Let Γ be a discrete and torsion-free subgroup of PU(n, 1), the group of biholomorphisms of the unit ball in \mathbb{C}^n , denoted by $\mathbb{H}^n_{\mathbb{C}}$. We show that if Γ is Abelian, then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold. If the critical exponent $\delta(\Gamma)$ of Γ is less than 2, a conjecture of Dey and Kapovich predicts that the quotient $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is Stein. We confirm this conjecture in the case where Γ is parabolic or geometrically finite. We also study the case of quotients with $\delta(\Gamma) = 2$ that contain compact complex curves and confirm another conjecture of Dey and Kapovich. We finally show that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is Stein when Γ is a parabolic or geometrically finite group preserving a totally real and totally geodesic submanifold of $\mathbb{H}^n_{\mathbb{C}}$, without any hypothesis on the critical exponent.

In this article we study the existence of non-constant holomorphic functions on quotients of the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ of dimension n, thought of as the unit ball in \mathbb{C}^n , by discrete and torsion-free subgroups Γ of PU(n,1). More precisely, we give sufficient conditions for $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ to be holomorphically convex or Stein. These conditions involve the group structure of Γ or its critical exponent $\delta(\Gamma)$, which is defined by

Holomorphic functions on geometrically finite quotients of

the ball

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Abstract

$$\delta(\Gamma) := \inf\{s \in \mathbb{R}_+ \mid \sum_{\gamma \in \Gamma} e^{-sd(o,\gamma o)} < \infty\},\tag{1}$$

where o is an arbitrary point of $\mathbb{H}^n_{\mathbb{C}}$ and d the complex hyperbolic distance on the ball, normalized so that the associated Riemannian metric has sectional curvature pinched between -4 and -1. This number $\delta(\Gamma)$, which does not depend on the choice of $o \in \mathbb{H}^n_{\mathcal{C}}$, has been first related with the analytical properties of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ by Dey and Kapovich in [DK20]. These authors have proposed the following conjecture, and have solved it for convex-cocompact subgroups of PU(n, 1).

Conjecture ([DK20]). Let Γ be a discrete and torsion-free subgroup of PU(n, 1). If $\delta(\Gamma) < 2$, then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold.

Our first main result confirms in particular this conjecture for geometrically finite subgroups of PU(n, 1).

Theorem 1. Let Γ be a geometrically finite and torsion-free subgroup of PU(n, 1).

- (a) If Γ is Gromov-hyperbolic, then $\mathbb{H}^n_{\Gamma}/\Gamma$ is holomorphically convex.
- (b) If Γ is a free group or if $\delta(\Gamma) < 2$, then $\mathbb{H}^n_{\Gamma}/\Gamma$ is a Stein manifold.
- (c) If Γ preserves a totally real and totally geodesic submanifold of $\mathbb{H}^n_{\mathbb{C}}$, then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold.

We recall that a parabolic subgroup of PU(n, 1) is a subgroup of PU(n, 1) that fixes a point at infinity and does not contain any hyperbolic element. To establish Theorem 1, we need to understand when the quotient of the complex hyperbolic space by a discrete parabolic subgroup is Stein. For unipotent parabolic subgroups this has been done in [Mie24], and we settle the general case, obtaining the following result.

Theorem 2. Let Γ be a discrete and torsion-free parabolic subgroup of PU(n, 1).

- (a) If $\delta(\Gamma) < 2$ or Γ preserves a totally real and totally geodesic submanifold $\mathbb{H}^k_{\mathbb{R}} \subset \mathbb{H}^n_{\mathbb{C}}$, then Γ is virtually Abelian.
- (b) If Γ is virtually Abelian, then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold.

The second point of Theorem 2 is a consequence of Theorem 13, stated in Section 2, which provides a complete characterisation of discrete and torsion-free parabolic subgroups Γ of PU(n, 1)for which $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is Stein, and whose proof consists in reducing the problem to the unipotent case solved earlier in [Mie24]. In complex dimension 2, this characterisation takes the following simpler form: a parabolic quotient $\mathbb{H}^2_{\mathbb{C}}/\Gamma$ is Stein if and only if Γ is virtually Abelian (Corollary 16).

