while properties (v) and (vi) are the contents of Proposition 2.2 and Lemma 2.3, respectively.

It is also notable that one can identify that the B(n,k) is expressed in a basis akin to the collection of PMM's. This is the content of Appendix B, that we now describe. For each fixed natural numbers $l \leq L < N$, each $0 \leq j \leq L$ and each $\eta \in \Omega_N$, let $m_j^L(\eta) := \sum_{y \in W_j^L} \eta(y)$, where $W_j^L := -j + [0, L+1] \setminus \{j, j+1\}$, and

$$\mathbf{p}_{L}^{l}(\eta) := \frac{1}{L+1} \sum_{j=0}^{L} \frac{\binom{m_{j}^{L}(\eta)}{l}}{\binom{L}{l}} \frac{\mathbf{1}_{\{m_{j}^{L}(\eta) \ge l\}}}{2^{L-l}}.$$
(1.6)

The constraint \mathbf{p}_L^l induces a gradient model, that we shall denote by $\text{PMM}_L(l)$, with generator that we write as $\mathcal{L}_N^{l:L}$. For each $n, k \geq 0$ it holds that

$$\mathcal{L}_N^{n,k} = \sum_{\ell=0}^k (-1)^\ell \binom{n+k}{k} \binom{k}{\ell} 2^{L-\ell} \mathcal{L}_N^{n+\ell:n+k}$$

This is the content of Lemma B.1. The factor 2^{L-l} in (1.6) normalizes the indicator function, so that \mathbf{p}_L^l is normalized in Ω_N , for each $0 \leq l \leq L$. This provides a combinatorial interpretation for $\mathbf{p}_{L,j}^l(\eta)$ as the probability of choosing l entries equal to 1, from the total of $m_j^L(\eta)$ entries, given that there are at least l entries equal to 1 in the window W_j . The normalization factor $\binom{L}{l}2^{L-l} = \sum_{i=0}^{L-l} \binom{L}{l}\binom{L-l}{i}$ corresponds to the total number of ways to choose l entries from L positions taking into account the possible configurations of the remaining entries, knowing that there are *at least* l entries equal to one. In Figure 4 below, the values 0 and 1 coincide with $\mathbf{1}_{\{m_i^L(\eta) > l\}}$, for different values of j.

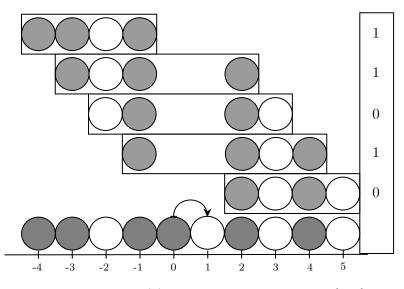


FIGURE 4. PMM₄(3) constraints for a jump in $\{0, 1\}$.

We note the interpolation $\text{PMM}(L) = \text{PMM}_L(L)$, and that, letting \mathcal{F} be the flip operator $\mathcal{F}\eta := \overline{\eta}$, one can see $\{\mathcal{F}\mathbf{p}_n^l\}_{0 \le l \le n}$ as a basis for the PMM(L):

$$\mathbf{c}^{L} = \sum_{\ell=0}^{L} (-1)^{\ell} {\binom{L}{\ell}} 2^{L-\ell} \mathcal{F} \mathbf{p}_{L}^{\ell},$$

with \mathbf{c}^{L} as in (1.1). We refer the reader to further properties in Lemma B.3, and note that the $\text{PMM}_{L}(l)$ is associated with the diffusivity $2^{-(L-l)}\rho^{l}$.

As a final note with respect to the definition of the main model B(n,k), the linear system (2.6) seems to not have a unique solution for general $n, k \in \mathbb{N}_+$. In Appendix C we present the linear system for the particular case of n, k = 2, and exemplify a non-uniform solution. More solutions were verified computationally.

In the last part of this manuscript, Section 3, we analyse the limiting behaviour of the *empirical measure* in the diffusive time-scaling N^2 ,

$$\pi^{N}(\eta_{N^{2}t}, \mathrm{d}u) = \frac{1}{N} \sum_{x \in \mathbb{T}_{N}} \eta_{N^{2}t}(x) \delta_{N^{-1}x}(\mathrm{d}u),$$

where δ_v is the Dirac measure at $v \in \mathbb{T}$, associated with the *perturbed process*, given by the generator

$$\mathcal{L}_N := N^2 \mathcal{L}_N^{n,k} + \Theta_{\mathfrak{p}}(N) \mathcal{L}_N^{\text{SSEP}}, \quad \text{for} \quad \mathfrak{p} \in (0,1) \quad \text{and} \quad \Theta_{\mathfrak{p}}(N) = N^{2-\mathfrak{p}}.$$

Precisely, we prove the theorem that we state next. We let the system start from some initial configuration distributed by μ_N , and let this initial state correspond macroscopically (as $N^{-1}\mathbb{T}_N \xrightarrow{N} \mathbb{T}$), to some measurable function $g: \mathbb{T} \to [0,1]$. If a sequence $\{\mu_N\}$ and a function g are related this way, we say that μ_N is a *local equilibrium measure* associated with the *profile* g. This is made more rigorous in Definition 3.2.

Theorem 1.2 (Hydrodynamic limit). Fixed $n, k \in \mathbb{N}$ and a finite time horizon T > 0, let $\{\mu_N\}_{N\geq 1}$ be a sequence of local equilibrium measures associated with a profile ρ^{ini} . Then, for

any $t \in [0,T]$ it holds the weak convergence

$$\int_{\mathbb{T}} G(u) \pi^{N}(\eta_{N^{2}t}, \mathrm{d}u) \xrightarrow{N \to \infty} \int_{\mathbb{T}} G(u) \rho_{t}(u) \mathrm{d}u \quad in \ probability \ w.r.t. \ \mathbb{P}_{\mu_{N}},$$

for any smooth test function $G : \mathbb{T} \to \mathbb{R}$, and where $\rho : (t, u) \mapsto \rho_t(u)$ is the unique weak solution of the generalized porous media equation

$$\begin{cases} \partial_t \rho = \binom{n+k}{k} \partial_u^2 \Phi^{n,k}(\rho), & in \ \mathbb{T} \times (0,T], \\ \rho_0 = \rho^{ini}, & in \ \mathbb{T} \end{cases}$$
(1.7)

in the sense of Definition 3.1, and with $\Phi^{n,k}$ as in (1.3).

We show this by following the entropy method, introduced in [12], and our main references are the works [11, 2, 1], on the PMM and the SSEP. The perturbation of order $N^{-\mathfrak{p}}$ allows for the irreducibility of the process, while being of an order small enough so that its effect does not contribute to the macroscopic evolution of the process.

Although the entropy method is quite delicate, a large portion of it is very robust when applied to one-dimensional, symmetric exclusion processes with periodic boundary conditions. In this way, we shall prove only the required adaptations, and direct the reader to appropriate references. The main novelty of Section 3 lies in the proof of Lemma 3.10, the so-called "two blocks estimate", involving the irreducibility of the (perturbed) process, the existence of mobile clusters, and the ability of treating configurations where, locally, there is an extreme density of particles. In order to be more precise, in [11, 2, 5] it was explored similar approaches in order to remove configurations with not *enough* particles in a microscopic box, in order to construct mobile clusters – which are then invoked in a mass-transportation argument. This low-density cutoff is important because the PMM mobile clusters are composed by specific boxes *completely* filled with particles. In our case, they are composed by clusters with a prescribed density, in this way requiring an equilibrium between the number of particles and vacant sites (see the proof of Lemma 2.3 and the discussion just before it). The arguments in the aforementioned references allows one to remove either low or high density cases, but not both. Our solution requires a more involved analysis.

1.2. Future work. A natural question is the extension of the dynamics to long-range interactions. This cannot be performed with the same reasoning as for the PMM in [4], since it breaks the gradient property. In a companion paper we perform this extension, deriving a fractional version of the hydrodynamic equation (1.7).

A motivation for the introduction of the B(n,k) is the work in [11], where the authors extend the PMM into a family continuously parametrized by $m \in (0,2]$, and derive the equation $\partial_t \rho = \partial_u^2 \rho^m$, in this manner describing the transition from slow (m > 1) to fast (m < 1) diffusion. A tool missing for the extension of the dynamics to m > 2 is precisely the collection of generalized Porous Media Models here introduced.

Lastly, this work serves as a first effort in the direction of the larger question of classifying the types of diffusivities $D(\rho)$ attainable through gradient dynamics. In a project currently in development, this is analysed in a constructive manner, aiming for a Weierstrass-like type of theorem in the context of gradient models.

1.3. Outline of the paper. We now summarize the results and reasoning followed in the next sections. The main results of Section 2 and 3 are the Propositions 2.13, 2.19, and the Theorem 3.3, respectively.

Section 2 is devoted to the definition and characterization of the main model. The B(n,k) is introduced in Definition 2.1 and the partition of the unity and existence of mobile clusters are proved in Proposition 2.2 and Lemma 2.3. In Subsection 2.1 we investigate the gradient property constructively. There is a combinatorial interplay associated with the gradient condition, that is encapsulated into Lemma 2.9. In Proposition 2.13 it is derived the linear system characterizing the gradient property, whose variables are the weights involved in the constraint in Definition 2.6. In Proposition 2.15 we identify the B(n,k) with the uniform solution. The resulting functions involved in the gradient property are simplified in Proposition 2.19. This is achieved through a study of the equivalence classes generated by the weights associated with our model, in the series of Lemmas A.1, A.3, A.4 and A.5.

Regarding Section 3, we analyse the B(n,k) superposed with the SSEP with rates of the order of $N^{-\mathfrak{p}}$, for any $\mathfrak{p} \in (0,1)$ fixed. Our main result is Theorem 3.3, where we establish the Hydrodynamic Limit for the empirical measure. We overview the approach for the proof, requiring minor adaptations, except for the *two-blocks estimate*, Lemma 3.10. For completeness, we present the step corresponding to the "characterization of the limit-points", Proposition 3.4.

In Appendix B we explain how the B(n,k) is expressed in a basis of gradient Porous Media Models, and prove some of its properties. This results in Lemmas B.1 and B.3, with the basis as in Definition B.2. In Appendix C we focus on the case n, k = 2. We present all the sets and linear systems involved in the proof of Proposition 2.13, for this particular case, in Subsection C.1. The linear system is in (C.3). In Subsection C.2, precisely in Lemma C.1, we show that, if an extended linear system is satisfied, for $n, k \ge 0$ fixed, with a non-uniform solution, then one can also remove from the potential the dependence on the complicated objects, as in the study in Appendix A in order to show Proposition 2.19. We provide a concrete non-uniform solution, for n, k = 2, satisfying the aforementioned extended system, in Table 20.

2. Bernstein model

Our first goal is to define a specific constraint such that the diffusion coefficient of the corresponding hydrodynamic equation of the process induced by it is given by $D_{n,k}(\rho) = \binom{n+k}{n}\rho^n(1-\rho)^k$, for $n, k \in \mathbb{N}_+$; and such that the induced model enjoys the gradient property. We start by fixing some notation and setup. We fix $N \in \mathbb{N}$ very large, and write $\mathbb{T}_N := \mathbb{T}/N\mathbb{Z}$ and \mathbb{T} for the discrete and continuous one-dimensional torus. The set \mathbb{T}_N is referred to as the "lattice", its elements by "sites", and a pair of neighboring sites as a "node". A configuration of particles is an element of the state-space $\Omega_N = \{0,1\}^{\mathbb{T}_N}$. We denote by $\eta(x) \in \{0,1\}$ the occupation value of η at the site x, and by $\overline{\eta}$ the (flipped) configuration, defined through $\overline{\eta}(x) = 1 - \eta(x)$, for each $x \in \mathbb{T}_N$. For any function $f : \Omega_N \to \mathbb{R}$ and operator $\mathcal{O} : \Omega_N \to \Omega_N$ we let $\mathcal{O}f$ be defined through $\mathcal{O}f(\eta) = f(\mathcal{O}\eta)$.

A symmetric exclusion process is characterized by its Markov generator \mathcal{L} , which is given, for any $f: \Omega_N \to \mathbb{R}$, by

$$\mathcal{L}f(\eta) = \frac{1}{2} \sum_{x \in \mathbb{T}_N} \mathbf{c}(\tau^x \eta) (\eta(x)\overline{\eta}(x+1) + \overline{\eta}(x)\eta(x+1)) (\nabla_{x,x+1}f)(\eta),$$

where $\nabla_{x,y} := \theta_{x,y} - \mathbf{1}$, with $\theta_{x,y}$ the operator that exchanges the occupations of the sites xand y; the operator $\mathbf{1}$ is the identity in Ω_N ; and $\tau^x \eta(z) := \eta(z+x)$ is the shift operator. The map $\tau^x \mathbf{c} := \mathbf{c}_{x,x+1} \ge 0$ corresponds to the kinetic constraint for a jump in the node $\{x, x+1\}$. The notion of gradient property that we use is as in [13]. For a symmetric constraint \mathbf{c} , this is reduced to existing some $\mathbf{H} : \Omega_N \to \mathbb{R}$ such that $\mathbf{j} = -\nabla^+ \mathbf{H}$, where $\mathbf{j}(\eta) = -\mathbf{c}(\eta)\nabla^+\eta(0)$ is the algebraic current between the sites 0 and 1. This implies that, if \mathbf{c} is associated with the infinitesimal generator \mathcal{L} , then $\mathcal{L}\eta(x) = \Delta \mathbf{H}(\tau^x \eta)$ for any site x.

As a brief motivation for our approach, in [11, Proposition 2.18] it was shown that for any fixed configuration η the sequence of Porous Media Model (gradient) constraints $(\mathbf{c}^n(\eta))_{n\geq 1}$, such that

$$\mathbf{c}^{n}(\eta) := \frac{1}{n+1} \sum_{i=0}^{n} \prod_{\substack{i=-n-1+j\\i\neq 0,1}}^{j} \eta(i),$$

is non-increasing and associated with the diffusivities $\{\rho^n\}_{n\geq 1}$. In this way, one expects the constraint $\mathbf{r}^{n-1,1} := \mathbf{c}^{n-1} - \mathbf{c}^n \geq 0$ to be related with a diffusion coefficient given by $\rho^{n-1} - \rho^n = \rho^{n-1}(1-\rho)$. Moreover, one can show the following curious rearrangements

$$\begin{aligned} 3\mathbf{r}^{1,1}(\eta) &= \eta(-2)\overline{\eta}(-1) + \frac{\eta(-1)\overline{\eta}(2) + \overline{\eta}(-1)\eta(2)}{2} + \overline{\eta}(2)\eta(3), \\ 4\mathbf{r}^{2,1}(\overline{\eta}) &= \eta(-3)\overline{\eta}(-2)\overline{\eta}(-1) + \frac{\overline{\eta}(-2)\overline{\eta}(-1)\eta(2)}{3} + 2\frac{\eta(-2)\overline{\eta}(-1)\overline{\eta}(2)}{3} \\ &+ 2\frac{\overline{\eta}(-1)\overline{\eta}(2)\eta(3)}{3} + \frac{\eta(-1)\overline{\eta}(2)\overline{\eta}(3)}{3} + \overline{\eta}(2)\overline{\eta}(3)\eta(4), \end{aligned}$$

and obtain a general formula for $n \ge 2$. It turns out that the rationale to define $\mathbf{r}^{n-1,1}$ does not generalize in the most natural way: defining $\mathbf{r}^{n-1,2} := \mathbf{r}^{n-1,1} - \mathbf{r}^{n,1}$ leads to possible negative rates because the sequence $\{\mathbf{r}^{n-1+i,1}\}_{i\ge 0}$ is not monotone. This led us to investigate if, extending the simple combinatorial mechanism of the PMM's constraints by flipping particular occupation variables, with an associated weight, may lead to a gradient model. The object that we have in mind is the "prototype" constraint in the forthcoming Definition 2.6.

Our extension of the collection $\{PMM(n)\}_{n\in\mathbb{N}_+}$ to a 2-parameter family is realized in the next definition.

Definition 2.1 (Bernstein model). For each $n, k \in \mathbb{N}$ fixed, let

$$W_j := \llbracket -j, -j+n+k+1 \rrbracket \backslash \{0,1\} \quad \text{for} \quad 0 \le j \le n+k$$

Define the constraint $\mathbf{b}^{n,k}:\Omega_N\to[0,1]$ as

$$\mathbf{b}^{n,k} := \frac{1}{n+k+1} \sum_{j=0}^{n+k} \mathbf{b}_j^{n,k} \quad \text{with} \quad \mathbf{b}_j^{n,k}(\eta) := \mathbf{1} \{ \langle \eta \rangle_{W_j} = \frac{n}{n+k} \}$$

where $\langle \eta \rangle_{W_j} = \frac{1}{|W_j|} \sum_{z \in W_j} \eta(z)$. We refer to the model induced by $\mathbf{b}^{n,k}$ as $\mathbf{B}(n,k)$, and its corresponding Markov generator as $\mathcal{L}_N^{n,k}$.

We refer the reader to Figure 3 for an example of the transition rates for n, k = 2. In general, $\mathbf{b}^{n,k}(\eta)$ corresponds to the proportion of windows of length n + k around the node $\{0,1\}$ with exactly n particles. We shall now prove some of the properties of the B(n,k) mentioned in the introduction section. The main result is the gradient property, for which Subsection 2.1 completely devoted to. Next result connects the Bernstein model with the SSEP, and the subsequent concerns the existence of mobile clusters.

Proposition 2.2. Fixed L < N, the collection of constraints $\{\mathbf{b}^{n,L-n}\}_{0 \le n \le L}$ forms a partition of the unity in Ω_N . In other words,

$$\mathcal{L}_N^{SSEP} = \sum_{n=0}^L \mathcal{L}_N^{n,L-n},$$

where \mathcal{L}_N^{SSEP} is the generator of the Symmetric Simple Exclusion Process, identified with the B(0,0). In this way the SSEP can be seen as a superposition of Bernstein models.

Proof. Simply note that for any $\eta \in \Omega_N$ fixed it holds that

$$\sum_{n=0}^{L} \mathbf{b}^{n,L-n}(\eta) = \frac{1}{L+1} \sum_{j=0}^{L} \left(\sum_{n=0}^{L} \mathbf{1} \left\{ \langle \eta \rangle_{W_j} = \frac{n}{L} \right\} \right) = 1.$$

The second equality is justified by observing that for any fixed η and $0 \leq j \leq L$ (hence, fixed a window $W_j = -j + [0, n + k + 1] \setminus \{j, j + 1\}$), there is exactly one density $\rho_j \in \{0, 1/L, 2/L, \ldots, 1\}$ such that $\langle W_j \rangle = \rho_j$. In this way, for each j, the summation over n in the previous display equals one.

We recall the notion of mobile cluster [9] that we use here. Let \blacksquare represent a fixed finite box in some discrete lattice, composed by particles and vacant sites. Representing a particle by • and a hole by •, we say that \blacksquare constitutes a mobile cluster if it enjoys the following.

- ▷ *Mobility*: the transitions $\blacksquare \bullet \leftrightarrow \bullet \blacksquare$ and $\blacksquare \circ \leftrightarrow \circ \blacksquare$ are possible with a finite number of jumps, and independent of the rest of the configuration;
- ▷ Mass transport: it is always possible for a jump to occur in a node, if there exists a cluster in the vicinity of the respective node, that is, $\blacksquare \circ \bullet \leftrightarrow \blacksquare \bullet \circ$ and $\circ \bullet \blacksquare \leftrightarrow \bullet \circ \blacksquare$.

Lemma 2.3. For each n, k fixed, any box of length n + k + 2 with exactly n + 1 particles constitutes a mobile cluster.

Proof. Let \Box represent a cluster of particles composed by a window of length n + k with exactly n particles, let \bullet represent a particle and \circ a hole, and let \blacksquare represent a box of length n + k + 2 with exactly n + 1 particles. We note that the particles in \blacksquare can always be reorganized into the clusters $\Box \circ \bullet$ or $\circ \bullet \Box$. Indeed, removing a pair of the form $\circ \bullet$ (or $\bullet \circ$) from \blacksquare results in the local configuration " $\blacksquare \setminus \bullet \circ$ " (or " $\blacksquare \setminus \circ \bullet$ ") that corresponds to a window of length n + k around the node where $\bullet \circ$ is located at, with exactly n particles. In this way, any exchange $\bullet \circ \leftrightarrow \circ \bullet$ is possible within \blacksquare , and this cluster can be reorganized in a sequence of steps to, for example, $\blacksquare = \Box \circ \bullet$ or $\circ \bullet \Box$.

Regarding the mobility of \blacksquare , the transition $\blacksquare \bullet \leftrightarrow \bullet \blacksquare$ is possible because the cluster \blacksquare can be reorganized into $\blacksquare' = \bullet \cdots$, leading to the transition $\blacksquare \bullet \mapsto \blacksquare' \bullet = \bullet (\cdots \bullet) := \bullet \blacksquare''$. Next, the cluster \blacksquare'' can be reorganized into \blacksquare . This reasoning is summarized in the next diagram.

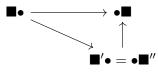


FIGURE 5. Mobility by a mixing argument.

Since \blacksquare has at least one empty site \circ , the reasoning for the transition $\blacksquare \circ \leftrightarrow \circ \blacksquare$ is completely analogous.

For mass transportation, because \blacksquare can be reorganized, starting w.l.o.g. from $\blacksquare = \circ \bullet \square$, because the jumps $\Box \bullet \circ \leftrightarrow \Box \circ \bullet$ are possible since \Box has length n + k and n particles, we see that

$$\blacksquare \bullet \circ = (\circ \bullet \Box) \bullet \circ \leftrightarrow (\circ \bullet \Box) \circ \bullet = \blacksquare \circ \bullet.$$

Starting from $\blacksquare = \Box \circ \bullet$, similarly we see that the jumps $\bullet \circ \Box \leftrightarrow \circ \bullet \Box$ are also possible.

2.1. Proof of the gradient property. The approach here presented to define the B(n,k)in such a way that it maintains the gradient property satisfied by the PMM(n) is indirect, and relies on solving a linear system associated with the constraints of an auxiliary model. In order to present this, we need to introduce some notation and several auxiliary sets.

Notation 2.4.

- \triangleright For $A \subseteq \mathbb{T}_N$ we write $\eta(A) \equiv \prod_{i \in A} \eta(i)$. If $A = \emptyset$ it is defined by convention that $\prod_{\emptyset} \equiv 1;$
- \triangleright For $A \subseteq \mathbb{T}_N$ we write η^A as the configuration where for any $x \in \mathbb{T}_N$ it holds $\eta^A(x) =$ $(1 - \eta(x))\mathbf{1}\{x \in A\} + \eta(x)\mathbf{1}\{x \notin A\};\$
- \triangleright Given two sets $A, B \subset \mathbb{T}_N$ such that $A \cap B = \emptyset$ we denote by $A \sqcup B$ their disjoint union;
- \triangleright For any $x \in \mathbb{T}_N$ we write $x + A = \{x + a : a \in A\};$
- \triangleright For any $r \in \mathbb{N}_+$ and $A \subseteq \mathbb{T}_N$ we write $rA = \{ra : a \in A\}$.

Definition 2.5. For $A \subseteq \mathbb{T}_N$, and $0 \le a \le N$, we denote the collection of subsets of A with a elements by

$$\mathcal{P}_a(A) = \{A': A' \subseteq A, |A'| = a\}.$$

In what follows, let $0 \le l \le k$. We introduce the following sets:

- $\triangleright I_{\ell} = \llbracket 1, \binom{n+k}{k-\ell} \rrbracket$. For $\ell = 0$ we write $I \equiv I_0$;
- $\triangleright J = [\![0, n+k]\!];$
- $> M_j = M \setminus \{j, j+1\}, \text{ for each } j \in J, \text{ where } M = \llbracket 0, n+k+1 \rrbracket;$ $> P_{ij}^{n+\ell}, \text{ for } (i,j) \in I_\ell \times J, \text{ are such that } \mathcal{P}_{n+\ell}(M_j) = \{P_{ij}^{n+\ell}\}_{i \in I_\ell}. \text{ For } n+\ell=k, \text{ we }$ shorten $P_{ij} = P_{ij}^k$;
- $\triangleright Q_{ijq}^{\ell}, \text{ with } (i,j) \in I \times J \text{ and } 1 \leq q \leq {k \choose \ell}, \text{ are such that } \mathcal{P}_{\ell}(P_{ij}) = \{Q_{ijq}^{\ell}\}_{1 \leq q \leq {k \choose \ell}};$
- $\triangleright p_{ij}^{\ell,x}: \text{ for each } (i,j) \in I_{\ell} \times J \text{ and } x = 0,1, \text{ we let } p_{ij}^{\ell,x} = \min\{P_{ij}^{n+\ell} \sqcup \{j+x\}\} \ge 0 \text{ and } A_{ij}^{n+\ell,x} = -p_{ij}^{\ell,x} + (P_{ij}^{n+\ell} \sqcup \{j+x\}).$

Although all the previously defined sets depend on n and k, we do not make that dependence always explicit in order to simplify the presentation. The reader should have in mind, however, that $n, k \in \mathbb{N}_+$ are fixed. We also note that I_{ℓ} corresponds to the indexes of the elements of $\mathcal{P}_{n+\ell}(M_i).$

We will focus on the following constraint.

Definition 2.6. Fixed $n, k \in \mathbb{N}_+$, let $\mathfrak{a} = \{a_{ij}\}_{(i,j)\in I\times J}$ be a collection of non-negative weights such that $\sum_{i\in I} a_{ij} = 1$, for each $j \in J$. The constraint $\mathbf{c}^{n,k}$ is defined as

$$\mathbf{c}^{n,k} = \frac{1}{n+k+1} \sum_{j=0}^{n+k} \mathbf{c}_j^{n,k} \quad \text{where} \quad \mathbf{c}_j^{n,k} = \sum_{i \in I} a_{ij} \eta^{-j+P_{ij}} (-j+M_j).$$
(2.1)

We note that for each $j \in J$, $\mathbf{c}_{j}^{n,k}(\eta)$ can be interpreted as the expected value of the projection of η in the box W_j , according to some law $\mathbb{P}_j^{n,k}$ concentrated on configurations with exactly n particles on W_j , described by \mathfrak{a} – in the sense that each $\xi \in \{0,1\}^{W_j}$ with exactly n particles is associated with an element of $\mathcal{P}_n(W_j)$, and one can express, $\mathbb{P}_j^{n,k}(\xi) = a_{ij} \geq 0$, for some $i \in I$. In this way, $\mathbf{c}^{n,k}$ corresponds to a second (now uniform) averaging over the boxes $\{W_j\}_{j=0,\dots,n+k+1}$. As such, the main goal of this subsection will be to investigate if, for each n, k, there is a "gradient distribution", that is, if for each j there is some non-trivial $\mathbb{P}_j^{n,k}$, such that the corresponding model induced by $\mathbf{c}^{n,k}$ enjoys the gradient property.

As a concrete example of (2.1), for the particular case of n, k = 2, we have the following, randomly labelled, sets:

		$-j + I_{ij}$				
$i W_{\cdot} = -i + M_{\cdot}$	i/j	0	1	2	3	4
$\begin{array}{c c} j & W_j = -j + M_j \\ \hline 0 & \{2, 3, 4, 5\} \end{array}$	1	$\{2,3\}$	$\{-1,2\}$	$\{-2, -1\}$	$\{-3, -2\}$	$\{-4, -3\}$
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	2	$\{2,4\}$	$\{-1,3\}$	$\{-2,2\}$	$\{-3, -1\}$	$\{-4, -2\}$
$2 \{-2, -1, 2, 3\}$	3	$\{2, 5\}$	$\{-1,4\}$	$\{-2,3\}$	$\{-3,2\}$	$\{-4, -1\}$
$2 \begin{bmatrix} 2 & 2 & 1 & 2 \end{bmatrix}$	4	$\{3,4\}$	$\{2,3\}$	$\{-1,2\}$	$\{-2, -1\}$	$\{-3, -2\}$
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	5	$\{3, 5\}$	$\{2,4\}$	$\{-1,3\}$	$\{-2,2\}$	$\{-3, -1\}$
$4 \ 1^{-4}, -3, -2, -1$	6	$\{4, 5\}$	$\{3, 4\}$	$\{2,3\}$	$\{-1,2\}$	$\{-2, -1\}$

TABLE 1. Constraints' windows.

TABLE 2. Sets corresponding to the sites with flipped occupation value.

 $i \perp D_{i}$

Using the distributive property we develop the products of occupation variables in $s_1 := \{\mathbf{c}_j^{n,k}(\eta)\eta(1)\}_{j\in J}$, generating a polynomial – a linear combination of terms of the form $\eta(A^1)$. Similarly, $s_0 := \{\mathbf{c}_j^{n,k}\eta(0)\}_{j\in J}$ generates linear combinations of terms of the form $\eta(A^0)$. The underlying idea is that a necessary condition for the model to be of gradient type is that the sets A^1 's, associated with s_1 , must be translations of the A^0 's, associated with s_0 . In this way, translating these sets to the same reference point, shows that the gradient property is expressed through a linear system on the coefficients of the aforementioned polynomials, that will be functions of \mathfrak{a} . A particular solution of this system is then obtained through combinatoric arguments. We provide in Appendix C the relevant sets and system for the case n, k = 2.

Now we present some technical results regarding the sets introduced in Definition 2.5. The next lemma will be used at the start of the proof of Proposition 2.13, in order to group the common terms arising from the aforementioned application of the distributive rule, and in Proposition 2.15, establishing that the uniform distribution leads to a gradient model.

Lemma 2.7. For each $j \in J$ and $0 \leq \ell \leq k$ there is a non-injective but surjective map $\psi_{j,\ell} : I \times [\![1, \binom{k}{\ell}]\!] \to I_\ell$ such that for each $(i,q) \in I \times \{1,\ldots,\binom{k}{\ell}\}$ there exists $i' \in I_\ell$ such that

$$[M_j \backslash P_{ij}] \sqcup Q_{ijq}^{\ell} = P_{i'j}^{n+\ell} \quad where \quad \psi_{j,\ell}(i,q) = i'.$$

Moreover, for any $(i', j) \in I_{\ell} \times J$ it holds that $\left| (\psi_{j,\ell})^{-1}(i') \right| = {n+\ell \choose \ell}$.

Proof. Let $0 \leq \ell \leq k$ be fixed and introduce the set M' = [[0, n + k - 1]]. We write $\mathcal{P}_k(M') = \{P_i\}_{i \in I}$ and $\mathcal{P}_\ell(P_i) = \{Q_{iq}^\ell\}_{1 \leq q \leq \binom{k}{\ell}}$. For each $j \in J$, introduce also the map Φ_j through

$$\Phi_j(A) = \{a \in A : a < j\} \sqcup \{a + 2 : a \in A, a \ge j\}, \quad \text{where} \quad A \subseteq \mathbb{T}_N.$$
(2.2)

Note that $|\Phi_j(A)| = |A|$ and that $M_j = \Phi_j(M')$, $P_{ij} = \Phi_j(P_i)$, $Q_{ijq}^{\ell} = \Phi_j(Q_{iq}^{\ell})$ and $P_{ij}^{n+\ell} = \Phi_j(P_i^{n+\ell})$, for any $0 \le \ell \le k$, $(i,j) \in I_\ell \times J$.

It is clear that

$$\mathcal{P}_{n+\ell}(M') \subseteq \{ [M' \setminus P_i] \sqcup Q_{iq}^\ell \}_{(i,q) \in I \times \{1, \dots, \binom{k}{\ell} \}}$$

where, for $\ell \neq k$, since $|\mathcal{P}_{n+\ell}(M_j)| < \binom{k}{\ell}|I|$ the right-hand-side above must have "repeated terms". Let then ψ_ℓ be the map such that $[M' \setminus P_i] \sqcup Q_{iq}^\ell = P_{i'}^{n+\ell} \Leftrightarrow \psi_\ell(i,q) = i'$ for some $i' \in I_\ell$. Then to compute $|(\psi_\ell)^{-1}(i')|$ for any particular $i' \in I_\ell$ one needs to count the number of pairs (P,Q) such that $P \in \mathcal{P}_k(M'), Q \in \mathcal{P}_\ell(P)$ and

$$(M' \backslash P) \sqcup Q = P_{i'}^{n+\ell}$$

In order to do so, we can pick any ℓ elements of $P_{i'}^{n+\ell}$ and construct Q, existing some set $A \subseteq M'$ such that $P' \setminus Q = A$. In particular, there exists a unique $P \in \mathcal{P}_k(M')$ such that $A = M' \setminus P$, that is, $P = M' \setminus A$. To see that $Q \subseteq P$ it is enough to note that $M' \setminus P = P' \setminus Q$ and so $Q \not\subset M' \setminus P$, hence $Q \subseteq P$. Since there are $\binom{n+\ell}{\ell}$ ways to choose ℓ elements of $P_{i'}^{n+\ell}$, we conclude that $|(\psi_\ell)^{-1}(i')| = \binom{n+\ell}{\ell}$.

we conclude that $|(\psi_{\ell})^{-1}(i')| = \binom{n+\ell}{\ell}$. In particular, it also holds that $(\Phi_j(M')/\Phi_j(P)) \sqcup \Phi_j(Q) = P_{i'j}^{n+\ell}$ and one can define $\psi_{j,\ell} := \psi_{\ell}$, where now $\psi_{j,\ell}(i,q) = i' \Leftrightarrow (\Phi_j(M')/\Phi_j(P_i)) \sqcup \Phi_j(Q_{iq}^{\ell}) = P_{i'j}^{n+\ell}$. Since $\Phi_j(M) = M_j$ and there is exactly one $i \in I$ such that $\Phi_j(P) = P_{ij}$ and $\Phi_j(Q_{iq}^{\ell}) = Q_{ijq}^{\ell}$, and $\Phi_j(M' \backslash P_i) = \Phi_j(M') \backslash \Phi_j(P_i)$, it is simple to see that

$$[M_j \backslash P_{ij}] \sqcup Q_{ijq}^{\ell} = P_{i'j}^{n+\ell} \Leftrightarrow [M' \backslash P_i] \sqcup Q_{iq}^{\ell} = P_{i'}^{n+\ell},$$

which concludes the proof.

The arguments in the forthcoming Proposition 2.13 revolve around translating appropriate sets in Definition 2.5, and organizing them in a specific manner. But to do so, we need to keep track of their respective indexes or, to be precise, of particular equivalence classes of indexes.

Definition 2.8. Define over the set composed by all the non-empty subsets of \mathbb{T}_N the equivalence relation \equiv through

$$A \equiv B \Leftrightarrow \exists a \in \mathbb{Z} : A = a + B, \quad A, B \subset \mathbb{T}_N.$$

For each $0 \leq \ell \leq k$ and x = 0, 1, define the equivalence relation $\overset{\ell,x}{\sim}$ over $I_{\ell} \times J$ as

$$(i,j) \stackrel{\ell,x}{\sim} (i',j') \Leftrightarrow P_{ij}^{n+\ell} \sqcup \{j+x\} \equiv P_{i'j'}^{n+\ell} \sqcup \{j'+x\}.$$

We shorten $\mathcal{C}_{\ell,x} = I_{\ell} \times J_{\ell,x}$.

The next lemma provides the essential ingredient for constructing a gradient model starting from the PMM mechanism. It will be invoked in the proof of Proposition 2.13.

Lemma 2.9. For each $0 \le \ell \le k$ and $(i, j) \in I_{\ell} \times J$ there exists $(i', j') \in I_{\ell} \times J$ such that

$$P_{ij}^{n+\ell} \sqcup \{j+1\} \equiv P_{i'j'}^{n+\ell} \sqcup \{j'\}.$$

Similarly, the converse also holds: for each $0 \leq \ell \leq k$ and $(i, j) \in I_{\ell} \times J$ there exists $(i', j') \in I_{\ell} \times J$ such that

$$P_{ij}^{n+\ell} \sqcup \{j\} \equiv P_{i'j'}^{n+\ell} \sqcup \{j'+1\}.$$

Proof. It is convenient to see the "*P*-sets" as binary strings. For each $A \subseteq \mathbb{T}_N$, let us consider the configuration $\xi_A \in \Omega_N$ where $\xi_A(y) = \mathbf{1}\{y \in A\}$.

Let us start by fixing $\ell = k$ and $j \in J$. Note that in this case $|I_{\ell}| = 1$ and $P_{1j}^{n+\ell} = M \setminus \{j, j+1\}$, where we recall from Definition 2.5 that M = [[0, n+k+1]]. Then

$$\xi_{P_{1,j}^{n+\ell}\sqcup\{j+1\}} = \begin{cases} \tau^{-1}\xi_{P_{1,j+\ell}^{n+\ell}\sqcup\{n+\ell\}}, & j=0, \\ \xi_{P_{1,j-1}^{n+\ell}\sqcup\{j-1\}}, & j\neq 0. \end{cases}$$

For $\ell \leq k-1$ the rationale is analogous. Fixed $(i, j) \in I_{\ell} \times J$, if $\xi_{P_{ij}^{n+\ell}}$ consists of $n+\ell$ consecutive particles inside the box M with $\xi_{P_{ij}^{n+\ell}}(n+k+1) = 1$, that is, $\xi_{P_{ij}^{n+\ell}}(\llbracket k+1-\ell, n+k+1 \rrbracket) = 1$, and $j+1=k-\ell$, then

$$\xi_{P_{i,k-1-\ell}^{n+\ell} \sqcup \{k-\ell\}} = \tau^{-1} \xi_{P_{i',n+k+1}^{n+\ell} \sqcup \{n+k+1\}}$$

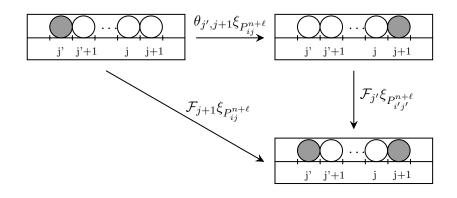
where $P_{i',n+k+1}^{n+\ell}$ corresponds to $\xi_{P_{i',n+k+1}^{n+\ell}}(\llbracket k-\ell, n+k-1 \rrbracket) = 1$. If $j+1 \neq k-\ell$ or if we do not have $n+\ell$ consecutive particles, then there exists a local configuration of the form (1,0)in the window M. Let us then say that we have (1,0) at $\{j', j'+1\}$ for some $j' \in \llbracket 0, n+k \rrbracket$. Since $\{P_{\tilde{i}\tilde{j}}^{n+\ell}\}_{\tilde{i}\in I_{\ell}} = \mathcal{P}_{n+\ell}(M_{\tilde{j}})$ for all $(\tilde{i}, \tilde{j}) \in I_{\ell} \times J$, there is also some $i' \in I_{\ell}$ such that

$$\theta_{j',j+1}\xi_{P_{ij}^{n+\ell}} = \xi_{P_{i'j'}^{n+\ell}},$$

where for any $x, y \in \mathbb{T}_N$, $\theta_{x,y}$ is the operator acting on Ω_N that exchanges the occupations at x, y, and \mathcal{F}_x is the operator that flips the occupation at x. In this case, specifically, holds that $\mathcal{F}_{j+1} = \mathcal{F}_{j'}\theta_{j',j+1}$, that is,

$$\xi_{P_{ij}^{n+\ell} \sqcup \{j+1\}} = \mathcal{F}_{j+1}\xi_{P_{ij}^{n+\ell}} = \mathcal{F}_{j'}\theta_{j',j+1}\xi_{P_{ij}^{n+\ell}} = \xi_{P_{i'j'}^{n+\ell} \sqcup \{j'\}}.$$

This rationale is represented in the following commutative diagram.



The converse is analogous.

The previous lemma has the following simple, but important corollary, which can be seen as the analogue of the previous lemma in terms of the indexes.

Corollary 2.10. For each $0 \le \ell \le k$ there exists a permutation ϕ_{ℓ} on the set of indexes $(i, j) \in I_{\ell} \times J$ such that

$$-p_{ij}^{\ell,1} + P_{ij}^{n+\ell} \sqcup \{j+1\} = -p_{j'i'}^{\ell,0} + P_{i'j'}^{n+\ell} \sqcup \{j'\} \quad where \quad \phi_{\ell}(i,j) = (i',j').$$
(2.3)

In particular, it holds that for each $\underline{c} \in C_{\ell,1}$

$$|\underline{c}| = |\phi_{\ell}(\underline{c})|. \tag{2.4}$$

Proof. The equality (2.3) is a consequence of shifting the equivalent sets (with respect to the equivalence relation \equiv) to the origin; while the equality (2.4) holds because from Lemma 2.9 each set of the form $P_{ij}^{n+\ell} \sqcup \{j+1\}$, with $(i,j) \in I_{\ell} \times J$, can be seen as a translation of some set $P_{i'j'}^{n+\ell} \sqcup \{j'\}$, for some $(i',j') \in I_{\ell} \times J$ (and, naturally, vice-versa). As such, the respective classes of those sets (therefore the classes of their indexes) must be of equal size.

Notation 2.11. Fixed $x \in \{0, 1\}$ and $0 \le \ell \le k$, given any $\underline{c} \in \mathcal{C}_{\ell,x}$ we shall write as $A_{\underline{c}}$ one set representative of the class \underline{c} , that is, such that $A_{\underline{c}} = A_{ij}^{n+\ell,x}$ for every $(i, j) \in \underline{c}$.

Remark 2.12. From Corollary 2.10, for any $\underline{c_1} \in \mathcal{C}_{\ell,1}$ there is a corresponding $\underline{c_0} \in \mathcal{C}_{\ell,0}$ such that $\underline{c_0} = \phi_{\ell}(\underline{c_1})$. The converse is also true since ϕ_{ℓ} is a bijection. In this way, $\mathcal{C}_{\ell,0} = \phi_{\ell}(\mathcal{C}_{\ell,1})$, and in particular $A_{\underline{c_1}} = A_{\phi_{\ell}(\underline{c_1})}$.

We are now ready to present the main result of this section. We recall Definitions 2.5 and 2.6.

Proposition 2.13. For each $(i, j) \in I_{\ell} \times J$ with $0 \le \ell \le k$ fixed, consider the quantity

$$b_{ij}^{\ell} = \sum_{(i',q)\in(\psi_{j,\ell})^{-1}(i)} a_{i'j}.$$
(2.5)

If \mathfrak{a} is such that the linear system

$$\left\{\sum_{(i,j)\in\underline{c}} b_{ij}^{\ell} = \sum_{(i,j)\in\phi_{\ell}(\underline{c})} b_{ij}^{\ell}\right\}_{0\leq\ell\leq k-1, \ \underline{c}\in\mathcal{C}_{\ell,1}}$$
(2.6)

has a solution, then it holds that $\mathbf{c}^{n,k}(\eta)\nabla^+\eta(0) = \nabla^+\mathbf{h}^{n,k}(\eta)$ with $\mathbf{h}^{n,k} = \mathbf{h}_0^{n,k} + \mathbf{h}_1^{n,k}$ where

$$\mathbf{h}_{0}^{n,k} = \frac{1}{n+k+1} \sum_{j=0}^{n+k} \sum_{y=1}^{j} \mathbf{c}_{j}^{n,k}(\tau^{j-y}\eta) \nabla^{+}\eta(j-y),$$

$$\mathbf{h}_{1}^{n,k} = \frac{1}{n+k+1} \sum_{\ell=0}^{k} (-1)^{\ell} \sum_{\underline{c} \in \mathcal{C}_{\ell,1}} \sum_{(i,j) \in \underline{c}} \left\{ b_{ij}^{\ell} \sum_{y=0}^{p_{ij}^{\ell,1}-1} (\tau^{y}\eta)(A_{\underline{c}}) - b_{\phi_{\ell}(i,j)}^{\ell} \sum_{y=0}^{p_{\phi_{\ell}(i,j)}^{\ell,0}-1} (\tau^{y}\eta)(A_{\underline{c}}) \right\}^{(2.7)}.$$

with $\mathbf{s}_{j}^{n,k}$ as in (2.1).

Proof. Recalling from Definition 2.5 that for each $(i, j) \in I \times J$ we write $\{Q_{ijq}^{\ell}\}_{1 \le q \le {k \choose \ell}} = \mathcal{P}_{\ell}(P_{ij})$, from the distributive rule we have

$$\mathbf{c}^{n,k}(\eta) = \frac{1}{n+k+1} \sum_{(i,j)\in I\times J} a_{ij}(\tau^{-j}\eta) (M_j \setminus P_{ij}) \tau^{-j} \left\{ (-1)^k \eta(P_{ij}) + \sum_{\ell=0}^{k-1} \sum_{q=1}^{\binom{k}{\ell}} (-1)^\ell \eta(Q_{ijq}^\ell) \right\}$$
$$= \frac{1}{n+k+1} \sum_{\ell=0}^{k-1} \sum_{(i,j)\in I\times J} a_{ij} \sum_{q=1}^{\binom{k}{\ell}} (-1)^\ell (\tau^{-j}\eta) ([M_j \setminus P_{ij}] \sqcup Q_{ijq}^\ell)$$
$$+ \frac{(-1)^k}{n+k+1} \sum_{j\in J} (\tau^{-j}\eta) (M_j) \sum_{i\in I} a_{ij}.$$
(2.8)

Note that since $\sum_{i \in I} a_{ij} = 1$ for each $j \in J$, the term in the last line of the previous display is identified with $(-1)^k \mathbf{c}^{n+k}(\eta)$.