In [DK20], Dey and Kapovich have shown on the one hand that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ does not contain any compact analytic subset of positive dimension if $\delta(\Gamma) < 2$ and on the other hand that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is holomorphically convex if Γ is convex-cocompact. By contrast, when Γ is geometrically finite, the manifold $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is not always holomorphically convex. Using techniques from [Che13], we get the following characterization.

Theorem 3. Let Γ be a geometrically finite and torsion-free subgroup of PU(n, 1). The following are equivalent:

- 1. The manifold $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ admits a plurisubharmonic exhaustion function.
- 2. For any parabolic subgroup $P < \Gamma$, the quotient $\mathbb{H}^n_{\mathbb{C}}/P$ is holomorphically convex, or equivalently a Stein manifold.
- 3. The manifold $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is holomorphically convex.

The strategy for proving Theorem 1 is to use Theorems 2 and 3 by showing that if Γ satisfies one of the assumptions of Theorem 1, then its parabolic subgroups are virtually Abelian. We do not recall the definition of a geometrically finite group in this article, but instead refer the reader to [Bow95], or to the description given in [Che13, p. 1031]. In connection with the conjecture of Dey and Kapovich, we also show that quotients $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ with $\delta(\Gamma) < 2$ always admit non-constant holomorphic functions, as a particular case of Proposition 6 below.

We provide in Section 2 an example showing that the constant 2 in Theorem 2 is optimal, and an example of a unipotent parabolic group Γ for which $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is biholomorphic to a bundle of punctured disks over a non-compact Cousin manifold, and is not holomorphically convex. By a Cousin manifold, we mean a quotient of \mathbb{C}^{n-1} by a discrete subgroup, which does not admit any non-constant holomorphic function, see [Cou10; Kop64; AK01]. The critical exponent of this example is equal to $\frac{5}{2}$.

A complex Fuchsian group is a discrete and torsion-free subgroup Γ of PU(n, 1) which acts cocompactly on a Γ -invariant complex geodesic. If Γ is a complex Fuchsian group, then its critical exponent is equal to 2, and the quotient $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ contains a compact subvariety¹ of positive dimension. When Γ is a convex-cocompact and torsion-free subgroup of PU(n, 1) with critical

¹Here and throughout all this article we use the word *subvariety* as a synonym of *closed analytic subset*.

exponent $\delta(\Gamma) = 2$, Dey and Kapovich conjecture that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is non-Stein if and only if Γ is a complex Fuchsian group, see [DK20, Conjecture 17]. Using techniques from [CMW23], we confirm this conjecture as follows.

Theorem 4. Let Γ be a discrete and torsion-free subgroup of PU(n, 1) with $\delta(\Gamma) = 2$. Suppose that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ contains a compact subvariety of positive dimension. Then Γ is a complex Fuchsian group.

We now discuss the relation of our results with earlier works. Theorem 1 applies in particular to representations of free groups seen as finite-index subgroups of the examples in [GP92; FP03]. They also apply to Schottky quotients, recovering [MO18, Theorem 4.3]. Theorem 2, together with the fact that the quotient of $\mathbb{H}^n_{\mathbb{C}}$ by a loxodromic cyclic group is Stein (see for example [dFab98], [Che13] or Section 1 below), implies that the quotient of the complex hyperbolic space by any discrete and torsion-free Abelian subgroup of PU(n, 1) is a Stein manifold. In [Che13], Chen asks whether the quotient of $\mathbb{H}^n_{\mathbb{C}}$ by a discrete and torsion-free subgroup of PO(n, 1) is Stein. Theorems 1 and 2 yields a positive answer to this question for geometrically finite or parabolic subgroups.

Here are some earlier results about the analytic properties of quotients of the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$. It is known that the quotient of $\mathbb{H}^n_{\mathbb{C}}$ by an infinite discrete cyclic group is a Stein manifold [dFab98; dFI01; Mie10]. The article [Che13] gives criteria for a quotient of $\mathbb{H}^n_{\mathbb{C}}$ by a discrete subgroup to be Stein, and in particular shows that a quotient of the complex hyperbolic space by a unipotent Abelian parabolic group is Stein. It also contains results in the more general setting of quotients of Kähler-Hadamard manifolds. The case of quotients by unipotent parabolic subgroups is completely solved in [Mie24]. Finally, as mentioned above, the article [DK20] in which the above conjecture appears contains the analogue of Theorem 1 in the case of convex-cocompact groups. Section 9 of the overview article [Kap22] also contains interesting results on the ends of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$, which are related to the analytic properties of this manifold.