Fixed ℓ in (2.8), there are repeated elements in $\{[M_j \setminus P_{ij}] \sqcup Q_{ijq}^\ell\}_{(i,j) \in I \times J, 1 \le q \le \binom{k}{\ell}}$. We want to group the coefficients associated with these repeated sets. Recalling Lemma 2.7 and Definition 2.8, introducing, for each $(i,j) \in I \times J$ and $0 \le \ell \le k$, the weights b_{ij}^ℓ is in (2.5), we can then express

$$\mathbf{c}^{n,k}(\eta)\nabla^+\eta(0) = \frac{1}{n+k+1}\sum_{x=0,1}^{k}(-1)^{x-1}\sum_{\ell=0}^k(-1)^\ell\sum_{(i,j)\in I_\ell\times J}b_{ij}^\ell(\tau^{-j}\eta)(P_{ij}^{n+\ell}\sqcup\{j+x\}).$$
(2.9)

We want to remove the translations τ^{-j} , having then to work only with subsets of $M_j = M \setminus \{j, j+1\}$, for each $j \in J$. Since $\mathbf{1} - \tau^{-j} = \nabla^+ \circ \sum_{y=1}^j \tau^{-y}$, summing and subtracting the appropriate terms, the expression in the right-hand side in the previous display equals

$$\nabla^{+}\mathbf{h}_{0}^{n,k}(\eta) + \sum_{x=0,1} (-1)^{x-1} \sum_{\ell=0}^{k} (-1)^{\ell} \sum_{(i,j)\in I_{\ell}\times J} b_{ij}^{\ell} \eta(P_{ij}^{n+\ell} \sqcup \{j+x\})$$
(2.10)

with $\nabla^+ \mathbf{h}_0^{n,k}$ as in (2.7). Indeed,

$$\begin{split} \sum_{x=0,1} (-1)^{x-1} \sum_{\ell=0}^{k} (-1)^{\ell} \sum_{(i,j)\in I\times J} a_{ij} \sum_{q=1}^{\binom{k}{\ell}} \sum_{y=1}^{j} (\tau^{-y}\eta) ([M_{j}\setminus P_{ij}] \sqcup Q_{ijq}^{\ell} \sqcup \{j+x\}). \\ &= \sum_{(i,j)\in I\times J} a_{ij} \sum_{q=1}^{\binom{k}{j}} \sum_{y=1}^{j} \nabla^{+} (\tau^{-y}\eta) (j) (\tau^{-y}\eta) (M_{j}\setminus P_{ij}) \sum_{\ell=0}^{k} (-1)^{\ell} (\tau^{-y}\eta) (Q_{ijq}^{\ell}) \\ &= \sum_{(i,j)\in I\times J} a_{ij} \sum_{y=1}^{j} \nabla^{+} (\tau^{-y}\eta) (j) (\tau^{-y}\eta^{P_{ij}}) (M_{j}) \\ &= \sum_{j=0}^{n+k} \sum_{y=1}^{j} \tau^{j-y} \left\{ \nabla^{+} \eta(0) \mathbf{c}_{j}^{n,k}(\eta) \right\} = \mathbf{h}_{0}^{n,k}(\eta) (n+k+1), \end{split}$$

where in the second equality we applied the distributive rule.

At this point, for each x fixed in the second term in (2.10) we want to translate every set to the origin, in this way facilitating the comparison between the terms associated with x = 0 with the terms associated with x = 1. Recall then from Definition 2.5 that for every $(i, j) \in I_{\ell} \times J$, $0 \le \ell \le k$ and x = 0, 1 we short-write

$$p_{ij}^{\ell,x} = \min\{P_{ij}^{n+\ell} \sqcup \{j+x\}\} \ge 0 \quad \text{and} \quad A_{ij}^{n+\ell,x} = -p_{ij}^{\ell,x} + (P_{ij}^{n+\ell} \sqcup \{j+x\}).$$

Naturally, for all the indexes i, j, ℓ, x it holds that $p_{ij}^{\ell,x} \ge 0$ and $0 \in A_{ij}^{n+\ell,x}$. With this, we can rewrite the second term in (2.10) as

$$(\nabla^{+}\mathbf{h}_{1}^{n,k})(\eta) + \frac{1}{n+k+1} \sum_{x=0,1}^{k} (-1)^{x-1} \sum_{\ell=0}^{k} (-1)^{\ell} \sum_{(i,j)\in I_{\ell}\times J} b_{ij}^{\ell} \eta(A_{ij}^{n+\ell,x})$$
(2.11)

where, recalling Notation 2.11,

$$\mathbf{h}_{1}^{n,k}(\eta) = \frac{1}{n+k+1} \sum_{\ell=0}^{k} (-1)^{\ell} \sum_{\underline{c_{1}} \in \mathcal{C}_{\ell,1}} \sum_{(i,j) \in \underline{c_{1}}} b_{ij}^{\ell} \sum_{y=0}^{p_{ij}^{\ell,1}-1} (\tau^{y}\eta) (A_{\underline{c_{1}}})$$

$$- \frac{1}{n+k+1} \sum_{\ell=0}^{k} (-1)^{\ell} \sum_{\underline{c_{0}} \in \mathcal{C}_{\ell,0}} \sum_{(i,j) \in \underline{c_{0}}} b_{ij}^{\ell} \sum_{y=0}^{p_{ij}^{\ell,0}-1} (\tau^{y}\eta) (A_{\underline{c_{0}}})$$

$$(2.12)$$

and where we used that $\tau^p - \mathbf{1} = \nabla^+ \circ \sum_{y=0}^{p-1} \tau^y$ for any $p \in \mathbb{N}$. We stress that $\sum_{\emptyset} := 0$ and in the previous display one has $p_{ij}^{\ell,x} > 0$ for all i, j, ℓ, x , as we only translate the sets that do not start at the origin. Due to the shifting of the sets $P_{ij}^{n+\ell} \sqcup \{j+x\}$ for all the indexes i, j, for each x and ℓ fixed the collection $\{A_{ij}^{n+\ell,x}\}_{(i,j)\in I_\ell \times J}$ may have repeated elements. Grouping the repeated sets we can rewrite the second term in (2.11) as

$$\frac{1}{n+k+1} \sum_{x=0,1} (-1)^{x-1} \sum_{\ell=0}^{k} (-1)^{\ell} \sum_{\underline{c_x} \in \mathcal{C}_{\ell,x}} \eta(A_{\underline{c_x}}) \sum_{(i,j)\in\underline{c_x}} b_{ij}^{\ell}$$
(2.13)

where we recall that $A_{\underline{cx}}$ is the set representative of the class \underline{cx} , that is, such that $A_{\underline{cx}} = A_{ij}^{n+\ell,x}$ for all $(i, j) \in c_{\underline{x}}$ with x = 0 and x = 1.

The final step consists in invoking Lemma 2.9, which provides, through its Corollary 2.10, the existence, for each ℓ , of a permutation, ϕ_{ℓ} , over the set of indexes, $(i, j) \in I_{\ell} \times J$, such that

$$\phi_{\ell}(j,i) = (j',i') \quad \text{where} \quad -p_{ij}^{\ell,1} + (P_{ij}^{n+\ell} \sqcup \{j+1\}) = -p_{j',i'}^{\ell,0} + (P_{i'}^{j',n+\ell} \sqcup \{j'\}).$$

With this, it is clear that a sufficient condition for the model to have the gradient property is (2.6), since it leads to the quantity (2.13) being equal to zero.

The map ϕ_{ℓ} allows us to rewrite (2.12) as in (2.7) as a consequence of the following. For $(i, j) \in I_{\ell} \times J$ it holds that $A_{ij}^{n+\ell,1} = A_{\phi_{\ell}(i,j)}^{n+\ell,1} = A_{\underline{c}}$ where $(i, j) \in \underline{c} \in \mathcal{C}_{\ell,1}$; and since ϕ_{ℓ} is a permutation over the indexes induced by a bijection between the classes, for each $\underline{c}_0 \in \mathcal{C}_{\ell,0}$ there exists one $\underline{c}_1 \in \mathcal{C}_{\ell,1}$ such that $\underline{c}_0 = \phi_{\ell}(\underline{c}_1)$. Therefore, one can replace the summation over $\mathcal{C}_{\ell,0}$ by a summation over $\mathcal{C}_{\ell,1}$ and apply the map ϕ_{ℓ} to the indexes:

$$\sum_{\underline{c_0} \in \mathcal{C}_{\ell,0}} \sum_{(i,j) \in \underline{c_0}} b_{ij}^{\ell} \sum_{y=0}^{p_{ij}^{\ell,0}-1} (\tau^y \eta) (A_{\underline{c_0}}) = \sum_{\underline{c_1} \in \mathcal{C}_{\ell,1}} \sum_{(i,j) \in \underline{c_1}} b_{\phi_{\ell}(i,j)}^{\ell} \sum_{y=0}^{p_{\phi_{\ell}(i,j)}^{\ell,0}-1} (\tau^y \eta) (A_{\underline{c_1}}),$$

thus concluding the proof.

As a simple observation, for each fixed $0 \le \ell \le k$ the system

$$\left\{\sum_{(i,j)\in\underline{c}} b_{ij}^{\ell} = \sum_{(i,j)\in\phi_{\ell}(\underline{c})} b_{ij}^{\ell}\right\}_{\underline{c}\in\mathcal{C}_{\ell,1}}$$
(2.14)

is associated with a gradient model.

Corollary 2.14. Fixed $0 \le l \le k$, if the linear system (2.14) is satisfied, then the constraint given through the following map satisfies the gradient condition

$$\eta \mapsto \frac{1}{n+k+1} \sum_{j=0}^{n+k} \sum_{i=1}^{\binom{n+k}{k-\ell}} b_{ij}^{\ell}(\tau^{-j}\eta)(P_{ij}^{n+\ell}).$$

It is now simple to see that for general $n, k \in \mathbb{N}_+$ the uniform distribution corresponds to a solution of (2.6). A non-uniform solution is presented in Table 20, for n, k = 2.

Proposition 2.15. For any fixed $n, k \in \mathbb{N}_+$, the uniform choice $a_{ij} = \frac{1}{|I|} = \binom{n+k}{k}^{-1}$, for $(i, j) \in I \times J$, is a solution of the system (2.6).

Proof. From Lemma 2.7 we know that for every $(i, j) \in I_{\ell} \times J$ the quantity $|(\psi_{j,\ell})^{-1}(i)|$ depends only on k and ℓ . This, coupled with the existence of a particular permutation on $(i, j) \in I_{\ell} \times J$, for each $0 \leq \ell \leq k$, as in Corollary 2.10, allow us to readily extract as a solution the uniform choice.

Remark 2.16. The constraint $\mathbf{b}^{n,k}$ in Definition 2.1 corresponds to letting $a_{ij} = \frac{1}{|I|} = {\binom{n+k}{k}}^{-1}$ in (2.1), then renormalizing the resulting constraint by a factor ${\binom{n+k}{n}}$.

Fixed the uniform weights, our main goal now is to express the gradient property of the B(n,k) in a more convenient manner, as the expression for $h^{n,k}$ from (2.7) depends on different maps and equivalence classes, making difficult the identification of the relevant terms. This is achieved with the Lemma 2.18, below. It is also worth noting that this same simplification can be performed for, existing, non-uniform solutions under an extra set of conditions, as in Lemma C.1.

Several technical results, characterizing the equivalence classes in the previous results, will be invoked in the next Lemma. This analysis is performed in Appendix A.

We now introduce a key map.

Definition 2.17. For each $A \subseteq M = \llbracket 0, n+k+1 \rrbracket$ introduce $s(A) = \max(A) - \min(A)$ and the quantity

$$s_M(A) = 1 - \frac{s(A)}{n+k+1}$$

Lemma 2.18. Fixed $n, k \ge 1$, let $M' = [\![1, n+k]\!]$ and $\mathbf{h}_1^{n,k}(\eta)$ be as in (2.7) with the uniform distribution, as in Proposition 2.15. One can express

$$\mathbf{h}_{1}^{n,k} = \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \left(\mathbf{P}_{n+k}^{n,k} + \nabla^{+} \mathbf{g}_{n+k}^{n,k} \right),$$

where, for each $0 \leq l \leq L$,

$$\mathbf{P}_{L}^{\ell}(\eta) = \frac{1}{\binom{L}{\ell}} \sum_{A \in \mathcal{P}_{\ell}(M')} s_{M}(\{0\} \sqcup A) \eta(\{0\} \sqcup A),$$

$$\mathbf{g}_{L}^{\ell}(\eta) = \frac{1}{\binom{L}{\ell}} \sum_{A \in \mathcal{P}_{\ell}(M')} \sum_{y=1}^{L-\max(A)} \left(s_{M}(\{0\} \sqcup A) - \frac{y}{L+1} \right) (\tau^{y-1}\eta)(\{0\} \sqcup A).$$
(2.15)

Proof. For the uniform choice the expression for $\mathbf{h}_1^{n,k}(\eta)$ in (2.7) simplifies to

$$\mathbf{h}_{1}^{n,k}(\eta) = \frac{1}{n+k+1} \sum_{\ell=0}^{k} (-1)^{\ell} \frac{\binom{n+\ell}{\ell}}{\binom{n+k}{k}} \sum_{\underline{c} \in \mathcal{C}_{\ell,1}} \sum_{(i,j) \in \underline{c}} \left\{ \sum_{y=0}^{p_{ij}^{\ell,1}-1} (\tau^{y}\eta) (A_{\underline{c}}) - \sum_{y=0}^{p_{\phi_{\ell}(i,j)}^{\ell,0}-1} (\tau^{y}\eta) (A_{\underline{c}}) \right\}.$$

Fixed ℓ and $\underline{c} \in C_{\ell,1}$, from the property (1) in Lemma A.4 the summation over $(i, j) \in \underline{c}$ in the previous display can be expressed as

$$\sum_{y=0}^{n+k+1-\max(A_{\underline{c}})-1} (\tau^{y}\eta)(A_{\underline{c}}) \times \\ \times \left\{ \sum_{(i,j)\in\underline{c}} \mathbf{1}\{p_{ij}^{\ell,1} = n+k+1-\max(A_{\underline{c}}) \neq 0\} - \sum_{(i,j)\in\underline{c}} \mathbf{1}\{p_{\phi_{\ell}(i,j)}^{\ell,0} = n+k+1-\max(A_{\underline{c}}) \neq 0\} \right\}$$

We stress that we can apply Lemma A.4 since for any $\underline{c} \in C_{\ell,1}$ in the summations above it holds that $\max(A_{\underline{c}}) \neq n+k+1$, since otherwise $p_{ij}^{\ell,1} = 0$ for any $(i, j) \in \underline{c}$, and these elements

are not present in those summations, as explained just after (2.12). Now we apply Lemma A.4 again, concretely, property (3), obtaining that

$$\mathbf{h}_{1}^{n,k}(\eta) = \frac{1}{n+k+1} \sum_{\ell=0}^{k} (-1)^{\ell} \frac{\binom{n+\ell}{\ell}}{\binom{n+k}{k}} \sum_{\underline{c} \in \mathcal{C}_{\ell,1}} \sum_{y=0}^{n+k-\max(A_{\underline{c}})} (\tau^{y}\eta)(A_{\underline{c}}),$$
(2.16)

and from Lemma A.5,

$$\mathbf{h}_{1}^{(n,k)}(\eta) = \frac{1}{n+k+1} \sum_{\ell=0}^{k} (-1)^{\ell} \frac{\binom{n+\ell}{\ell}}{\binom{n+k}{k}} \sum_{A \in \mathcal{P}_{n+\ell}(M \setminus \{0,n+k+1\})} \sum_{y=0}^{n+k-\max(A)} (\tau^{y}\eta)(\{0\} \sqcup A).$$

In order to end the proof, we use that, for any $w \in \mathbb{N}_+$,

$$\sum_{y=0}^{w} \tau^{y} = (\nabla^{+} \circ \sum_{y=0}^{w} \sum_{z=0}^{y-1} \tau^{z}) + w\mathbf{1} = (w+1)\mathbf{1} + \nabla^{+} \circ \sum_{y=1}^{w} (w+1-y)\tau^{y-1},$$

then $n + k + 1 - \max(A) = (n + k + 1)s_M(\{0\} \sqcup A)$ and

$$\frac{\binom{n+\ell}{\ell}}{\binom{n+k}{k}\binom{k}{\ell}} = \frac{1}{\binom{n+k}{n+\ell}}.$$

We conclude this subsection by presenting the full expression for the potential associated with the Bernstein model, with the terms collected from the previous results.

Proposition 2.19. Fixed $n, k \in \mathbb{N}$, for every $\eta \in \Omega_N$ it holds that

$$\mathbf{b}^{n,k}(\eta)\nabla^+\eta(0) = \nabla^+\mathbf{H}^{n,k} \quad where \quad \mathbf{H}^{n,k} = \mathbf{P}^{n,k} + \nabla^+\mathbf{g}^{n,k} + \mathbf{f}^{n,k},$$

with

$$\mathbf{f}^{n,k}(\eta) = \frac{1}{L+1} \sum_{j=0}^{n+k} \sum_{y=1}^{j} \nabla^+ \eta(j-y) \mathbf{c}_j^{n,k}(\tau^{j-y}\eta),$$
$$\iota^{n,k} = \sum_{\ell=0}^{k} (-1)^{\ell} \binom{n+k}{k} \binom{k}{\ell} \iota_{n+k}^{n+\ell}, \qquad for \quad \iota \equiv \mathbf{g}, \mathbf{P},$$

and $\mathbf{g}_{n+k}^{n+\ell}$, $\mathbf{P}_{n+k}^{n+\ell}$ as in (2.15).

We observe that $\int_{\Omega_N} \mathbf{f}^{n,k} d\nu_{\alpha} = 0$ for any $\alpha \in (0,1)$.

3. Hydrodynamic Limit for the empirical measure

In this section we are going to prove a law of large numbers for the *perturbed* process, under the diffusive time-scaling. Precisely, we study the Markov process defined through the generator

$$\mathcal{L}_N := N^2 \mathcal{L}_N^{n,k} + \Theta_{\mathfrak{p}}(N) \mathcal{L}_N^{\text{SSEP}}, \quad \text{for} \quad \mathfrak{p} \in (0,1) \quad \text{and} \quad \Theta_{\mathfrak{p}}(N) = N^{2-\mathfrak{p}}.$$

In order to present the main result of this section we need to introduce some definitions.

Fix a finite time horizon [0, T], let μ_N be an initial probability measure on Ω_N , and let $\{\eta_{N^2t}\}_{t\geq 0}$ be the process generated by \mathcal{L}_N . For any $\eta \in \Omega_N$, the *Empirical measure* $\pi^N(\eta, du)$ on the one-dimensional, continuous torus $\mathbb{T} = [0, 1]$, is defined by

$$\pi^{N}(\eta, \mathrm{d} u) = \frac{1}{N} \sum_{x \in \mathbb{T}_{N}} \eta(x) \delta_{N^{-1}x}(\mathrm{d} u),$$

where δ_v is the Dirac measure at $v \in \mathbb{T}$. Its (diffusive) time evolution is defined as $\pi_t^N(\eta, du) = \pi^N(\eta_{N^2t}, du)$, and for any function $G : \mathbb{T} \to \mathbb{R}$, shorten the integral of G with respect to the empirical measure as

$$\langle \pi_t^N, G \rangle = \int_{\mathbb{T}} G(u) \pi_t^N(\eta, \mathrm{d}u).$$

We denote by \mathcal{M}_+ the space of positive measures on [0,1] with total mass at most 1 and endowed with the weak topology. The Skorokhod space of trajectories induced by $\{\eta_{N^2t}\}_{t\in[0,T]}$ with initial measure μ_N is denoted by $\mathcal{D}([0,T],\Omega_N)$, and we denote by \mathbb{P}_{μ_N} the induced probability measure on it. Moreover, $\mathbb{Q}_N := \mathbb{P}_{\mu_N} \circ (\pi^N)^{-1}$ is the probability measure on $\mathcal{D}([0,T],\mathcal{M}_+)$ induced by $\{\pi_t^N\}_{t\in[0,T]}$ and μ_N .

For $p \in \mathbb{N}_+ \cup \{\infty\}$, let $C^p(\mathbb{T})$ be the set of p times continuously differentiable, real-valued functions defined on \mathbb{T} ; and let $C^{q,p}([0,T] \times \mathbb{T})$ be the set of all real-valued functions defined on $[0,T] \times \mathbb{T}$ that are q times differentiable on the first variable and p times differentiable on the second variable, with continuous derivatives. For $f, g \in L^2(\mathbb{T})$, we denote by $\langle f, g \rangle$ their standard Euclidean product in $L^2(\mathbb{T})$ and $\|\cdot\|_{L^2(\mathbb{T})}$ its induced norm. The repeated notation $\langle \cdot, \cdot \rangle$ will be clear from the context.

We now aim to introduce the relevant weak formulation of the formal equation in (1.2). To that end, for any pair $G, H \in C^{\infty}(\mathbb{T})$ let $\langle G, H \rangle_1 = \langle \partial_u G, \partial_u H \rangle$ be their semi inner-product on $C^{\infty}(\mathbb{T})$, and $\|\cdot\|_1$ its associated semi-norm. The space $\mathcal{H}^1(\mathbb{T})$ is the Sobolev space on \mathbb{T} , defined as the completion of $C^{\infty}(\mathbb{T})$ for the norm $\|\cdot\|_{\mathcal{H}^1(\mathbb{T})}^2 = \|\cdot\|_{L^2}^2 + \|\cdot\|_1^2$. We write as $L^2([0,T];\mathcal{H}^1(\mathbb{T}))$ the set of measurable functions $f:[0,T] \to \mathcal{H}^1(\mathbb{T})$ such that $\int_0^T \|f_s\|_{\mathcal{H}^1(\mathbb{T})}^2 ds < \infty$.

3.1. Main result. We are now ready to introduce the notion of weak solution used in this manuscript, and state and prove the hydrodynamic limit for the empirical measure, Theorem 3.3.

Definition 3.1 (Weak solution). For $\rho^{\text{ini}} : \mathbb{T} \to [0,1]$ a measurable function, we say that $\rho : [0,T] \times \mathbb{T} \mapsto [0,1]$ is a weak solution of the Generalized Porous Media Equation

$$\begin{cases} \partial_t \rho = \binom{n+k}{k} \partial_u^2 \Phi^{n,k}(\rho), & \text{in } \mathbb{T} \times (0,T], \\ \rho_0 = \rho^{\text{ini}}, & \text{in } \mathbb{T} \end{cases}$$

with $n, k \in \mathbb{N}_+$ arbitrarily fixed and Φ as in (1.3), if

- (1) $\Phi^{n,k}(\rho) \in L^2([0,T]; \mathcal{H}^1(\mathbb{T}));$
- (2) for any $t \in [0,T]$ and $G \in C^{1,2}([0,T] \times \mathbb{T})$, ρ satisfies the formulation $F_t(\rho^{\text{ini}}, \rho, G) = 0$, where

$$F_t(\rho^{\text{ini}},\rho,G) := \langle \rho_t, G_t \rangle - \langle \rho^{\text{ini}}, G_0 \rangle - \int_0^t \left\{ \langle \rho_s, \partial_s G_s \rangle + \binom{n+k}{k} \langle \Phi^{n,k}(\rho_s), \partial_u^2 G_s \rangle \right\} \mathrm{d}s.$$
(3.1)

From item (1) above, the uniqueness of solutions of the weak formulation in last definition follows by Oleinik's method (see, for instance, [2, Subsection 7.1]). The hydrodynamic limit establishes the existence of solutions as the density of the limiting empirical measure governed by the dynamics of B(n,k), under the space-time scaling $(x,t) \mapsto (N^{-1}x, N^2t)$. We will state this precisely shortly.

Definition 3.2 (Local equilibrium distribution). Let $\{\mu_N\}_{N\geq 1}$ be a sequence of probability measures on Ω_N , and let $g : \mathbb{T} \to [0,1]$ be a measurable function. If, for any continuous function $G : \mathbb{T} \to \mathbb{R}$ and every $\delta > 0$, it holds

$$\lim_{N \to +\infty} \mu_N \left(\eta \in \Omega_N : \left| \langle \pi^N, G \rangle - \langle g, G \rangle \right| > \delta \right) = 0,$$

we say that the sequence $\{\mu_N\}_{N\geq 1}$ is a local equilibrium measure associated with the profile g.

Theorem 3.3 (Hydrodynamic limit). Fix $n, k \in \mathbb{N}_+$. Let $\rho^{\text{ini}} : \mathbb{T} \to [0, 1]$ be a measurable function and let $\{\mu_N\}_{N\geq 1}$ be a local equilibrium measure associated with it. Then, for any $t \in [0, T]$ and $\delta > 0$, it holds

$$\lim_{N \to +\infty} \mathbb{P}_{\mu_N} \left(\left| \langle \pi_t^N, G \rangle - \langle \rho_t, G \rangle \right| > \delta \right) = 0,$$

where ρ is the unique solution of (1.2) in the sense of Definition 3.1, with initial data ρ^{ini} .

In order to prove the main theorem, we apply the entropy method, first introduced in [12], following the method as presented in [11]. Let us overview the approach.

The link between our process and the weak formulation (3.1) is given through Dynkin's martingale (see [13, Appendix 1, Lemma 5.1]),

$$M_t^N(G) := \langle \pi_t^N, G_t \rangle - \langle \pi_0^N, G_0 \rangle - \int_0^t (\partial_s + \mathcal{L}_N) \langle \pi_s^N, G_s \rangle \mathrm{d}s.$$
(3.2)

for $G \in C^{2,1}(\mathbb{T} \times [0,T])$. Indeed, one can show that the sequence of probability measures $(\mathbb{Q}_N)_{N \in \mathbb{N}}$ is tight with respect to the Skorokhod topology of $\mathcal{D}([0,T], \mathcal{M}_+)$ by resorting to Aldous' conditions [10, proof of Proposition 4.1], that can be shown to be satisfied due to the gradient property, uniformly boundedness, in N, of the rates, and the quadratic variation of $M_T^N(G)$ vanishing as $N \to \infty$. Indeed, from the formula in [13] and computations in [11], it is simple to see that

$$\mathbb{E}_{\mu_N}\left[\left(M_T^N(G)\right)^2\right]^{\frac{1}{2}} \lesssim \frac{1}{N^2} \sum_{x \in \mathbb{T}_N} \left\{N(G(\frac{x+1}{N}) - G(\frac{x}{N}))\right\}^2 \xrightarrow{N \to +\infty} 0, \tag{3.3}$$

where we write $f \leq g$ whenever there is some constant c > 0 independent of N such that $f \leq cg$. With this, we conclude that the sequence of empirical measures is tight, existing then weakly convergent subsequences. Since there is at most one particle per site, one can show that the limiting measure, that we write as \mathbb{Q} , is concentrated on paths of absolutely continuous measures with respect to the Lebesgue measure. One can prove this with small adaptations from [13, page 57].

At this point, we have that the sequence $(\pi^N(\eta, du))_{N \in \mathbb{N}}$ converges weakly, with respect to \mathbb{Q}_N , to an absolutely continuous measure $\pi_{\cdot}(du) = \rho_{\cdot}(u)du$. The next step is to characterize the limiting points. Precisely, we aim to show the following.

Proposition 3.4. For any limit point \mathbb{Q} of $(\mathbb{Q}_N)_{N \in \mathbb{N}}$ it holds

$$\mathbb{Q}\bigg(\pi \in \mathcal{D}([0,T],\mathcal{M}_{+}): \begin{cases} \Phi^{n,k}(\rho) \in L^{2}([0,T];\mathcal{H}^{1}(\mathbb{T})) \\ F_{t}(\rho^{\mathrm{ini}},\rho,G) = 0, \ \forall t \in [0,T], \ \forall G \in C^{1,2}([0,T] \times \mathbb{T}) \end{cases} \bigg) = 1,$$

where $F_t(\rho^{\text{ini}}, \rho, G)$ is given in (3.1).

The standard machinery enables one to show that $\Phi^{n,k}(\rho) \in L^2([0,T]; \mathcal{H}^1(\mathbb{T}))$ holds \mathbb{Q} almost surely, from the fact that the lattice is the one dimensional torus, the constraints are uniformly bounded in N, the model satisfies the gradient property, and the Replacement Lemma 3.8 and the equality in (3.6) (see [11, Section 5 and Proposition 5.5]). All there is left to do is to show that $F_t(\rho^{\text{ini}}, \rho, G) = 0 - \mathbb{Q}$ a.s.. We note that the sequence $(\mathbb{Q}_N)_N$ converges completely, because the solution of the hydrodynamic equation is unique.

Because the lattice is finite, the process is of exclusion type and satisfies the gradient condition, as we shall see below, it is enough to show the so-called *Replacement Lemmas*. This is the content of Subsection 3.2, and the precise results are the Lemmas 3.8,3.9 and 3.10. Our contribution in Subsection 3.2 lies in the proof of the two-blocks estimate, Lemma 3.10. The difference between our argument and what is present in the literature is discussed in more detail throughout the proof of the aforementioned lemma.

For the convenience of the reader, we present the main steps of the characterization of the limit points. For $\epsilon > 0$ and $u, v \in \mathbb{T}$, define the cutoff function

$$\iota^u_\epsilon(v) := \frac{1}{\epsilon} \mathbf{1}_{v \in [u, u+\epsilon)}.$$

From Lebesgue's differentiation theorem one can show the following.

Lemma 3.5. For any $\epsilon > 0$, any $p \in \mathbb{N}_+$, a.e. $u \in \mathbb{T}$ and $s \in [0, T]$, it holds that

$$\limsup_{\epsilon \to 0} \left| (\rho_s(u))^p - \prod_{j=0}^{p-1} \langle \pi_s, \iota_{\epsilon}^{u+j\epsilon} \rangle \right| = 0.$$

Proof. Observing that, for any a_0, b_0, a_1, b_1 , one can express $a_0a_1 - b_0b_1 = a_0(a_1 - b_1) + (a_0 - b_0)b_1$, proceeding inductively and applying the triangle inequality, it is enough to analyse

$$\begin{aligned} \left| \rho_s(u) - \langle \pi_s, \iota_{\epsilon}^{u+j\epsilon} \rangle \right| &\leq \left| \rho_s(u) - \frac{1}{(j+1)\epsilon} \int_u^{u+(j+1)\epsilon} \rho_s(v) \mathrm{d}v \right| \\ &+ j \left| \rho_s(u) - \frac{1}{(j+1)\epsilon} \int_u^{u+(j+1)\epsilon} \rho_s(v) \mathrm{d}v \right| \\ &+ j \left| \rho_s(u) - \frac{1}{j\epsilon} \int_u^{u+j\epsilon} \rho_s(v) \mathrm{d}v \right|. \end{aligned}$$

The proof ends by an application of Lebesgue's differentiation theorem.

Proof of Proposition 3.4. Recalling $B_{n+1,k+1}$ from (1.3), we are going to show that, for any $\delta > 0$, it holds

$$\sup_{t \in [0,T]} \left| \langle G_t, \rho_t \rangle - \langle G_0, \rho^{\mathrm{ini}} \rangle - \int_0^t \langle \rho_s, \partial_s G_s \rangle \mathrm{d}s + \sum_{\ell=0}^k (-1)^\ell b_\ell \left\langle \partial_u^2 G_s, \rho^{n+\ell+1} \right\rangle \mathrm{d}s \right| \le \delta \quad \mathbb{Q} - \mathrm{a.s.},$$

where we shortened, for n, k fixed, $b_{\ell} := (n + \ell + 1)^{-1} \binom{k}{\ell} \binom{n+k}{k}$, from which Proposition 3.4 then follows, assuming that $\Phi^{n,k}(\rho) \in L^2([0,T]; \mathcal{H}^1(\mathbb{T}))$ already holds. This can be straightforwardly reduced to estimating

$$\begin{split} & \mathbb{Q}\left(\sup_{t\in[0,T]}\left|\sum_{\ell=0}^{k}(-1)^{\ell}b_{\ell}\int_{0}^{t}\left\langle\partial_{u}^{2}G_{s},\rho_{s}^{n+\ell+1}-\prod_{j=0}^{n+\ell}\left\langle\pi_{s},\iota_{\epsilon}^{\cdot+j\epsilon}\right\rangle\right\rangle\mathrm{d}s\right|>\frac{\delta}{2}\right)\\ &+\mathbb{Q}\left(\sup_{t\in[0,T]}\left|\left\langle G_{t},\rho_{t}\right\rangle-\left\langle G_{0},\rho_{0}\right\rangle-\int_{0}^{t}\left\langle\rho_{s},\partial_{s}G_{s}\right\rangle-\sum_{\ell=0}^{k}(-1)^{\ell}b_{\ell}\left\langle\partial_{u}^{2}G_{s},\prod_{j=0}^{n+\ell}\left\langle\pi_{s},\iota_{\epsilon}^{\cdot+j\epsilon}\right\rangle\right\rangle\mathrm{d}s\right|>\frac{\delta}{2^{2}}\right)\\ &+\mathbb{Q}\left(\left|\left\langle G_{0},\rho_{0}-\rho^{\mathrm{ini}}\right\rangle\right|>\frac{\delta}{2^{2}}\right). \end{split}$$

From Lemma 3.5 and the fact that μ_N is a local equilibrium measure associated with the profile ρ^{ini} , it remains to study only the second probability in the previous display. The starting point for this analysis is to relate the microscopic and macroscopic scales through Portmanteau's Theorem. This cannot be performed directly, due to the discontinuity introduced through the cutoff functions ι_{ε} , but it is standard in the literature [2, 8, 11] to approximate them by continuous functions. With these justifications, applying Portmanteau's Theorem, then replacing back the cutoff functions by their discontinuous versions, reduces us to the study of

$$\begin{split} \liminf_{N \to +\infty} \mathbb{Q}_N \left(\sup_{t \in [0,T]} \left| \langle \pi_t^N, G_t \rangle - \langle G_0, \rho_0 \rangle - \int_0^t \langle \pi_s^N, \partial_s G_s \rangle \mathrm{d}s \right. \\ \left. - \sum_{\ell=0}^k (-1)^\ell b_\ell \int_0^t \left\langle \partial_u^2 G_s, \prod_{j=0}^{n+\ell} \langle \pi_s^N, \iota_\epsilon^{\cdot + j\epsilon} \rangle \right\rangle \mathrm{d}s \right| > \frac{\delta}{2^4} \end{split}$$
(3.4)

Recalling the expression for the martingale $M_t^N(G)$ in (3.2), from the gradient property as in Proposition 2.19,

$$\langle \mathcal{L}_N \pi^N, G \rangle = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \Delta^N G(\frac{x}{N}) \left(\mathbf{f}^{n,k} + \nabla^+ \mathbf{g}^{n,k} + \sum_{\ell=0}^k (-1)^\ell \binom{n+k}{k} \binom{k}{\ell} \mathbf{P}_{n+k}^{n+\ell} \right) (\tau^x \eta)$$

where $\Delta^N G(\frac{x}{N}) := N^2 \{ G(\frac{x+1}{N}) - 2G(\frac{x}{N}) + G(\frac{x-1}{N}) \}$. Summing and subtracting the appropriate terms and then applying Markov's inequality, we obtain that the probability in (3.4)

is bounded from above by

$$\begin{aligned} \mathbb{P}_{\mu_{N}}\left(\sup_{t\in[0,T]}\left|M_{t}^{N}(G)\right| > \frac{\delta}{2^{5}}\right) \\ &+ \mathbb{P}_{\mu_{N}}\left(\sup_{t\in[0,T]}\left|\sum_{\ell=0}^{k}(-1)^{\ell}b_{\ell}\int_{0}^{t}\left\langle\partial_{u}^{2}G_{s}-\Delta^{N}G_{s},\prod_{j=0}^{n+\ell}\langle\pi_{s}^{N},\iota_{\epsilon}^{+j\epsilon}\rangle\right\rangle\mathrm{d}s\right| > \frac{\delta}{2^{6}}\right) \\ &+ \mathbb{P}_{\mu_{N}}\left(\sup_{t\in[0,T]}\left|\frac{1}{N^{\mathfrak{p}}}\int_{0}^{t}\frac{1}{N}\sum_{x\in\mathbb{T}_{N}}\Delta^{N}G_{s}(\frac{x}{N})\eta_{N^{2}s}(x)\right\}\mathrm{d}s\right| > \frac{\delta}{2^{7}}\right) \\ &+ \mathbb{P}_{\mu_{N}}\left(\sup_{t\in[0,T]}\left|\int_{0}^{t}\frac{1}{N}\sum_{x\in\mathbb{T}_{N}}\Delta^{N}G_{s}(\frac{x}{N})(\mathbf{f}^{n,k}+\nabla^{+}\mathbf{g}^{n,k})(\tau^{x}\eta_{N^{2}s})\right\}\mathrm{d}s\right| > \frac{\delta}{2^{8}}\right) \\ &+ \mathbb{P}_{\mu_{N}}\left(\sup_{t\in[0,T]}\left|\int_{0}^{t}\frac{1}{N}\sum_{x\in\mathbb{T}_{N}}\Delta^{N}G_{s}(\frac{x}{N})\sum_{\ell=0}^{k}(-1)^{\ell}b_{\ell}\times\right. \\ &\times\left\{\prod_{j=0}^{n+\ell}\langle\pi_{s}^{N},\iota_{\epsilon}^{\frac{x}{N}+j\epsilon}\rangle - (n+\ell+1)\mathbf{P}_{n+k}^{n+\ell}(\tau^{x}\eta_{N^{2}s})\right\}\mathrm{d}s\right| > \frac{\delta}{2^{9}}\right) \tag{3.5}$$

From Doob's inequality and the first limit in (3.3), the $\limsup_{N\to+\infty}$ of the first probability in (3.5) vanishes; while because $G_s \in C^2(\mathbb{T})$ for all $s \in [0, t]$ so does the second. This, coupled with the fact that $\mathfrak{p} > 0$, implies that on the limit $N \to +\infty$ the third probability vanishes too. We now analyse the fourth term in (3.5).

From a summation by parts and the fact that $G_s \in C^2(\mathbb{T})$ for each $0 \leq s \leq t$, it holds that

$$\frac{1}{N}\sum_{x\in\mathbb{T}_N}\Delta^N G_s(\frac{x}{N})\nabla^+ \mathbf{g}_{n+k}^{n+\ell}(\tau^x\eta_{N^2s})\lesssim \frac{1}{N}.$$

Moreover, from the forthcoming Lemma 3.8, we can infer that

$$\limsup_{N \to +\infty} \mathbb{E}_{\mu_N} \left[\sup_{t \in [0,T]} \left| \int_0^t \frac{1}{N} \sum_{x \in \mathbb{T}_N} \Delta^N G_s(\frac{x}{N}) \mathbf{f}^{n,k}(\tau^x \eta_{N^2 s}) \mathrm{d}s \right| \right] = 0.$$
(3.6)

We now focus on $\mathbf{P}_{n+k}^{n+\ell}$, as in (2.15). Having in mind that for any sequences of real numbers $a = (a_i)_{i\geq 0}$ and $b = (b_i)_{i\geq 0}$, one can express

$$\prod_{j=0}^{n+\ell} a_j - \prod_{j=0}^{n+\ell} b_j = \sum_{j=0}^{n+\ell} \left(\prod_{i=0}^{j-1} b_i\right) (a_j - b_j) \left(\prod_{i=j+1}^{n+\ell} a_i\right),\tag{3.7}$$

for each ℓ one can reorganize, for each $A = \{a_1, \ldots, a_{n+\ell}\} \subset M'$ fixed with $a_i < a_{i+1}$ and $|A| = n + \ell$,

$$\eta(A) = \eta(\llbracket 1, n+\ell \rrbracket) + \sum_{y=1}^{n+\ell} \left[\prod_{i=1}^{y-1} \eta(i) \right] (\eta(a_y) - \eta(y)) \left[\prod_{j=y+1}^{n+\ell} \eta(a_j) \right].$$

Recalling \mathbf{P}_{L}^{l} as in (2.15) and also (1.3), the rearrangement above coupled with the observation that it must be that

$$\sum_{A \in \mathcal{P}_{\ell}(M')} s_M(\{0\} \sqcup A) = \frac{1}{\ell + 1} \binom{L}{\ell},$$

because $\int_{\Omega_N} \mathbf{b}^{n,k} d\nu_{\alpha} = \alpha^n (1-\alpha)^k$, and again Lemma 3.8, makes it so that it is now enough to show that, for each $\ell \in \{0, \ldots, k\}$ the $\limsup_{\epsilon \to 0} \limsup_{N \to +\infty} of$

$$\mathbb{E}_{\mu_N} \left[\sup_{t \in [0,T]} \left| \int_0^t \frac{1}{N} \sum_{x \in \mathbb{T}_N} \Delta^N G_s(\frac{x}{N}) \left\{ \prod_{j=0}^{n+\ell} \langle \pi_s^N, \iota_{\epsilon}^{\frac{x}{N}+j\epsilon} \rangle - \eta(x + \llbracket 0, n+\ell \rrbracket) \right\} \mathrm{d}s \right| \right]$$

is zero. The proof of this fact is done in several steps, and is consequence of the Lemmas 3.8, 3.9 and 3.10, as we now explain, with l in the latter two lemmas fixed as $l = N^{\mathfrak{p}/2}$. We note that in the aforementioned lemmas, the sup over $t \in [0, T]$ is absent. This is justified from [3, Lemma 4.3.2.].

For each $l \in \mathbb{N}_+$ consider the ball $B_l = [0, l-1]$, and shorten for any $\eta \in \Omega_N$ the average

$$\langle \eta \rangle_l = \frac{1}{l} \sum_{y \in B_l} \eta(y).$$

From (3.7), one can rewrite

$$\prod_{j=0}^{n+\ell} \langle \pi_s^N, \iota_{\epsilon}^{j\epsilon} \rangle - \prod_{j=0}^{n+\ell} \eta(j) = \sum_{j=0}^{n+\ell} \varphi_j^{\epsilon,\epsilon N} \bigg\{ \langle \pi_s^N, \iota_{\epsilon}^{j\epsilon} \rangle - \langle \tau^{j\epsilon N} \eta \rangle_{\epsilon N} \bigg\} \\
+ \sum_{j=0}^{n+\ell} \varphi_j^{\epsilon N,l}(\eta) \bigg\{ \langle \tau^{j\epsilon N} \eta \rangle_{\epsilon N} - \langle \tau^{jl} \eta \rangle_l \bigg\}$$
(3.8)

$$+\sum_{j=0}^{n+\ell}\varphi_j^l(\eta)\bigg\{\langle\tau^{jl}\eta\rangle_l - \eta(jl)\bigg\}$$
(3.9)

$$+\sum_{j=0}^{n+\ell} \psi_j^l(\eta) \bigg\{ \eta(jl) - \eta(j) \bigg\},$$
(3.10)

where $\varphi_j^{\epsilon N,l}$ depends on the occupation value at the sites $[\![0, j\epsilon N - 1]\!] \cup [\![jl, (j+1)l]\!]; \varphi_j^l$ depends on the occupation value at the sites $[\![0, jl - 1]\!] \cup \{il\}_{i=j+1,\dots,n+l};$ and ψ_j^l depends on the occupation value at the sites $\{il\}_{i=0,\dots,j-1} \cup \{i\}_{i=j+1,\dots,n+l}$.

The term associated with (3.10) is analysed with the upcoming Lemma 3.8. Lemma 3.9 treats the term associated with (3.9), while Lemma 3.10 is used to treat (3.8). All of these lemmas are proved in the next subsection.

3.2. **Replacement Lemmas.** In order to prove the subsequent results, we introduce the next objects.

Definition 3.6. Fixed $n, k \in \mathbb{N}$, the Carré du Champ operator $\Gamma_N^{n,k}$ is defined through the equality

$$\frac{1}{2}\Gamma_N^{n,k}g:=g(-\mathcal{L}_N^{n,k})g+\frac{1}{2}\mathcal{L}_N^{n,k}g,$$

for any $g: \Omega_N \to \mathbb{R}$ and with $\mathcal{L}_N^{n,k}$ as in Definition 2.1. Moreover, for each probability measure ν on Ω_N , we shorten $\mathcal{D}_N^{n,k}(g|\nu) := \int_{\Omega_N} \Gamma_N^{n,k} g d\nu$. Since n, k are fixed, we will shorten

$$\mathfrak{D}_{N,\mathfrak{p}}(g|\nu_{\alpha}^{N}) := \mathcal{D}_{N}^{n,k}(g|\nu_{\alpha}^{N}) + N^{-\mathfrak{p}}\mathcal{D}_{N}^{0,0}(g|\nu_{\alpha}^{N}).$$

In what follows, we say $f: \Omega_N \to \mathbb{R}^+$ is a *density* (with respect to ν_{α}^N) if $\int_{\Omega_N} f \nu_{\alpha}^N = 1$. We remark that rewriting $-a(b-a) = (a-b)^2/2 + (a^2-b^2)/2$ and using the fact that the B(n,k) is reversible w.r.t. ν_{α}^N , for any density f it holds that

$$\frac{1}{2}\mathcal{D}_{N}^{n,k}(f|\nu_{\alpha}^{N}) = \frac{1}{4}\int_{\Omega_{N}}\sum_{x\in\mathbb{T}_{N}}\tau^{x}\mathbf{c}^{n,k}|\nabla_{x,x+1}f|^{2}\mathrm{d}\nu_{\alpha}^{N} = -\int_{\Omega_{N}}f\mathcal{L}_{N}^{n,k}f\mathrm{d}\nu_{\alpha}^{N}.$$
(3.11)

In the same context, it is a simple computation to see that for any f density and, fixed $x, y \in \mathbb{T}_N$, for any $\varphi : \Omega_N \to \mathbb{R}$ that is independent of the transformation $\eta \mapsto \theta_{x,y}\eta$, it holds that

$$\int_{\Omega_N} \varphi(\eta) \nabla_{x,y} \eta(x) f(\eta) d\nu_\alpha^N = -\frac{1}{2} \int_{\Omega_N} \varphi(\eta) \nabla_{x,y} \eta(x) \nabla_{x,y} f(\eta) d\nu_\alpha^N.$$
(3.12)

Now we recall the relative entropy between two measures.

Definition 3.7. For μ and ν probability measures on Ω_N , the *relative entropy* of μ with respect to ν is defined as

$$H(\mu|\nu) := \sup_{g:\Omega_N \to \mathbb{R}} \bigg\{ \int_{\Omega_N} g \mathrm{d}\mu - \log \int_{\Omega_N} e^g \mathrm{d}\nu \bigg\}.$$

One can see [13, Theorem 8.3] that if μ is absolutely continuous with respect to ν , then

$$H(\mu|\nu) = \int_{\Omega_N} f \log f d\nu, \quad \text{where} \quad f = \frac{d\mu}{d\nu}$$
(3.13)

and otherwise, $H(\mu|\nu) = +\infty$. The entropy inequality in [13, Page 338] will be of importance.

One can see that the proof of the next lemma follows the same reasoning as [11, Lemma 4.6]. We present it for the convenience of the reader, and for future reference of some steps.

Lemma 3.8. For each $x \in \mathbb{T}_N$ and $s \in [0,T]$, let $\varphi_x(\cdot, s) \mapsto \varphi(\eta, s)$ be some map independent of the occupation value at the sites $[\![0,r]\!]$, and uniformly bounded from above. For any $t \in [0,T]$, it holds that

$$\mathbb{E}_{\mu_N} \left[\left| \int_0^t \frac{1}{N} \sum_{x \in \mathbb{T}_N} \varphi_x(\eta_{N^2}, s) (\eta_{N^2}(x+r) - \eta_{N^2}(x)) \mathrm{d}s \right| \right] \lesssim T \sqrt{\frac{r}{\Theta_{\mathfrak{p}}(N)}}.$$

Proof. The standard procedure relies on considering a reference measure "close" enough to the invariant measure, applying the entropy inequality, using the fact that $e^{|x|} \leq e^x + e^{-x}$, and then applying Feynmann Kac's formula as in [1, Page 14]. This shows that it is enough to estimate

$$\frac{H(\mu_N|\nu_{\alpha}^N)}{NA} + \sup_{f \text{ density}} \left\{ \int_0^t \left| \int_{\Omega_N} \frac{1}{N} \sum_{x \in \mathbb{T}_N} \varphi_x(\eta, s)(\eta(x+r) - \eta(x))f(\eta) \mathrm{d}\nu_{\alpha}^N(\eta) \right| \mathrm{d}s - \frac{N^2}{2NA} \mathfrak{D}_{N,\mathfrak{p}}(\sqrt{f}|\nu_{\alpha}^N) \right\}, \quad (3.14)$$

where we used also the observation (3.11).

Let us consider the following sequence of transformations, representing the transport of a particle or hole from the site 0 to the site r > 0, and from the site r > 0 to the site 0:

$$\begin{cases} \gamma_{i+1}^{0,r} = \theta_{i,i+1} \gamma_i^{0,r}, & 0 \le i \le r-1, \\ \gamma_0^{0,r} := \mathbf{1}, & \text{and} & \begin{cases} \gamma_{i+1}^{r,0} = \theta_{r-i,r-i-1} \gamma_i^{r,0}, & 0 \le i \le r-1, \\ \gamma_0^{r,0} := \mathbf{1}. & \end{cases} \end{cases}$$

where we recall that $\theta_{x,y} : \eta \mapsto \theta_{x,y}\eta$ corresponds to the operator that exchanges the occupation value between the sites x and y. One can then decompose

$$\nabla_{0,r} = \sum_{i=0}^{r-1} \nabla_{i,i+1} \gamma_i^{0,r} + \sum_{i=0}^{r-1} \nabla_{r-i,r-i-1} \gamma_i^{r,0} \gamma_r^{0,r}.$$
(3.15)

Because $\nabla_{x,x+r} = \tau^x \nabla_{0,r} \tau^{-x} f$, from (3.12), the first equality in (3.11), Young's inequality, f being a density with respect to ν_{α}^N , and ν_{α}^N being translation invariant, the integral term in (3.14) is \lesssim than

$$\begin{split} \frac{1}{N} \sum_{x \in \mathbb{T}_N} \int_{\Omega_N} |\tau^x \nabla_{0,r} \tau^{-x} f| \mathrm{d}\nu_\alpha^N \\ \lesssim \frac{1}{N} \sum_{i=0}^{r-1} \sum_{x \in \mathbb{T}_N} \int_{\Omega_N} \tau^x \Big\{ |\nabla_{i,i+1} \gamma_i^{0,r} \tau^{-x} f| + |\nabla_{r-i,r-i-1} \gamma_i^{r,0} \gamma_r^{0,r} \tau^{-x} f| \Big\} \mathrm{d}\nu_\alpha^N \\ \lesssim \frac{r}{NA_0} \mathcal{D}_N^{0,0} (\sqrt{f} |\nu_\alpha^N) + A_0 r. \end{split}$$

From the inequality

$$\mathcal{D}_{N}^{0,0}(\sqrt{f}|\nu_{\alpha}^{N}) \leq N^{\mathfrak{p}}\mathfrak{D}_{N,\mathfrak{p}}(\sqrt{f}|\nu_{\alpha}^{N}), \qquad (3.16)$$

we see that (3.14) is bounded from above by some constant times

$$\frac{H(\mu_N|\nu_{\alpha}^N)}{NA} + \left(\frac{N^{\mathfrak{p}}rT}{NA_0} - \frac{N}{A}\right)\mathfrak{D}_{N,\mathfrak{p}}(\sqrt{f}|\nu_{\alpha}^N) + A_0rT.$$

Fixing $A_0 = c_0 ArT N^{p-2}$ and $A = T^{-1} (r/\Theta_{\mathfrak{p}}(N))^{-1/2}$, in order to finish the proof it is enough to crudely bound from above $H(\mu_N | \nu_{\alpha}^N) \leq N$, which can be achieved from (3.13) with the fact that $\alpha \in (0, 1)$.

We now provide a one-block estimate.

Lemma 3.9. For each $x \in \mathbb{T}_N$, $t \in [0,T]$, and $l \geq 1$, let $\varphi_x(\cdot,t) \mapsto \varphi(\eta,t)$ be some map independent of the occupation value of the sites x + [[jl, jl + l]], and uniformly bounded from above. For any $t \in [0,T]$ and $j,l \geq 1$ fixed, it holds that

$$\mathbb{E}_{\mu_N}\left[\left|\int_0^t \frac{1}{N} \sum_{x \in \mathbb{T}_N} \varphi_x(\eta_{N^2}, s) \left(\langle \tau^{x+jl} \eta_{N^2 s} \rangle_l - \eta_{N^2 s}(x+jl)\right) \mathrm{d}s\right|\right] \lesssim T \sqrt{\frac{l^2}{\Theta_{\mathfrak{p}}(N)}}$$

Proof. Expressing

$$\langle \tau^{jl}\eta\rangle_l - \eta(jl) = \frac{1}{l} \sum_{y \in B_l(0)} \left(\eta(jl+y) - \eta(jl) \right),$$

it is simple to obtain the target estimate following the proof of the previous lemma.

We conclude this subsection by providing the following two-blocks estimate.

Lemma 3.10. For each $x \in \mathbb{T}_N$, $t \in [0,T]$, and $l \geq 1$, let $\varphi_x(\cdot,t) \mapsto \varphi(\eta,t)$ be some map independent of the occupation value of the sites $x + [[jl, j \in N + \epsilon N]]$, and uniformly bounded from above. Then, it holds that

$$\mathbb{E}_{\mu_N}\left[\left|\int_0^t \frac{1}{N} \sum_{x \in \mathbb{T}_N} \varphi_x(\eta_{N^2 s}) \left(\langle \tau^{x+j\epsilon N} \eta_{N^2 s} \rangle_{\epsilon N} - \langle \tau^{x+jl} \eta_{N^2 s} \rangle_l\right) \mathrm{d}s\right|\right] \lesssim T \sqrt{(j\epsilon)^2 (1 + \frac{N^{\mathfrak{p}}}{l^2}) + \frac{l^2}{\Theta_{\mathfrak{p}}}}.$$

Before proceeding with the proof, let us make some comments regarding our approach in the present manuscript. To the best of our knowledge, a two-blocks estimate in order to treat non-linear terms as in (3.8) was derived in [2, 4, 11, 5]. A fundamental step in the proof of this estimate is a mass-transportation argument, often referred to as a moving particle lemma, that allows for a comparison between $\sum_x \int_{\Omega_N} |\nabla_{y(x),x}f| d\nu$ and the energy term $\mathfrak{D}_N(g|\nu)$, where ν is a reference measure and f a density with respect to it. This comparison is achieved by defining a path in Ω_N that exchanges the occupation values of the sites y(x) and x through a sequence of jumps occurring with positive rate, with respect to the underlying dynamics. This is equivalent to decomposing $\nabla_{y(x),x} = \sum_{i=0}^{L} \nabla_{x_i,x_{i+1}}$, where $(x_i, x_{i+1})_{i=0,\dots,L}$ corresponds to the ordered sequence of bonds where the occupation values are exchanged – in this way characterizing a path in Ω_N , of some length L.

Because the SSEP perturbation, of order $N^{-\mathfrak{p}}$, is invisible in the macroscopic scale, the exchange cannot be performed solely through the SSEP dynamics if y(x) and x are too far away. Indeed, from Lemma 3.8, it should be at most that $|y(x) - x| = o(\Theta_{\mathfrak{p}}(N))$.

In [2], where the perturbed PMM(2) was analysed, the strategy was then to take advantage of the existence of mobile clusters and use the perturbation to construct one, in this manner reducing the state-space. Then, these clusters are used to perform the exchange. Coincidentally, one must be able to argue that there are enough particles in some discrete box of length of at most $o(\Theta_{\mathfrak{p}}(N))$, in order to bring the particles together and form a mobile cluster. This argument was then adapted in [11, 4, 5], where a linear combination of PMM and a long-range PMM in general dimension were analysed. In order to perform the cut-off of small densities, the argument in these works starts by decomposing the box $B_{\epsilon N}$ located at $\tau^{j\epsilon N}$ into the smaller boxes $\{il + B_l\}_{i=0,...,\epsilon N/l}$, effectively rewriting $\langle \tau^{il}\eta \rangle_l - \langle \tau^{jl}\eta \rangle_l$, for each *i* as previously. Then, it is considered the event where there are at least the needed number of particles in the box $il + B_l$ or $jl + B_l$, and it is noted that in the complementary event the quantity $\langle \tau^{il}\eta \rangle_l - \langle \tau^{jl}\eta \rangle_l$ is very small. This is the step where the argument fails in our case.

Because our mobile clusters are composed of a mix of particles and vacant sites, we must condition on having at least n particles and k holes. In this way, it is possible that there are at least n particles in one box and at least k holes in the other. Since these boxes can be far away from each other, the SSEP dynamics cannot be used to bring the holes and particles together, so a mobile cluster cannot be straightforwardly constructed. For the same reason, these boxes cannot be brought closer to each other, as this yields a very poor estimate.

Our approach relies on a first reorganization of the boxes, as in (3.18) below, and then on the ability to increase the box $B_l \mapsto B_{\epsilon N}$ iteratively, on each step by an extra length l. This leads us to be able to condition on the existence of the required number of particles and holes in the "extra" part of length l (see (3.20) and the discussion just after it). It turns out that this argument still does not provide the desired estimate, and it must be complemented with another mass-transportation argument, where a path is considered, exchanging the position of the particles and holes in the aforementioned box, as in (3.21). It is relevant to note that our proof holds for n = 0 < k and k = 0 < n, in this way also being valid for the PMM.

Proof of Lemma 3.10. Following the same reasoning as in the previous two lemmas, and bounding from above $H(\mu_N | \nu_{\alpha}^N) \lesssim N$, one needs to estimate

$$\frac{1}{A} + \sup_{f \text{ density}} \left\{ \int_{0}^{t} \left| \int_{\Omega_{N}} \frac{1}{N} \sum_{x \in \mathbb{T}_{N}} \varphi_{x}(\eta, s) \left(\langle \tau^{x+j\epsilon N} \eta \rangle_{\epsilon N} - \langle \tau^{x+jl} \eta \rangle_{l} \right) f(\eta) \mathrm{d}\nu_{\alpha}^{N}(\eta) \right| \mathrm{d}s - \frac{N^{2}}{2NA} \mathfrak{D}_{N,\mathfrak{p}}(\sqrt{f} | \nu_{\alpha}^{N}) \right\}. \quad (3.17)$$

We split

$$\langle \tau^{j\epsilon N} \eta \rangle_{\epsilon N} - \langle \tau^{jl} \eta \rangle_l = \left(\langle \tau^{j\epsilon N} \eta \rangle_{\epsilon N} - \langle \tau^{j\epsilon N} \eta \rangle_l \right) + \left(\langle \tau^{j\epsilon N} \eta \rangle_l - \langle \tau^{jl} \eta \rangle_l \right), \tag{3.18}$$

and analyse each of the terms in the right-hand side above. Let us focus on the first term. In what follows, let $L := \epsilon N$ and suppose that L is a multiple of l. For a clearer presentation, we present the arguments without the translation $\tau^{j\epsilon N}$. We focus on

$$\langle \eta \rangle_L - \langle \eta \rangle_l. \tag{3.19}$$

Let us consider, for each $x \in \mathbb{T}_N$, the set

$$\Omega^{n,k}(x) := \left\{ \eta \in \Omega_N : \exists i \in \llbracket 1, L/l \rrbracket : \left(\langle \tau^{x+il} \eta \rangle_l \ge \frac{n+k+1}{l} \text{ or } \langle \tau^{x+(i-1)l} \eta \rangle_l \ge \frac{n+k+1}{l} \right) \\ \text{and} \quad \left(\langle \tau^{x+il} \overline{\eta} \rangle_l \ge \frac{n+k+1}{l} \text{ or } \langle \tau^{x+(i-1)l} \overline{\eta} \rangle_l \ge \frac{n+k+1}{l} \right) \right\}.$$

If L is not a multiple of l, one replaces above L/l by |L/l| and obtain an extra term of the order of l^{-1} in the next display, (3.20). In order to simplify the presentation, we assume from now on that L/l is an integer. It is important to see that for events not in the set $\Omega^{n,k}(0)$, the random variable $\langle \eta \rangle_L - \langle \eta \rangle_l$ takes very small values. In order to see this, we now express alternatively,

$$\begin{split} \langle \eta \rangle_L - \langle \eta \rangle_l &= \sum_{i=1}^{L/l} \left\{ \langle \eta \rangle_{il+l} - \langle \eta \rangle_{il} \right\} \\ &= \sum_{i=1}^{L/l} \frac{1}{i+1} \left\{ \langle \tau^{il} \eta \rangle_l - \langle \eta \rangle_{il} \right\} \\ &= \sum_{i=1}^{L/l} \frac{1}{(i+1)i} \sum_{m=0}^{i-1} \left\{ \langle \tau^{il} \eta \rangle_l - \langle \tau^{ml} \eta \rangle_l \right\} \\ &= \sum_{i=1}^{L/l} \frac{1}{(i+1)i} \sum_{m=0}^{i-1} \sum_{q=m}^{i-1} \left\{ \langle \tau^{ql+l} \eta \rangle_l - \langle \tau^{ql} \eta \rangle_l \right\}. \end{split}$$
(3.20)

The factor (i+1)i above will be important in what follows. The second equality comes from

expressing $\langle \eta \rangle_{il+l} = \frac{i}{i+1} \langle \eta \rangle_{il} + \frac{1}{i+1} \langle \tau^{il} \eta \rangle_l$, while the third from $\langle \eta \rangle_{il} = \frac{1}{i} \sum_{m=0}^{i-1} \langle \tau^{ml} \eta \rangle_l$. Because for any pair of sites x_0, x_1 we have $\eta(x_0) - \eta(x_1) = -(\overline{\eta}(x_0) - \overline{\eta}(x_1))$, for $\xi \notin \Omega^{n,k}(0)$ it holds, for every $i \leq L/l$, that $\langle \tau^{il+l} \xi \rangle_l - \langle \tau^{il} \xi \rangle_l \in [-\frac{n+k+1}{l}, \frac{n+k+1}{l}]$. This can be used to

estimate the absolute value of the l.h.s. of (3.20) for configurations not in $\Omega^{n,k}(0)$, leading to an upper bound of the order of L/l^2 . However, l cannot be large enough such that $L/l^2 \to 0$ as $N \to +\infty$. This, together with the fact that $\langle \tau^{il+l}\xi \rangle_l - \langle \tau^{il}\xi \rangle_l$ is very small, motivates the next reasoning instead.

Fixed x, l, i, m, q and $\xi \notin \Omega^{n,k}(x + j\epsilon N)$, we need to analyse both the low and high density cases. We start by the former, that is, $\langle \tau^{x+j\epsilon N+ql}\eta \rangle_l \leq (n+k+1)l^{-1}$ and $\langle \tau^{x+j\epsilon N+ql+l}\eta \rangle_l \leq (n+k+1)l^{-1}$. We will define a random path. Let us denote by P_0 and P_1 the sets composed by all the occupied sites in the ball $B^0 := x + j\epsilon N + ql + B_l$ and $B^1 := x + j\epsilon N + (q+1)l + B_l$, respectively; and by H_0 and H_1 the sets composed by all the empty sites in the ball B^0 and B^1 . For l large enough, we pair each particle in P_0 with an arbitrary hole in H_1 ; and each particle in P_1 with an arbitrary hole in H_0 .

We can then bound from above

$$\frac{1}{N} \sum_{x \in \mathbb{T}_{N}} \left| \int_{(\Omega^{n,k}(x+j\epsilon N))^{c}} \varphi_{x}(\eta,s) \left(\langle \tau^{x+j\epsilon N+ql+l} \eta \rangle_{l} - \langle \tau^{x+j\epsilon N+ql} \eta \rangle_{l} \right) f d\nu_{\alpha}^{N} \right| \\
\lesssim \frac{1}{lN} \sum_{x \in \mathbb{T}_{N}} \int_{(\Omega^{n,k}(x+j\epsilon N))^{c}} \frac{1}{|P_{0}| \binom{|H_{1}|}{|P_{0}|}} \sum_{H \in \mathcal{P}_{|P_{0}|}(H_{1})} \sum_{(p,h) \in P_{0} \times H} |\nabla_{p,h} f| d\nu_{\alpha}^{N} \\
+ \frac{1}{lN} \sum_{x \in \mathbb{T}_{N}} \int_{(\Omega^{n,k}(x+j\epsilon N))^{c}} \frac{1}{|P_{1}| \binom{|H_{0}|}{|P_{1}|}} \sum_{H \in \mathcal{P}_{|P_{1}|}(H_{0})} \sum_{(p,h) \in P_{1} \times H} |\nabla_{p,h} f| d\nu_{\alpha}^{N}, \quad (3.21)$$

where we recall that for a a natural number and A a set, $\mathcal{P}_a(A)$ corresponds to all subsets of A with a elements. Moreover, we note that P_0, P_1, H_0 and H_1 depend on $x, j \in N$ and η . Now we decompose $\nabla_{p,h}$ through a sequence of transformations expressing the exchange of the occupation value at the sites p and h, as in (3.15).

With this argument, the right-hand side in the previous display is bounded from above by some constant independent of l and L, times

$$\frac{1}{lN}\frac{l}{A_0}\mathcal{D}_N^{(0,0)}(\sqrt{f}|\nu_{\alpha}^N) + \frac{1}{l}A_0l.$$

In particular, from (3.20) we obtain for any $y \in \mathbb{T}_N$ the estimate

$$\left|\frac{1}{N}\sum_{x\in\mathbb{T}_{\mathbb{N}}}\int_{(\Omega^{n,k}(x+y))^{c}}\varphi_{x}(\eta,s)\big(\langle\tau^{x+y}\eta\rangle_{L}-\langle\tau^{x+y}\eta\rangle_{l}\big)f\mathrm{d}\nu_{\alpha}^{N}\right|\lesssim\frac{L}{lNA_{0}}\mathcal{D}_{N}^{0,0}(\sqrt{f}|\nu_{\alpha}^{N})+A_{0}\frac{L}{l}.$$

The above concludes the low-density case. The high-density, that is, both $\langle \tau^{x+j\epsilon N+ql}\overline{\eta}\rangle_l \leq (n+k+1)l^{-1}$ and $\langle \tau^{x+j\epsilon N+ql+l}\overline{\eta}\rangle_l \leq (n+k+1)l^{-1}$, can be studied analogously. The difference being that one now pairs, whenever they exist, holes in one box, with arbitrary particles in the other box. This leads to an upper bound of the same order as the one in the previous display.

Noting that (3.19) still needs to be analysed for $\eta \in \Omega^{n,k}(0)$, let us now postpone this study and focus on the second term in (3.18). We split

$$\langle \tau^{jL}\eta\rangle_l - \langle \tau^{jl}\eta\rangle_l = \left(\langle \tau^{jL}\eta\rangle_l - \langle \tau^{jl}\eta\rangle_{jL-jl+l}\right) + \left(\langle \tau^{jl}\eta\rangle_{jL-jl+l} - \langle \tau^{jl}\eta\rangle_l\right).$$

The second difference in the previous display can be treated in the same way as the one in (3.19), since we aim to enlarge the box $B_l \mapsto B_{jL-jl+l}$. We focus on the first term, $\langle \tau^{jL} \eta \rangle_l - \langle \tau^{jl} \eta \rangle_{jL-jl+l}$, that can be analysed with a similar reasoning. Denoting $B_{-p} = [\![-p, 0]\!]$

for $p \ge 1$, we see that

$$\langle \tau^{jL} \eta \rangle_l - \langle \tau^{jl} \eta \rangle_{jL-jl+l} = \langle \tau^{jL+l} \eta \rangle_{-l} - \langle \tau^{jL+l} \rangle_{-(jL-jl+l)}$$
(3.22)

and, by a reflection symmetry of the problem, this can be studied just like (3.19). All that is left to do is to analyse (3.19). We can do this by applying the path argument developed in [2, Lemma 5.7], there for the case of n = 2 and k = 0, but, as we shall see, can be straightforwardly adapted to our case. We present now the arguments.

Recall from Lemma 2.3 that a box of length n+k+2 with exactly n+1 particles constitutes a mobile cluster. A mobile cluster can transport mass throughout the lattice, as explained just before Figure 5, allowing for the exchange of the occupation values at any pair of sites xand y. Concretely, if a mobile cluster is in the vicinity of some site x, it can incorporate the occupation value at x into a larger cluster, then move to some arbitrary site x + r, exchange the particle/hole that was at the site x with the occupation at x + r, then move back to the site x and replace its occupation with the one that was previously in x + r.

Let then $\eta \in \Omega^{n,k}(0)$, and express further

$$\langle \eta \rangle_L - \langle \eta \rangle_l = \frac{1}{L} \sum_{i=1}^{L/l} \sum_{y \in B_l} \left\{ \eta(il+y) - \eta(y) \right\},$$

and fix $1 \leq i \leq L/l$, $y \in B_l$ and $\eta \in \Omega^{n,k}(0)$. Because $\eta \in \Omega^{n,k}(0)$, there is some site $1 \leq i^* \leq L/l-l$ such that the box i^*+B_{2l} contains at least n+k+1 particles and holes. Let us denote the set of all the configurations with at least one mobile cluster in the aforementioned box, by Ω^* . Let then $\gamma^* := (\gamma_i^*)_{i=0,\ldots,l^*}$ be a path of length $l^* \leq (n+k+1)l$ defined inductively by $\gamma_i^* = \theta_{x_i^*, x_i^*+1}\gamma_{i-1}^*$, with $\gamma_0^*\eta := \eta$ and $\gamma_{l^*}^*\eta =: \eta^* \in \Omega^*$, where $\{x_i^*, x_i^*+1\}_{i=0,l^*}$ is the sequence of nodes characterizing the exchange of the occupation value between the sites x_i and $x_i + 1$.

Starting from η^* , we proceed similarly and consider another path, $\gamma' := (\gamma'_i)_{i=0,\dots,L'}$, of length L', describing the rationale previously discussed in order exchange the occupation value of η^* at the sites il + y and y with B(n, k) jumps only, by taking advantage of the presence of mobile clusters. Concretely, $\gamma'_0\eta^* = \eta^*$ and $\gamma'_{L'}\eta = \theta_{il+y,y}\eta^*$, with $\gamma'_i = \theta_{x'_i,x'_i+1}\gamma'_{i-1}$ for $1 \leq i \leq L' \leq L$. After the exchange, the cluster is then transported back to its original position. This procedure yields the configuration $\theta_{y,y+il}\eta^*$, being described precisely by a sequence of bonds, $\{x'_i, x'_i+1\}_{i=0,\dots,L'}$, where the exchange of occupations are to be performed.

Lastly, starting from $\theta_{y,y+il}\eta^*$, we define a path arriving at $\theta_{y,y+il}\eta$ by considering γ^* in reverse order, yielding

$$\theta_{y,il+y}f(\eta^{\star}) - \theta_{y,il+y}f(\eta) = -\sum_{i=1}^{l^{\star}} \nabla_{x_{L-i'},x_{L-i'}+1}\gamma_{L-i}^{\star}(\eta^{\star}), \quad \text{with} \quad l^{\star} \lesssim L.$$

Repeating these arguments for all $x \in \mathbb{T}_N$ and proceeding as in Lemma 3.8, the integral term in (3.17) is bounded from above by a constant times

$$\begin{split} &\frac{1}{LN} \sum_{i=1}^{L/l} \sum_{y \in B_l} \sum_{x \in \mathbb{T}_N} \int_{\Omega^{n,k}(x+j\epsilon N)} \frac{1}{A^{\star}} \sum_{i=0}^{l^{\star}} \left| \nabla_{x_i^{\star}, x_i^{\star}+1} f(\gamma_i^{\star} \eta) \right|^2 \mathrm{d}\nu_{\alpha}^N + A^{\star}l \\ &+ \frac{1}{LN} \sum_{i=1}^{L/l} \sum_{y \in B_l} \sum_{x \in \mathbb{T}_N} \int_{\Omega^{n,k}(x+j\epsilon N)} \frac{1}{A'} \sum_{i=0}^{L'} \tau^{x_i} \mathbf{c}^{(n,k)}(\gamma_i \eta) \left| \nabla_{x_i', x_i'+1} f(\gamma_i' \eta^{\star}) \right|^2 \mathrm{d}\nu_{\alpha}^N + A'L \\ &+ \frac{1}{LN} \sum_{i=1}^{L/l} \sum_{y \in B_l} \sum_{x \in \mathbb{T}_N} \int_{\Omega^{n,k}(x+j\epsilon N)} \sum_{i=0}^{l^{\star}} \frac{1}{A^{\star}} \left| \nabla_{x_{L-i}^{\star}, x_{L-i}^{\star}+1} f(\gamma_{L-i}^{\star} \eta^{\star}) \right|^2 \mathrm{d}\nu_{\alpha}^N + A^{\star}l \\ &\lesssim \frac{l}{A^{\star}N} \mathcal{D}_N^{0,0}(\sqrt{f} | \nu_{\alpha}^N) + \frac{L}{A'} \mathcal{D}_N^{n,k}(\sqrt{f} | \nu_{\alpha}^N) + A^{\star}l + A'L, \end{split}$$

where we remark that $x_i^{\star}, x_i', l^{\star}$ and L' depend on $x + j\epsilon N$ and $\eta \in \Omega^{n,k}(x + j\epsilon N)$. Repeating all of these arguments in order to treat (3.22), yields an analogous upper bound for the corresponding term, with L in the previous display replaced by jL - jl + l. With this, and applying (3.16) and $jL - jl + l \leq jL$, one obtains that (3.18) is no larger than some constant times

$$\frac{1}{A} + T \Big[A_0 \frac{jL}{l} + A^* l + A' jL \Big] + \left(\frac{jLN^{\mathfrak{p}}T}{lNA_0} + \frac{lN^{\mathfrak{p}}T}{A^*N} + \frac{jLT}{A'N} - \frac{N^2}{2NA} \right) \mathfrak{D}_{N,\mathfrak{p}}(\sqrt{f}|\nu_{\alpha}^N).$$

Fixing

$$\frac{jLT}{lNA_0} = \frac{\Theta_{\mathfrak{p}}(N)}{6NA}, \quad \frac{lT}{A^*N} = \frac{\Theta_{\mathfrak{p}}(N)}{6NA} \quad \text{and} \quad \frac{jLT}{A'N} = \frac{N^2}{6NA},$$

and recalling that $L = \epsilon N$, one then fixes $A = T^{-1}((j\epsilon)^2(1 + N^{\mathfrak{p}}l^{-2}) + l^2\Theta_p^{-1})^{-\frac{1}{2}}$ in order to obtain the upper bound in the statement of the current lemma.

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APPENDIX A. DESCRIPTION OF THE EQUIVALENCE CLASSES

Fixed $n, k \ge 0$, the forthcoming Lemma A.1 will be invoked in Lemmas A.3 and A.4, as it allows us to identify all the sets equal to either $P_{ij}^{n+\ell} \sqcup \{j+1\}$ or $P_{ij}^{n+\ell} \sqcup \{j\}$, for any fixed $0 \le \ell \le k$ and $(i, j) \in I_{\ell} \times J$, by analysing their particular structure. This identification, in turn, facilitates the study of the numbers $\{p_{ij}^{\ell,x}\}_{(i,j)\in I_\ell\times J, x=0,1}$ (see Lemma A.4) and a better understanding of the classes in $C_{\ell,x}$, for x = 0, 1.

Lemma A.1. Fixed $0 \le \ell \le k$ and $(i, j) \in I_{\ell} \times J$, for each integer $0 \le w \le n + k$ such that $w \notin P_{ij}^{n+\ell} \sqcup \{j+1\} \ni w+1 \text{ there exists } i' \in I_\ell \text{ such that } P_{i'w}^{n+\ell} \sqcup \{w+1\} = P_{ij}^{n+\ell} \sqcup \{j+1\}.$

Analogously, for each integer $0 \le w \le n+k$ such that $w+1 \notin P_{ij}^{n+\ell} \sqcup \{j+1\} \ni w$ there exists $i' \in I_{\ell}$ such that it holds $P_{i'w}^{n+\ell} \sqcup \{w\} = P_{ij}^{n+\ell} \sqcup \{j\}.$

Proof. Let $0 \le w \le n+k$ be such that $w \notin P_{ij}^{n+\ell} \sqcup \{j+1\} \ni w+1$. It is enough to note that because

$$P_{ij}^{n+\ell} \sqcup \{j+1\} = \left[(P_{ij}^{n+\ell} \sqcup \{j+1\}) \setminus \{w+1\} \right] \sqcup \{w+1\}$$

and since $(P_{ij}^{n+\ell} \sqcup \{j+1\}) \setminus \{w+1\} \in \mathcal{P}_{n+\ell}(M_w)$, there is a unique $i' \in I_\ell$ such that

$$(P_{ij}^{n+\ell} \sqcup \{j+1\}) \setminus \{w+1\} = P_{i'w}^{n+\ell}.$$

To see the second assertion in the statement the proof is analogous.

The goal now is to characterize the sets associated with a fixed equivalence class of the indexes.

Definition A.2. For each $j \in J$, $0 \le \ell \le k$ fixed and x = 0, 1 introduce the families of sets $\mathcal{P}^{\ell,x} = \{\mathcal{P}^{\ell,x}_j\}_{j\in J}$, where $\mathcal{P}^{\ell,x}_j := \{\{j+x\} \sqcup A : A \in \mathcal{P}_{n+\ell}(M \setminus \{j, j+1\})\}$.

Note that if $P \in \mathcal{P}_j^{\ell,x}$ for some j, ℓ, x as in the previous definition, then there exists $i \in I_\ell$ such that $P = P_{ij}^{n+\ell} \sqcup \{j+x\}$. Moreover, note that $\{0, \ldots, n+\ell\} \notin \mathcal{P}^{\ell,1}$ and $\{k+1-\ell, \ldots, n+\ell\}$ $k+1\} \notin \mathcal{P}^{\ell,0}.$

We are going to show that fixed some ℓ and given some set $P_{ij}^{n+\ell}$ with index $(i, j) \in \underline{c_x} \in \mathcal{C}_{\ell,x}$ with x = 0 or x = 1, if $P_{ij}^{n+\ell} \sqcup \{j + x\} \not\equiv \{0, \dots, n+\ell\}$ then all the sets corresponding to translations of $P_{ij}^{n+\ell} \sqcup \{j+1\}$ inside M are of the form $P_{i'j'}^{n+\ell} \sqcup \{j'+x\}$ where the index (i', j')is in the same class $\underline{c_x}$; while if $P_{ij}^{n+\ell} \sqcup \{j+x\} \equiv \{0, \ldots, n+\ell\}$ then one has all the translations except a particular one for x = 0 and another one for x = 1. This will be important for the proof of Lemmas A.4 and A.5.

Lemma A.3. Let $0 \le \ell \le k$ and $x \in \{0,1\}$ be fixed, and consider some class $\underline{c_x} \in \mathcal{C}_{\ell,x}$. (1) If $P_{ij}^{n+\ell} \sqcup \{j+x\} \equiv \{0, \ldots, n+\ell\}$ then for each $z \in \mathbb{Z} \setminus \{0\}$ such that

$$z + (P_{ij}^{n+\ell} \sqcup \{j+x\}) \subset M$$
 and $z \neq \begin{cases} -\min(P), & x = 1, \\ n+k+1 - \max(P), & x = 0, \end{cases}$

there exists $(i', j') \in \underline{c_x}$ such that $z + P_{ij}^{n+\ell} = P_{i'j'}^{n+\ell}$. (2) If $P_{ij}^{n+\ell} \sqcup \{j+x\} \not\equiv \{0, \dots, n+\ell\}$ then for each $z \in \mathbb{Z} \setminus \{0\}$ such that $z + (P_{ij}^{n+\ell} \sqcup \{j+1\})$ $x\}) \subset M$, there exists $(i', j') \in c_x$ such that $z + P_{ij}^{n+\ell} = P_{i'j'}^{n+\ell}$.

Proof. The proof is split in several cases. Let x = 1 and take some set $P \in \mathcal{P}^{\ell,1}$. Assume first that P is not a translation of $\{0, \ldots, n+\ell\}$. There is some $j \in J$ such that $P \in \mathcal{P}_i^1 =$ $\{j+1\} \sqcup \mathcal{P}_{n+\ell}(M \setminus \{j\})$. Let z be as in the statement and note that $z+P \in \{z+j+1\} \sqcup \mathcal{P}_{n+\ell}(z+M \setminus \{j\})$. Then there is some set $Q^1 \in \mathcal{P}_{n+\ell}(z+M \setminus \{j\})$ with the property that $\{z+j+1\} \sqcup Q^1 \subseteq M$. Therefore, we want to show that there exists some $j' \in J$ such that $Q_z^1 := \{z+j+1\} \sqcup Q^1 \in \mathcal{P}_{j'}^{\ell,1}$. From Lemma A.1 such j' exists as long as there is some $q \ge 0$ such that $q \notin Q_z^1 \ni q + 1 \leq n + k + 1$, and in that case it is then enough to fix j' = q. Since $P \in \mathcal{P}^1$ and $P \neq \{0, \ldots, n+\ell\}$, there exists $0 \leq p \notin P \ni p+1 \neq \min(P)$ and as such we can fix q = z + p = j'.

If P is a translation of $\{0, \ldots, n+\ell\}$, then the previous argument holds for all z such that $z + P \subset M$ except for $z = -\min(P)$, since $-\min(P) + P = \{0, \dots, n+\ell\} \notin \mathcal{P}^{\ell,1}$.

For x = 0 the argument is identical and as such we provide only the main steps. If P is not a translation of $\{k+1-\ell,\ldots,n+k+1\}$, letting z be as in the statement, one needs to argue that there is some $j' \in J$ such that $Q_z^0 := \{z+j\} \sqcup Q^0 \in \mathcal{P}_{j'}^{\ell,0}$, where $Q^0 \in \mathcal{P}_{n+\ell}(z+M \setminus \{j+1\})$. It is then enough to show that there is some $q+1 \notin Q_z^0 \ni q$ with $q+1 \leq n+k+1 \neq \max(P)$, which is done as previously, and invoking Lemma A.1. If P is a translation of $\{k+1-\ell,\ldots,n+k\}$, then the previous argument holds for all z in the statement except for $z = n + k + 1 - \max(P)$, since $(n + k + 1 - \max(P)) + P = \{k + 1 - \ell, \dots, n + k\} \notin \mathcal{P}^{\ell, 0}$.

In order to simplify the quantities arising from the gradient property it will be important to characterize the set $\{p_{ij}^{\ell,x}\}_{(i,j)\in I_{\ell}\times J, x=0,1}$ which will, in turn, allows us to fix the map ϕ_{ℓ} conveniently, with respect to each class. We note that for any i, j, ℓ, x as previously, $p_{ij}^{\ell,x} \in \{0, \ldots, n+k+1-\max(A_{ij}^{\ell,x})\}$, with $A_{ij}^{\ell,x}$ as in Definition 2.5.

Lemma A.4. Let $0 \le \ell \le k$ and $\underline{c} \in \mathcal{C}_{\ell,1}$ be fixed. If $\max(A_c) \ne n + k + 1$ then: (1) For any $p \neq 0$, $n + k + 1 - \max(A_c)$ it holds that

$$\begin{aligned} \left| \{ (i,j) \in \underline{c} : p_{ij}^{\ell,1} = p \} \right| &= \left| \{ (i,j) \in \phi_{\ell}(\underline{c}) : p_{ij}^{\ell,0} = p \} \right|; \\ (2) \left| \{ (i,j) \in \underline{c} : p_{ij}^{\ell,1} = 0 \} \right| &= \left| \{ (i,j) \in \phi_{\ell}(\underline{c}) : p_{ij}^{\ell,0} = 0 \} \right| - 1; \\ (3) \left| \{ (i,j) \in \underline{c} : p_{ij}^{\ell,1} = n + k + 1 - \max(A_{\underline{c}}) \} \right| \\ &= \left| \{ (i,j) \in \phi_{\ell}(\underline{c}) : p_{ij}^{\ell,0} = n + k + 1 - \max(A_{\underline{c}}) \} \right| + 1. \end{aligned}$$

Proof. In order to show (1) we will follow a "diagonal" argument. Let \underline{c} be a fixed class as in the statement of the current lemma. We start by showing that for each $(i, j) \in \underline{c}$ such that $p_{ij}^{\ell,1} \ge 1$ there is some $(i',j') \in \phi_{\ell}(\underline{c})$ such that $p_{ij}^{\ell,1} - 1 = p_{i'j'}^{\ell,0}$; and its converse. Next, we show that for any $1 \le p \le n + k + 1 - \max(A_c) - 1$ it holds that

$$\{(i,j) \in \underline{c} : p_{ij}^{\ell,1} = p\} = \{(i,j) \in \underline{c} : p_{ij}^{\ell,1} = p+1\}.$$
(A.1)

This directly implies the property (1). In particular, it also implies that

$$\left|\{(i,j)\in\underline{c}:p_{ij}^{\ell,1}=0\}\right| = \left|\{(i,j)\in\phi_{\ell}(\underline{c}):p_{ij}^{\ell,0}=n+k+1-\max(A_{\underline{c}})\}\right|,$$

which will then be used to show (2) and (3).

In this way, recall Definition A.2. If $p_{ij}^{\ell,1} + A_{ij}^{n+\ell,1} = P_{ij}^{n+\ell} \sqcup \{j+1\} \in \mathcal{P}_j^1$, then $P' := (p_{ij}^{\ell,1}-1) + A_{ij}^{n+\ell,1} \in \mathcal{P}^0$ since one can take $j' = \max(P')$ and in this way $n+k+1 \ge j'+1 \notin \mathcal{P}_j^0$

 $P' \ni j' \ge 0$ and from Lemma A.1 there is some $i' \in I_{\ell}$ such that $P' = p_{i'j'}^{\ell,0} + A_{i'j'}^{n+\ell,0}$ (note that $(i', j') \in \phi_{\ell}(\underline{c})$). The argument to show the converse: that for each $(i, j) \in \phi_{\ell}(\underline{c})$ there is one $(i', j') \in \underline{c}$ such that $p_{ij}^{\ell,0} + 1 = p_{i'j'}^{\ell,1}$ is identical and so we omit it.

We now aim to show (A.1), which is consequence of Lemma A.3. Fix $1 \le p \le n+k+1 - \max(A_{\underline{c}}^{n+\ell,1}) - 1$. If $p_{ij}^{\ell,1} = p$ then $p + A_{ij}^{n+\ell,1} \in \mathcal{P}_j^1$ and from the aforementioned lemma, $(p+1) + A_{ij}^{n+\ell,1} \in \mathcal{P}_j^1$, because $p+1 \le n+k+1 - \max(A_{\underline{c}}^{n+\ell,1})$. Likewise, if $p_{ij}^{\ell,1} = p+1$ then $(p+1) + A_{ij}^{n+\ell,1} \in \mathcal{P}_j^1$ and $p + A_{ij}^{n+\ell,1} \in \mathcal{P}_j^1$ because $p \ge 1$. This concludes the proof of (A.1). As previously explained, this reduces the proof of (2) to that of

$$\left| \{ (i,j) \in \phi_{\ell}(\underline{c}) : p_{ij}^{\ell,0} = n + k + 1 - \max(A_{\underline{c}}) \} \right| = \left| \{ (i,j) \in \phi_{\ell}(\underline{c}) : p_{ij}^{\ell,0} = 0 \} \right| - 1.$$

This is also consequence of Lemma A.3. Suppose that $p_{ij}^{\ell,0} = n + k + 1 - \max(A_{\underline{c}})$ where $(i, j) \in \phi_{\ell}(\underline{c})$ and \underline{c} is such that $A_{ij}^{n+\ell,1} \equiv \{0, \ldots, n+\ell\}$. Since $p_{ij}^{\ell,0} + A_{ij}^{n+\ell,0} \in \mathcal{P}_j^0$ and all the translations of $p_{ij}^{\ell,0} + A_{ij}^{n+\ell,0}$ inside M correspond to sets whose index is in the same class $\phi_{\ell}(\underline{c})$, for each $(i, j) \in \phi_{\ell}(\underline{c})$ such that $p_{ij}^{\ell,0} = n + k + 1 - \max(A_{\underline{c}})$ there is one $(i', j') \in \phi_{\ell}(\underline{c})$ such that $p_{i'j'}^{\ell,0} = 0$. The only set without this correspondence is the set $(n+k+1-(n+\ell))+\{0,\ldots,n+\ell\}$. To see (3) the argument is also analogous.

To conclude, we are going to characterize the sets associated with each element of $\mathcal{C}_{\ell,1}$.

Lemma A.5. It holds that

$$\{A_{\underline{c}}\}_{\underline{c}\in\mathcal{C}_{\ell,1}} = \{\{0\} \sqcup P : P \in \mathcal{P}_{n+\ell}(M \setminus \{0\})\}.$$
(A.2)

Proof. Clearly the collection on the left-hand side is contained on the collection in the righthand side. To see the converse, it is enough to note that any particular set of the collection on the right-hand side of (A.2) that can be translated (non-trivially) inside M corresponds to a set in $\mathcal{P}^{\ell,1}$ (introduced in Definition A.2), since for any particular $P \in \mathcal{P}_{n+\ell}(M \setminus \{0\})$ with $\max(P) \neq n + k + 1$ it holds that $z - 1 \notin z + (\{0\} \sqcup P) \ni z$ for $z \in \mathbb{N}_+$ and such that $z + (\{0\} \sqcup P) \subseteq M$. In this way, each $\{0\} \sqcup P$ that can be translated inside M can be seen as a shift to the origin of some set in $\mathcal{P}^{\ell,1}$, and as such for each $P \in \mathcal{P}_{n+\ell}(M \setminus \{0\})$ with $\max(P) \neq n + k + 1$ there is some class $\underline{c} \in \mathcal{C}_{\ell,1}$ such that $\{0\} \sqcup P = A_{\underline{c}}^{n+\ell+1}$.

To conclude the proof, we see that if $P' \in \{\{0\} \sqcup P : P \in \mathcal{P}_{n+\ell}(M \setminus \{0\}), \max(P) = n+k+1\}$ then $P' \in \{A_{\underline{c}}\}_{\underline{c} \in \mathcal{C}_{\ell,1}}$ also, since $\exists z \in M$ such that $z \notin \{0\} \sqcup P \ni z+1$ (recall that $0 \leq \ell \leq k$), and as such the proof is done by invoking Lemma A.1. \Box

Appendix B. Porous Media basis

The collection $\{\text{PMM}(n)\}_{n\geq 0}$ cannot be used directly as a basis to model $\rho^n(1-\rho)^k = \sum_{\ell=0}^k (-1)^\ell {k \choose \ell} \rho^{n+\ell}$, leading to possibly negative rates. In this manner, it is interesting to see that the B(n,k) can be seen as performing the aforementioned binomial development under a different basis. This is the content of the next lemma.

Lemma B.1. For each fixed natural numbers $l \leq L < N$, each $0 \leq j \leq L$ and each $\eta \in \Omega_N$, let $m_j^L(\eta) := \sum_{y \in W_j^L} \eta(y)$, where $W_j^L := -j + [0, L+1] \setminus \{j, j+1\}$, and

$$\mathbf{p}_{L}^{l} := \frac{1}{L+1} \sum_{j=0}^{L} \mathbf{p}_{L,j}^{l} \quad with \quad \mathbf{p}_{L,j}^{l}(\eta) := \frac{\binom{m_{J}^{L}(\eta)}{\ell}}{\binom{L}{l} 2^{L-l}} \mathbf{1}_{\{m_{J}^{L}(\eta) \ge l\}}.$$
 (B.1)

The constraint \mathbf{p}_L^l satisfies the gradient condition, and for each $n, k \ge 0$ it holds that

$$\mathbf{b}^{n,k} = \sum_{\ell=0}^{k} (-1)^{\ell} \binom{n+k}{k} \binom{k}{\ell} 2^{L-\ell} \mathbf{p}_{n+k}^{n+\ell}.$$
 (B.2)

Proof. From (2.9) and Lemma 2.7, yielding, $b_{ij}^{\ell} = \binom{n+\ell}{\ell} \binom{n+k}{k}^{-1}$ for every $(i,j) \in I_{\ell} \times J$ we can express

$$\mathbf{b}^{n,k}(\eta) = \frac{\binom{n+\ell}{\ell}}{n+k+1} \sum_{\ell=0}^{k} (-1)^{\ell} \sum_{(i,j)\in I_{\ell}\times J} (\tau^{-j}\eta) (P_{ij}^{n+\ell}).$$

It is now enough to see that for each $0 \le \ell \le k$,

$$\sum_{i \in I_{\ell}} (\tau^{-j} \eta)(P_{ij}^{n+\ell}) = \sum_{P \in \mathcal{P}_{n+\ell}(W_j^{n+k})} \eta(P) = \binom{m_j^L(\eta)}{n+\ell} \mathbf{1}_{\{m_j^L(\eta) \ge n+\ell\}},$$

then use that $\binom{k}{\ell} = \binom{n+\ell}{\ell} \binom{n+k}{n+\ell} \binom{n+k}{k}^{-1}$, yielding the collection of maps $(\mathbf{p}_{n+k}^{n+\ell})_{0 \leq l \leq k}$. The constraint \mathbf{p}_{L}^{ℓ} satisfies the gradient condition according to Corollary 2.14, and the particular function associated to it can be identified from the ones in Proposition 2.19, through the identification of the map $\mathbf{H}_{n+k}^{n+\ell}$ in the decomposition $\mathbf{H}^{n,k} = \sum_{\ell=0}^{k} (-1)^{\ell} \binom{n+k}{k} \binom{k}{\ell} 2^{L-\ell} \mathbf{H}_{n+k}^{n+\ell}$, consequence of (B.2).

Definition B.2. For each fixed natural numbers $\ell \leq L < N$, denote the process induced by the constraint \mathbf{p}_L^{ℓ} by $\mathrm{PMM}_L(\ell)$, and define the collection $\mathrm{PMM}_L := {\mathrm{PMM}_L(\ell)}_{0 \leq \ell \leq L}$. The Markov generator of the $\mathrm{PMM}_L(\ell)$ is denoted by $\mathcal{L}_N^{\ell;L}$.

We conclude this section by providing some properties of the collection PMM_L

Lemma B.3. For each natural number L < N fixed and any $0 \le \ell \le L$, it holds that

- (i) Interpolation: $PMM_L(L) = PMM(L)$
- (ii) The $PMM_L(l)$ enjoys of mobile clusters and blocked configurations;
- (iii) Monotony: the (renormalized) sequence $(2^{L-l}\mathbf{p}_L^l)_{0 \le l \le L}$ is non-increasing;
- (iv) Normalized in Ω_N : $\int_{\Omega_N} \mathbf{p}_L^n(\eta) d\eta = 1$;
- (v) Equilibrium expected value: $\int_{\Omega_N} \mathbf{p}_L^\ell d\nu_\alpha^N = 2^{-(L-\ell)} \alpha^\ell$.

Proof. The property (i) is direct from the definition of the constraint \mathbf{p}_L^l in (B.1); (iii) can be checked directly by decomposing $\mathbf{1}_{\{m_i^L(\eta) \ge l+1\}} = \mathbf{1}_{\{m_i^L(\eta) \ge l\}} - \mathbf{1}_{\{m_i^L(\eta) = l\}}$:

$$2^{L-l}\mathbf{p}_{L}^{l}(\eta) - 2^{L-(l+1)}\mathbf{p}_{L}^{l+1}(\eta) = \frac{1}{L+1} \sum_{j=0}^{L} \frac{\binom{m_{j}^{L}(\eta)}{l}}{\binom{L}{l}} \mathbf{1}_{\{m_{j}^{L}(\eta) \ge l\}} \left(1 - \frac{m_{j}^{L}(\eta)}{L-l}\right) + \frac{1}{L+1} \sum_{j=0}^{L} \frac{\binom{m_{j}^{L}(\eta)}{l+1}}{\binom{L}{l+1}} \mathbf{1}_{\{m_{j}^{L}(\eta)=l\}} \ge 0.$$

To see property (iv), we compute

$$\int_{\Omega_N} \mathbf{p}_L^n(\eta) d\eta = \frac{1}{\binom{L}{l} 2^{L-l}} \sum_{m=l}^{L} \binom{m}{l} \binom{L}{m} = \frac{1}{2^{L-l}} \sum_{m=l}^{L} \binom{L-l}{L-m} = 1.$$

Property (v) can be checked by expressing

$$\binom{m_j^L(\eta)}{n+\ell} \mathbf{1}_{\{m_j^L(\eta) \ge n+\ell\}} = \sum_{P \in \mathcal{P}_{n+\ell}(W_i^{n+k})} \eta(P).$$

It remains to see *(ii)*. It is clear that there are blocked configurations, since for a jump to occur it is required a window of length L with at least $l \leq L$ particles; moreover, regarding mobile clusters, following the same argument as in Lemma 2.3, we see that $\blacksquare_L := \square_L \circ \bullet$ constitutes a mobile cluster, where \square_L represent a cluster of particles composed by a window of length L with at least $l \leq L$ particles, \bullet represent a particle and \circ a hole.

Appendix C. Case n, k = 2

C.1. Linear system characterizing the gradient property. Here we exemplify, for the case n = 2 and k = 2, the approach to derive the linear system characterizing the gradient condition, (2.6). Recall from Definition 2.6 that the kinetic constraints take the form of

$$\mathbf{c}^{n,k}(\eta) = \sum_{j=0}^{n+k} \sum_{i \in I} a_{ij} \eta^{-j+P_{ij}}(-j+M_j),$$
(C.1)

and that for the particular case of n, k = 2 the sets in the previous display are given by

j	$W_j = -j + M_j$
0	$\{2, 3, 4, 5\}$
1	$\{-1, 2, 3, 4\}$
2	$\{-2, -1, 2, 3\}$
3	$\{-3, -2, -1, 2\}$
4	$\{-4, -3, -2, -1\}$

		$-j + P_{ij}$						
i/j	0	1	2	3	4			
1	$\{2,3\}$	$\{-1,2\}$	$\{-2, -1\}$	$\{-3, -2\}$	$\{-4, -3\}$			
2	$\{2,4\}$	$\{-1,3\}$	$\{-2,2\}$	$\{-3, -1\}$	$\{-4, -2\}$			
3	$\{2,5\}$	$\{-1,4\}$	$\{-2,3\}$	$\{-3,2\}$	$\{-4, -1\}$			
4	$\{3,4\}$	$\{2,3\}$	$\{-1,2\}$	$\{-2, -1\}$	$\{-3, -2\}$			
5	$\{3, 5\}$	$\{2,4\}$	$\{-1,3\}$	$\{-2,2\}$	$\{-3, -1\}$			
6	$\{4, 5\}$	$\{3,4\}$	$\{2,3\}$	$\{-1,2\}$	$\{-2, -1\}$			

TABLE 3. Constraints' windows.

TABLE 4. Sets corresponding to the sites with flipped occupation value.

We also find worth recalling Definition 2.5 and Lemma 2.7. With the aid of the two previous tables we can see, for example, that for (i, j) = (1, 0) we have $\eta^{P_{10}}(-0 + M_0) = (1 - \eta(2))(1 - \eta(3))\eta(4)\eta(5)$.

The starting point is to distribute the products of functions of the occupation variables in (C.1). This leads, in general, to

$$\mathbf{c}^{n,k}(\eta) = \frac{1}{\binom{n+k}{k}} \sum_{\ell=0}^{k} \sum_{(i,j)\in I \times J} \sum_{q=1}^{\binom{k}{\ell}} (-1)^{\ell} (\tau^{-j}\eta) ([M_j \setminus P_{ij}] \sqcup Q_{ijq}^{\ell}),$$

with the sets of the form Q_{ijq}^{ℓ} as in Definition 2.5. We will focus on each term of the summation over ℓ separately. The first goal is to identify each set of the form $[M_j \setminus P_{ij}] \sqcup Q_{ijq}^{\ell}$

as an element of $\mathcal{P}_{n+\ell}(M_j)$. In order to do so, recalling the introduction of the map Φ in (2.2), it is enough to consider the sets $M = \Phi_j^{-1}(M_j)$, $P_i = \Phi_j^{-1}(P_{ij})$ and $Q_{iq} = \Phi_j^{-1}(Q_{ijq})$ and to identify $[M \setminus P_i] \sqcup Q_{iq}^{\ell}$ as an element of $\mathcal{P}_{n+\ell}(M)$. This will induce a map ψ_{ℓ} defined through $\psi_{\ell}(i,q) = i' \Leftrightarrow [M \setminus P_i] \sqcup Q_{iq}^{\ell} = P_{i'}^{n+\ell} \in \mathcal{P}_{n+\ell}(M)$. The rationale is the following

$$P_i \to Q_{iq}^{\ell} \in \mathcal{P}_{\ell}(P_i) \to P_{i'}^{n+\ell} = [M \setminus P_i] \sqcup Q_{iq}^{\ell} \in \mathcal{P}_{n+\ell}(M) \to \psi_{\ell}(i,q) = i'.$$
(C.2)

One can then "introduce" the *j*-th window by an application of the map Φ_j . Concretely, by observing that

$$[M \setminus P_i] \sqcup Q_{iq}^{\ell} = P_{i'}^{n+\ell} \in \mathcal{P}_{n+\ell}(M) \Leftrightarrow [M_j \setminus P_{ij}] \sqcup Q_{ijq}^{\ell} = P_{i'j}^{n+\ell} \in \mathcal{P}_{n+\ell}(M_j),$$

where $P_{i'j}^{n+\ell} = \Phi_j(P_i^{n+\ell})$, and as such one can extend the map ψ_ℓ by defining $\psi_{j,\ell}(i,q) = \psi_\ell(i,q) = i'$.

C.1.1. Term $\ell = 0$. Because $\ell = 0$ we have that $\mathcal{P}_{\ell}(P_i) = \{\emptyset\}$ and as consequence every "Q-set" is identified with \emptyset , hence $M \setminus P_i$ is simply the complement of P_i in M. One then needs only to fix some index for the elements of $\mathcal{P}_{n+\ell}(M)$. The rationale in (C.2) is presented in the next figure.

$\begin{tabular}{ c c c c c } \hline $P_1 = \{0,1\}$ & $Q_{1,1} = \{\emptyset\}$ & $P_6^{n+0} = \{2,3\}$ & $\psi_0(1,1) = 6$ \end{tabular}$
$P_2 = \{0,2\} - Q_{2,1} = \{\emptyset\} - P_5^{n+0} = \{1,3\} - \psi_0(2,1) = 5$
$\begin{tabular}{ c c c c c } \hline $P_3 = \{0,3\}$ & $Q_{3,1} = \{\emptyset\}$ & $P_4^{n+0} = \{1,2\}$ & $\psi_0(3,1) = 4$ \end{tabular}$
$P_4 = \{1, 2\} - Q_{4,1} = \{\emptyset\} - P_3^{n+0} = \{0, 3\} - \psi_0(4, 1) = 3$
$P_5 = \{1,3\} - Q_{5,1} = \{\emptyset\} - P_2^{n+0} = \{0,2\} - \psi_0(5,1) = 2$
$P_6 = \{2,3\} - Q_{6,1} = \{\emptyset\} - P_1^{n+0} = \{0,1\} - \psi_0(6,1) = 1$

FIGURE 6. $\ell = 0$: Construction of the map ψ_0 .

Identifying the map $\psi_{j,\ell}$ leads to the sets in the next figure.

	$-j + P_{ij}^{n+0}$						
$i \backslash j$	0	1	2	3	4		
1	$\{2,3\}$	$\{-1,2\}$	$\{-2, -1, \}$	$\{-3, -2, \}$	$\{-4, -3, \}$		
2	$\{2,4\}$	$\{-1,3\}$	$\{-2,2\}$	$\{-3, -1, \}$	$\{-4, -2, \}$		
3	$\{2,5\}$	$\{-1,4\}$	$\{-2,3\}$	$\{-3,2\}$	$\{-4, -1, \}$		
4	$\{3,4\}$	$\{2,3\}$	$\{-1,2\}$	$\{-2, -1, \}$	$\{-3, -2, \}$		
5	$\{3, 5\}$	$\{2,4\}$	$\{-1,3\}$	$\{-2,2\}$	$\{-3, -1, \}$		
6	$\{4,5\}$	$\{3, 4\}$	$\{2, 3\}$	$\{-1,2\}$	$\{-2, -1, \}$		

TABLE 5. $\ell = 0$: Sets generated by $\{-j + [M_j \setminus P_{ij}] \sqcup Q_{ijq}^\ell\}_{(i,j) \in I_\ell \times J, 1 \leq q \leq \binom{k}{\ell}}$.

A multiplication of the rates by $\eta(1) - \eta(0)$ leads then to the sets in the next two figures.

	$[-j + P_{ij}^{n+0}] \sqcup \{0\} = -j + (P_{ij}^{n+0} \sqcup \{j\})$						
$i \backslash j$	0	1	2	3	4		
1	$\{0, 2, 3\}$	$\{-1, 0, 2\}$	$\{-2, -1, 0\}$	$\{-3, -2, 0\}$	$\{-4, -3, 0\}$		
2	$\{0, 2, 4\}$	$\{-1, 0, 3\}$	$\{-2, 0, 2\}$	$\{-3, -1, 0\}$	$\{-4, -2, 0\}$		
3	$\{0, 2, 5\}$	$\{-1, 0, 4\}$	$\{-2, 0, 3\}$	$\{-3, 0, 2\}$	$\{-4, -1, 0\}$		
4	$\{0, 3, 4\}$	$\{0, 2, 3\}$	$\{-1, 0, 2\}$	$\{-2, -1, 0\}$	$\{-3, -2, 0\}$		
5	$\{0, 3, 5\}$	$\{0, 2, 4\}$	$\{-1, 0, 3\}$	$\{-2, 0, 2\}$	$\{-3, -1, 0\}$		
6	$\{0, 4, 5\}$	$\{0, 3, 4\}$	$\{0, 2, 3\}$	$\{-1, 0, 2\}$	$\{-2, -1, 0\}$		

TABLE 6. $\ell = 0$: Sets resulting from the multiplication with $\eta(0)$.

	$[-j + P_{ij}^{n+0}] \sqcup \{1\}$						
$i \backslash j$	0	1	2	3	4		
1	$\{1, 2, 3\}$	$\{-1, 1, 2\}$	$\{-2, -1, 1\}$	$\{-3, -2, 1\}$	$\{-4, -3, 1\}$		
2	$\{1, 2, 4\}$	$\{-1, 1, 3\}$	$\{-2, 1, 2\}$	$\{-3, -1, 1\}$	$\{-4, -2, 1\}$		
3	$\{1, 2, 5\}$	$\{-1, 1, 4\}$	$\{-2, 1, 3\}$	$\{-3, 1, 2\}$	$\{-4, -1, 1\}$		
4	$\{1, 3, 4\}$	$\{1, 2, 3\}$	$\{-1, 1, 2\}$	$\{-2, -1, 1\}$	$\{-3, -2, 1\}$		
5	$\{1, 3, 5\}$	$\{1, 2, 4\}$	$\{-1, 1, 3\}$	$\{-2, 1, 2\}$	$\{-3, -1, 1\}$		
6	$\{1, 4, 5\}$	$\{1, 3, 4\}$	$\{1, 2, 3\}$	$\{-1, 1, 2\}$	$\{-2, -1, 1\}$		

TABLE 7. $\ell = 0$: Sets resulting from the multiplication with $\eta(1)$.

The first layer of translations corresponds to translating each window (with its respective constraints) to the origin. With this, one obtains the sets in the next two figures.

	$P_{ij}^{n+0} \sqcup \{j+1\}$						
$i \backslash j$	0	1	2	3	4		
1	$\{1, 2, 3\}$	$\{0, 2, 3\}$	$\{0, 1, 3\}$	$\{0, 1, 4\}$	$\{0, 1, 5\}$		
2	$\{1, 2, 4\}$	$\{0, 2, 4\}$	$\{0, 3, 4\}$	$\{0, 2, 4\}$	$\{0, 2, 5\}$		
3	$\{1, 2, 5\}$	$\{0, 2, 5\}$	$\{0, 3, 5\}$	$\{0, 4, 5\}$	$\{0, 3, 5\}$		
4	$\{1, 3, 4\}$	$\{2, 3, 4\}$	$\{1, 3, 4\}$	$\{1, 2, 4\}$	$\{1, 2, 5\}$		
5	$\{1, 3, 5\}$	$\{2, 3, 5\}$	$\{1, 3, 5\}$	$\{1, 4, 5\}$	$\{1, 3, 5\}$		
6	$\{1, 4, 5\}$	$\{2, 4, 5\}$	$\{3, 4, 5\}$	$\{2, 4, 5\}$	$\{2, 3, 5\}$		

TABLE 8. $\ell = 0$: Sets resulting from the removal of the first layer of translations, each associated with multiplying by $\eta(1)$.

	$P_{ij}^{n+0} \sqcup \{j\}$						
$i \backslash j$	0	1	2	3	4		
1	$\{0, 2, 3\}$	$\{0, 1, 3\}$	$\{0, 1, 2\}$	$\{0, 1, 3\}$	$\{0, 1, 4\}$		
2	$\{0, 2, 4\}$	$\{0, 1, 4\}$	$\{0, 2, 4\}$	$\{0, 2, 3\}$	$\{0, 2, 4\}$		
3	$\{0, 2, 5\}$	$\{0, 1, 5\}$	$\{0, 2, 5\}$	$\{0, 3, 5\}$	$\{0, 3, 4\}$		
4	$\{0, 3, 4\}$	$\{1, 3, 4\}$	$\{1, 2, 4\}$	$\{1, 2, 3\}$	$\{1, 2, 4\}$		
5	$\{0, 3, 5\}$	$\{1, 3, 5\}$	$\{1, 2, 5\}$	$\{1, 3, 5\}$	$\{1, 3, 4\}$		
6	$\{0, 4, 5\}$	$\{1, 4, 5\}$	$\{2, 4, 5\}$	$\{2, 3, 5\}$	$\{2, 3, 4\}$		

TABLE 9. $\ell = 0$: Sets resulting from the removal of the first layer of translations, each associated with multiplying by $\eta(0)$.

The second layer of translations corresponds to translating to the origin each set in the previous two figures. This leads to the sets in the next two figures.

	$A_{ij}^{n+0,0}$						
$i \setminus j$	0	1	2	3	4		
1	$\{0, 2, 3\}$	$\{0, 1, 3\}$	$\{0, 1, 2\}$	$\{0, 1, 3\}$	$\{0, 1, 4\}$		
2	$\{0, 2, 4\}$	$\{0, 1, 4\}$	$\{0, 2, 4\}$	$\{0, 2, 3\}$	$\{0, 2, 4\}$		
3	$\{0, 2, 5\}$	$\{0, 1, 5\}$	$\{0, 2, 5\}$	$\{0, 3, 5\}$	$\{0, 3, 4\}$		
4	$\{0, 3, 4\}$	$\{0, 2, 3\}$	$\{0, 1, 3\}$	$\{0, 1, 2\}$	$\{0, 1, 3\}$		
5	$\{0, 3, 5\}$	$\{0, 2, 4\}$	$\{0, 1, 4\}$	$\{0, 2, 4\}$	$\{0, 2, 3\}$		
6	$\{0, 4, 5\}$	$\{0, 3, 4\}$	$\{0, 2, 3\}$	$\{0, 1, 3\}$	$\{0, 1, 2\}$		

TABLE 10. $\ell = 0$: Sets associated with the multiplication by $\eta(0)$ translated to the origin.

	$A_{ij}^{n+0,1}$					
$i \backslash j$	0	1	2	3	4	
1	$\{0, 1, 2\}$	$\{0, 2, 3\}$	$\{0, 1, 3\}$	$\{0, 1, 4\}$	$\{0, 1, 5\}$	
2	$\{0, 1, 3\}$	$\{0, 2, 4\}$	$\{0, 3, 4\}$	$\{0, 2, 4\}$	$\{0, 2, 5\}$	
3	$\{0, 1, 4\}$	$\{0, 2, 5\}$	$\{0, 3, 5\}$	$\{0, 4, 5\}$	$\{0, 3, 5\}$	
4	$\{0, 2, 3\}$	$\{0, 1, 2\}$	$\{0, 2, 3\}$	$\{0, 1, 3\}$	$\{0, 1, 4\}$	
5	$\{0, 2, 4\}$	$\{0, 1, 3\}$	$\{0, 2, 4\}$	$\{0, 3, 4\}$	$\{0, 2, 4\}$	
6	$\{0, 3, 4\}$	$\{0, 2, 3\}$	$\{0, 1, 2\}$	$\{0, 2, 3\}$	$\{0, 1, 3\}$	

TABLE 11. $\ell = 0$: Sets associated with the multiplication by $\eta(1)$ translated to the origin.

In the next table, the first column corresponds to all the *unique* sets in Figures 10 and 11; the second (resp. third) column corresponds to the indexes $(i, j) \in I_{\ell} \times J$ of the sets in Figure 10 (resp. Figure 11) associated with the set in the first column. A concrete example is the following. Consider x = 0, (i, j) = (3, 0) and the set $A_{ij}^{n+0,x} = A_{3,0}^{n+0,0} = \{0, 2, 5\}$, presented in Figure 10. For x = 0 still, we also have that (i, j) = (3, 2) corresponds to $A_{3,2}^{n+0,0} = \{0, 2, 5\}$ and so the indexes $(i, j) \in \{(3, 0), (3, 2)\}$ all correspond to the set $\{0, 2, 5\}$, for x = 0, and in this way they belong to the same class in the quotient space $C_{\ell,0}$. For x = 1 we have that $(i, j) \in \{(3, 1), (2, 4)\}$ is also associated with the set $\{0, 2, 5\}$ (see Figure 11).

A^ℓ	$\mathcal{C}_{\ell,0}$	$\mathcal{C}_{\ell,1}$
$\{0, 2, 3\}$	$\{(1,0),(4,1),(6,2),(2,3),(5,4)\}$	$\{(4,0),(1,1),(6,1),(4,2),(6,3)\}$
$\{0, 2, 4\}$	$\{(2,0), (5,1), (2,2), (5,3), (2,4)\}$	$\{(5,0), (2,1), (5,2), (2,3), (5,4)\}$
$\{0, 2, 5\}$	$\{(3,0),(3,2)\}$	$\{(3,1),(2,4)\}$
$\{0, 3, 4\}$	$\{(4,0),(6,1),(3,4)\}$	$\{(6,0),(2,2),(5,3)\}$
$\{0, 3, 5\}$	$\{(5,0),(3,3)\}$	$\{(3,2),(3,4)\}$
$\{0, 4, 5\}$	$\{(6,0)\}$	$\{(3,3)\}$
$\{0, 1, 3\}$	$\{(1,1),(4,2),(1,3),(6,3),(4,4)\}$	$\{(2,0), (5,1), (1,2), (4,3), (6,4)\}$
$\{0, 1, 4\}$	$\{(2,1),(5,2),(1,4)\}$	$\{(3,0),(1,3),(4,4)\}$
$\{0, 1, 5\}$	$\{(3,1)\}$	$\{(1,4)\}$
$\{0, 1, 2\}$	$\{(1,2),(4,3),(6,4)\}$	$\{(1,0),(4,1),(6,2)\}$

TABLE 12. $\ell = 0$: Equivalence classes of indexes and the corresponding "A-set".

In conclusion, in the second (resp. third) column above we have the equivalence classes of the indexes that originate from multiplying the constraints by $\eta(0)$ (resp. $\eta(1)$); in the first column the unique sets that are obtained by translating every element of $\mathcal{P}_{n+0}(M_j)$, for each $0 \leq j \leq n + k = 4$, to the origin; and the correspondence between the second and third columns provides a bijection between $\mathcal{C}_{\ell,0}$ and $\mathcal{C}_{\ell,1}$, which can be extended into a permutation ϕ_{ℓ} over $I_{\ell} \times J$.

C.1.2. Term $\ell = 1$. Following the same procedure, the rationale in (C.2) provides the map ψ_{ℓ} which, in turn, provides $\psi_{j,\ell}$.

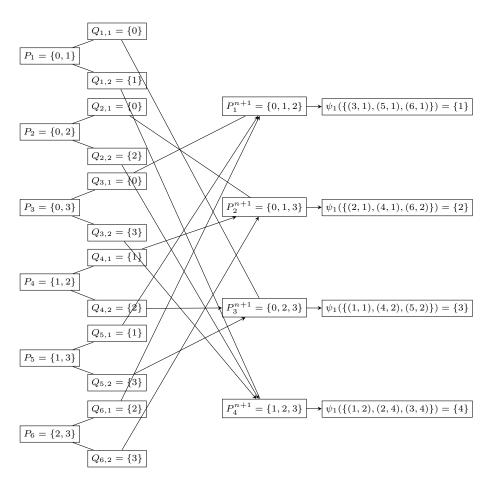


FIGURE 7. $\ell = 1$: Construction of the map ψ_1 .

The sets appearing in the product $\mathbf{c}^{(n,k;\mathfrak{a})}(\eta)(\eta(1) - \eta(0))$ are presented in the next two figures.

_	$[-j + P_{ij}^{n+1}] \sqcup \{0\}$						
$i \setminus j$	0	1	2	3	4		
1	$\{0, 2, 3, 4\}$	$\{-1, 0, 2, 3\}$	$\{-2, -1, 0, 2\}$	$\{-3, -2, -1, 0\}$	$\{-4, -3, -2, 0\}$		
2	$\{0, 2, 3, 5\}$	$\{-1, 0, 2, 4\}$	$\{-2, -1, 0, 3\}$	$\{-3, -2, 0, 2\}$	$\{-4, -3, -1, 0\}$		
3	$\{0, 2, 4, 5\}$	$\{-1, 0, 3, 4\}$	$\{-2, 0, 2, 3\}$	$\{-3, -1, 0, 2\}$	$\{-4, -2, -1, 0\}$		
4	$\{0, 3, 4, 5\}$	$\{0, 2, 3, 4\}$	$\{-1, 0, 2, 3\}$	$\{-2, -1, 0, 2\}$	$\{-3, -2, -1, 0\}$		

TABLE 13. $\ell = 1$: Sets resulting from the multiplication with $\eta(0)$.

		$[-j+P_{ij}^{n+1}]\sqcup\{1\}$					
$i \setminus j$	0	1	2	3	4		
1	$\{1, 2, 3, 4\}$	$\{-1, 1, 2, 3\}$	$\{-2, -1, 1, 2\}$	$\{-3, -2, -1, 1\}$	$\{-4, -3, -2, 1\}$		
2	$\{1, 2, 3, 5\}$	$\{-1, 1, 2, 4\}$	$\{-2, -1, 1, 3\}$	$\{-3, -2, 1, 2\}$	$\{-4, -3, -1, 1\}$		
3	$\{1, 2, 4, 5\}$	$\{-1, 1, 3, 4\}$	$\{-2, 1, 2, 3\}$	$\{-3, -1, 1, 2\}$	$\{-4, -2, -1, 1\}$		
4	$\{1, 3, 4, 5\}$	$\{1, 2, 3, 4\}$	$\{-1, 1, 2, 3\}$	$\{-2, -1, 1, 2\}$	$\{-3, -2, -1, 1\}$		

TABLE 14. $\ell = 1$: Sets resulting from the multiplication with $\eta(1)$.

The first layer of translations leads to the following.

	$P_{ij}^{n+1} \sqcup \{j\}$						
$i \backslash j$	0	1	2	3	4		
1	$\{0, 2, 3, 4\}$	$\{0, 1, 3, 4\}$	$\{0, 1, 2, 4\}$	$\{0, 1, 2, 3\}$	$\{0, 1, 2, 4\}$		
2	$\{0, 2, 3, 5\}$	$\{0, 1, 3, 5\}$	$\{0, 1, 2, 5\}$	$\{0, 1, 3, 5\}$	$\{0, 1, 3, 4\}$		
3	$\{0, 2, 4, 5\}$	$\{0, 1, 4, 5\}$	$\{0, 2, 4, 5\}$	$\{0, 2, 3, 5\}$	$\{0, 2, 3, 4\}$		
4	$\{0, 3, 4, 5\}$	$\{1, 3, 4, 5\}$	$\{1, 2, 4, 5\}$	$\{1, 2, 3, 5\}$	$\{1, 2, 3, 4\}$		

TABLE 15. $\ell = 1$: Sets resulting from the removal of the first layer of translations, each associated with multiplying by $\eta(0)$.

_	$P_{ij}^{n+1} \sqcup \{j+1\}$						
$i \backslash j$	0	1	2	3	4		
1	$\{1, 2, 3, 4\}$	$\{0, 2, 3, 4\}$	$\{0, 1, 3, 4\}$	$\{0, 1, 2, 4\}$	$\{0, 1, 2, 5\}$		
2	$\{1, 2, 3, 5\}$	$\{0, 2, 3, 5\}$	$\{0, 1, 3, 5\}$	$\{0, 1, 4, 5\}$	$\{0, 1, 3, 5\}$		
3	$\{1, 2, 4, 5\}$	$\{0, 2, 4, 5\}$	$\{0, 3, 4, 5\}$	$\{0, 2, 4, 5\}$	$\{0, 2, 3, 5\}$		
4	$\{1, 3, 4, 5\}$	$\{2, 3, 4, 5\}$	$\{1, 3, 4, 5\}$	$\{1, 2, 4, 5\}$	$\{1, 2, 3, 5\}$		

TABLE 16. $\ell = 1$: Sets resulting from the removal of the first layer of translations, each associated with multiplying by $\eta(1)$.

The "A-sets" corresponding to the second layer of translations are presented in the next two figures.

	$A_{ij}^{n+1,0}$					
$i \setminus j$	0	1	2	3	4	
1	$\{0, 2, 3, 4\}$	$\{0, 1, 3, 4\}$	$\{0, 1, 2, 4\}$	$\{0, 1, 2, 3\}$	$\{0, 1, 2, 4\}$	
2	$\{0, 2, 3, 5\}$	$\{0, 1, 3, 5\}$	$\{0, 1, 2, 5\}$	$\{0, 1, 3, 5\}$	$\{0, 1, 3, 4\}$	
3	$\{0, 2, 4, 5\}$	$\{0, 1, 4, 5\}$	$\{0, 2, 4, 5\}$	$\{0, 2, 3, 5\}$	$\{0, 2, 3, 4\}$	
4	$\{0, 3, 4, 5\}$	$\{0, 2, 3, 4\}$	$\{0, 1, 3, 4\}$	$\{0, 1, 2, 4\}$	$\{0, 1, 2, 3\}$	

TABLE 17. $\ell = 1$: Sets associated with the multiplication by $\eta(0)$ translated to the origin.

_	$A_{ij}^{n+1,1}$						
$i \backslash j$	0	1	2	3	4		
1	$\{0, 1, 2, 3\}$	$\{0, 2, 3, 4\}$	$\{0, 1, 3, 4\}$	$\{0, 1, 2, 4\}$	$\{0, 1, 2, 5\}$		
2	$\{0, 1, 2, 4\}$	$\{0, 2, 3, 5\}$	$\{0, 1, 3, 5\}$	$\{0, 1, 4, 5\}$	$\{0, 1, 3, 5\}$		
3	$\{0, 1, 3, 4\}$	$\{0, 2, 4, 5\}$	$\{0, 3, 4, 5\}$	$\{0, 2, 4, 5\}$	$\{0, 2, 3, 5\}$		
4	$\{0, 2, 3, 4\}$	$\{0, 1, 2, 3\}$	$\{0, 2, 3, 4\}$	$\{0, 1, 3, 4\}$	$\{0, 1, 2, 4\}$		

TABLE 18. $\ell = 1$: Sets associated with the multiplication by $\eta(1)$ translated to the origin.

Comparing the previous two figures leads to the identification of the equivalence classes of the indexes.

<u>A_c</u>	$\mathcal{C}_{\ell,0}$	$\mathcal{C}_{\ell,1}$
$\{0, 1, 2, 3\}$	$\{(1,3),(4,4)\}$	$\{(1,0),(4,1)\}$
$\{0, 1, 2, 4\}$	$\{(1,2),(1,4),(4,3)\}$	$\{(2,0),(1,3),(4,4)\}$
$\{0, 1, 2, 5\}$	$\{(2,2)\}$	$\{(1,4)\}$
$\{0, 1, 3, 4\}$	$\{(1,1),(2,4),(4,2)\}$	$\{(1,2),(3,0),(4,3)\}$
$\{0, 1, 3, 5\}$	$\{(2,1),(2,3)\}$	$\{(2,2),(2,4)\}$
$\{0, 1, 4, 5\}$	$\{(3,1)\}$	$\{(2,3)\}$
$\{0, 2, 3, 4\}$	$\{(1,0),(4,1),(3,4)\}$	$\{(1,1),(4,0),(4,2)\}$
$\{0, 2, 3, 5\}$	$\{(2,0),(3,3)\}$	$\{(2,1),(3,4)\}$
$\{0, 2, 4, 5\}$	$\{(3,0),(3,2)\}$	$\{(3,1),(3,3)\}$
$\{0, 3, 4, 5\}$	$\{(4,0)\}$	$\{(3,2)\}$

TABLE 19. $\ell = 1$: Equivalence classes of indexes and the corresponding "A-set".

The correspondence between the equivalence classes in Figure 12 and 19 provide a linear system for the "*b*-coefficients" (as in (2.5)). From the map $\psi_{j,\ell}$, for each $j \in J$, $0 \leq \ell \leq n+k$, the original "*a*-coefficients" (as in Definition 2.6) can be recovered, yielding a linear system

that can be reduced to

$$a_{1,0} + a_{2,3} = a_{1,1} + a_{1,2}$$

$$a_{1,0} + a_{2,0} + a_{2,2} + a_{2,4} = 2a_{1,1} + a_{1,3} + a_{2,1}$$

$$2a_{1,2} + a_{3,1} = a_{1,1} + a_{1,3} + a_{1,4}$$

$$a_{1,0} + a_{1,2} + a_{2,0} + a_{2,2} + a_{3,0} + a_{3,2} = 2a_{1,1} + 2a_{1,4} + 2a_{2,1}$$

$$a_{3,4} + a_{4,0} = a_{1,2} + a_{2,2}$$

$$a_{4,3} = a_{1,0}$$

$$a_{4,2} + a_{4,4} = a_{1,0} + a_{2,0}$$

$$a_{1,0} + 2a_{1,2} + a_{2,0} + a_{2,2} + a_{3,0} + a_{3,3} + a_{5,0} = 3a_{1,1} + 2a_{1,3} + a_{1,4} + a_{2,1} + a_{3,4}$$

$$a_{1,2} + a_{4,2} + a_{5,2} = a_{1,0} + a_{2,0} + a_{3,0}$$

$$a_{1,2} + a_{3,3} + a_{4,1} + a_{5,1} + a_{5,3} = 2a_{1,1} + a_{1,3} + a_{1,4} + a_{2,1}$$

$$a_{3,4} + a_{4,1} + a_{5,4} = a_{1,2} + a_{2,2} + a_{4,2}$$

$$a_{1,1} + a_{1,3} + a_{6,0} = 2a_{1,2} + a_{3,3}$$

$$a_{4,1} + a_{5,1} + a_{6,1} = a_{1,1} + a_{1,2} + a_{1,3}$$

$$a_{6,2} = a_{1,2}$$

$$a_{1,2} + a_{6,3} = a_{4,1} + a_{5,1}$$

$$a_{6,4} = a_{4,1}$$
(C.3)

C.2. Linear system characterizing the potential's invariance. In this subsection we prove Lemma C.1, where we derive an additional set of conditions on the weights \mathfrak{a} where the potential $\mathbf{h}^{n,k}$, as in Proposition 2.13, is related with the potential corresponding to the uniform choice (as in Proposition 2.19, modulo the factor $\binom{n+k}{k}$). Moreover, we present a non-uniform solution for the particular case n, k = 2, for the linear system characterizing the gradient condition, (2.6), when extended with the aforementioned conditions. For that reason, Lemma C.1 is not empty for n, k = 2 only. Other non-uniform solutions were found computationally, for different values of n and k.

Lemma C.1. Let \mathfrak{a} be such that (2.6) holds. If, for each $0 \leq \ell \leq k$ and $\underline{c} \in \mathcal{C}_{\ell,1}$ such that $\max(A_{\underline{c}}) \neq n+k+1$, the following equations are also satisfied, for $1 \leq p \leq n+k-\max(A_{\underline{c}})$,

$$\sum_{(i,j)\in\underline{c}} \mathbf{1}\{p_{ij}^{\ell,1} = p\}b_{ij}^{\ell} = \sum_{(i,j)\in\underline{c}} \mathbf{1}\{p_{\phi_{\ell}(i,j)}^{\ell,0} = p\}b_{\phi_{\ell}(i,j)}^{\ell}$$

$$\sum_{(i,j)\in\underline{c}} \mathbf{1}\{p_{ij}^{\ell,1} = n+k+1-\max(A_{\underline{c}})\}b_{ij}^{\ell} = \frac{\binom{n+\ell}{\ell}}{\binom{n+k}{k}} + \sum_{(i,j)\in\underline{c}} \mathbf{1}\{p_{\phi_{\ell}(i,j)}^{\ell,0} = n+k+1-\max(A_{\underline{c}})\}b_{ij}^{\ell}$$
(C.4)

then

$$\mathbf{h}_{1}^{n,k} = \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} \left(\mathbf{P}_{n+k}^{n,k} + \nabla^{+} \mathbf{g}_{n+k}^{n,k} \right),$$

with $\mathbf{P}_{n+k}^{n,k}$ and $\mathbf{g}_{n+k}^{n,k}$ as in (2.15).

Proof. Recalling the expression for $\mathbf{h}_1^{n,k}(\eta)$ from (2.7), fixed ℓ and $\underline{c} \in \mathcal{C}_{\ell,1}$, from the property (1) in Lemma A.4 the summation over $(i, j) \in \underline{c}$ in $\mathbf{h}_1^{(n,k;\mathfrak{a})}(\eta)$ can be expressed as

$$\sum_{p=1}^{n+k-\max(A_{\underline{c}})} \sum_{y=0}^{p-1} (\tau^{y}\eta)(A_{\underline{c}}) \left\{ \sum_{(i,j)\in\underline{c}} \mathbf{1}\{p_{ij}^{\ell,1} = p\}b_{ij}^{\ell} - \sum_{(i,j)\in\underline{c}} \mathbf{1}\{p_{\phi_{\ell}(i,j)}^{\ell,0} = p\}b_{\phi_{\ell}(i,j)}^{\ell} \right\}$$
$$+ \sum_{y=0}^{n+k+1-\max(A_{\underline{c}})-1} (\tau^{y}\eta)(A_{\underline{c}}) \times \left\{ \sum_{(i,j)\in\underline{c}} \mathbf{1}\{p_{ij}^{\ell,1} = n+k+1-\max(A_{\underline{c}})\neq 0\}b_{ij}^{\ell} - \sum_{(i,j)\in\underline{c}} \mathbf{1}\{p_{\phi_{\ell}(i,j)}^{\ell,0} = n+k+1-\max(A_{\underline{c}})\neq 0\}b_{\phi_{\ell}(i,j)}^{\ell} \right\}.$$

If the weights \mathfrak{a} are such that (C.4) is satisfied, one obtains that $\mathbf{h}_1^{n,k}$ in (2.7) equals (2.16), which concludes the proof.

We now present the additional linear system in Lemma C.1 for the case n, k = 2. We omit the case $\ell = 2$ as it is associated with the PMM(4). The equations in the first line of (C.4) are, in this case,

$$a_{2,4} + a_{3,1} = a_{3,0} + a_{3,2}$$

$$a_{3,2} + a_{3,4} = a_{3,3} + a_{5,0}$$

$$a_{1,4} + a_{3,3} = a_{3,1} + a_{6,0}$$

$$a_{1,4} + a_{2,4} + a_{3,4} = a_{3,0} + a_{5,0} + a_{6,0},$$
(C.5)

with the first 3 equations corresponding to $\ell = 0$ and the last to $\ell = 1$. The equations in the second line in (C.4) are

$$1/6 + a_{1,2} = a_{1,1} + a_{1,3}$$

$$1/6 + a_{2,1} + a_{2,3} = a_{2,0} + a_{2,2} + a_{2,4}$$

$$1/6 + a_{1,1} = a_{1,0} + a_{2,3}$$

$$1/6 + a_{1,3} = a_{1,4} + a_{2,1}$$

$$1/6 + a_{2,2} = a_{3,4} + a_{4,0}$$

$$1/6 = a_{1,2}$$

$$1/2 + a_{1,1} + a_{2,1} + a_{3,1} = a_{1,0} + a_{1,2} + a_{2,0} + a_{2,2} + a_{3,0} + a_{3,2}$$

$$1/2 + a_{1,2} + a_{2,2} + a_{3,2} = a_{1,0} + a_{1,3} + a_{2,3} + a_{3,3} + a_{4,0} + a_{5,0}$$

$$1/2 + a_{1,3} + a_{2,3} + a_{3,3} = a_{1,4} + a_{2,0} + a_{2,4} + a_{3,4} + a_{4,0} + a_{6,0}$$

$$1/2 = a_{1,1} + a_{2,1} + a_{3,1}$$
(C.6)

and the first 6 equations correspond to $\ell = 0$. The *extended* system composed by the equations in (C.3),(C.5) and (C.6) can be reduced to

$$\begin{aligned} a_{3,1} &= a_{1,4} \\ a_{4,3} &= a_{1,0} \\ a_{6,0} &= a_{3,3} \\ a_{6,4} &= a_{4,1} \\ a_{4,2} + a_{4,4} &= a_{1,0} + a_{2,0} \\ a_{1,2} &= 1/6 \\ a_{6,2} &= 1/6 \\ a_{1,0} + a_{2,3} &= 1/6 + a_{1,1} \\ a_{3,4} + a_{4,0} &= 1/6 + a_{2,2} \\ a_{4,1} + a_{5,1} &= 1/6 + a_{6,3} \\ a_{1,0} + a_{2,0} + a_{3,0} &= 1/6 + a_{4,2} + a_{5,2} \\ a_{3,4} + a_{4,1} + a_{5,4} &= 1/6 + a_{2,2} + a_{4,2} \\ a_{1,1} + a_{1,3} &= 1/3 \\ a_{1,1} + a_{1,4} + a_{2,1} &= 1/2 \\ a_{4,1} + a_{5,1} + a_{6,1} &= 1/2 \\ a_{3,3} + a_{4,1} + a_{5,1} + a_{5,3} &= 2/3 \\ a_{1,0} + a_{1,4} + a_{2,0} + a_{2,2} + a_{2,4} &= 5/6 \\ a_{1,0} + a_{2,0} + a_{2,2} + a_{3,0} + a_{3,3} + a_{5,0} &= 5/6 + a_{3,4}. \end{aligned}$$

A particular solution of the system above yielding a gradient model is

		a_{ij}				
i	i/j	0	1	2	3	4
	1	0	0	1/6	1/3	0
	2	0	1/2	0	1/6	5/6
	3	1/6	0	2/3	0	0
	4	1/6	0	0	0	0
	5	2/3	1/2	0	1/6	1/6
	6	0	0	1/6	1/3	0

TABLE 20. n, k = 2: Particular solution of the extended system.

where we recall that each (i, j) above is associated with a set $-j + P_{ij}$, as in Table 4.