In another direction, we generalise the results of Dey and Kapovich to non-symmetric spaces. In the following two propositions and the subsequent corollaries, which provide an alternative proof of the results of [DK20], (X, ω) is a simply connected complete Kähler manifold with sectional curvature bounded above by -1, and Γ a group acting freely and properly discontinuously by holomorphic isometries on X. One can define the critical exponent $\delta(\Gamma)$ of Γ by the same Formula (1) as for discrete subgroups of PU(n, 1), by choosing an arbitrary point o of X and with d the Riemannian distance associated with ω . This number does not depend on the choice of $o \in X$. Moreover, in this context, *pinched* means, when referring to the sectional curvature of (X, ω) , that it is bounded below by -b, and when referring to the Ricci curvature, that it is bounded below by $-b\omega(\cdot, J \cdot)$, for some constant b > 1.

Proposition 5. Let C be a Γ -invariant closed and geodesically convex subset of X. Then the compact connected subvarieties of positive dimension of X/Γ are included in C/Γ . Moreover, if the action of Γ on C is cocompact, then X/Γ is holomorphically convex.

Proposition 6. Assume that $\delta(\Gamma) < 2$. Then X/Γ admits a strictly plurisubharmonic function and in particular does not contain any compact subvariety of positive dimension. If moreover Xhas pinched Ricci curvature, then holomorphic functions on X/Γ separate points and define local coordinates at all points of X/Γ .

Corollary 7. Assume that $\delta(\Gamma) < 2$ and Γ is convex-cocompact. Then X/Γ is a Stein manifold.

The proof of Proposition 6 uses Patterson-Sullivan theory, of which we give a brief account in the article, referring to [Rob03; Nic89] for more details. Our next corollary, proven in Section 1, also involves an assumption on the *Patterson-Sullivan measure* $(\mu_x)_{x \in X}$. This is a family of finite measures on ∂X , indexed by points $x \in X$ whose construction is recalled in Section 1. It is said to have subexponential growth if for all $\eta > 0$, there is a constant $C_{\eta} > 0$ such that

$$\forall x \in X, \ \mu_x(\partial X) \le C_n e^{\eta d(x,o)},$$

for some basepoint $o \in X$, see [CMW23, §1.4]. For instance, if X has pinched sectional curvature, X/ Γ has positive injectivity radius and if the Bowen-Margulis measure ν associated with $(\mu_x)_{x \in X}$ is finite, then the total masses $\mu_x(\partial X)$ are uniformly bounded, see [CMW23, Theorem 1.15]. We also refer to the latter article for a definition of the Bowen-Margulis measure ν .

Corollary 8. Assume that $\delta(\Gamma) < 2$, X has pinched sectional curvature and the Patterson-Sullivan measure $(\mu_x)_{x \in X}$ has subexponential growth. Then X/Γ is a Stein manifold.

This article is organised as follows. In the first section, we recall what holomorphically convex and Stein manifolds are, we give the definition and some basic properties of Patterson-Sullivan measures, we prove Propositions 5, 6 and Corollaries 7, 8 and we give criteria for asserting that a quotient of the form X/Γ does not admit any compact subvariety of positive dimension, with X and Γ as above. Theorem 2 is proved in Section 2. Section 3 contains the proof of Theorem 3 from which is deduced the proof of Theorem 1. Section 4 is independent from Sections 2 and 3, and contains two proofs of Theorem 4.

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1 Quotients of negatively curved Kähler-Hadamard manifolds

We begin by recalling some definitions from complex analysis and then review some results of negative curvature geometry. After that, we prove Propositions 5 and 6 in Subsections 1.3 and 1.4, then prove Corollaries 7 and 8 in Subsection 1.5. Finally, we summarize known criteria for asserting that X/Γ does not admit a compact subvariety of positive dimension in Subsection 1.6.

1.1 Generalities about Stein manifolds

We first recall the definition of plurisubharmonic and strictly plurisubharmonic functions, and we refer to [Dem] for more details. Let X be a complex manifold. A C^2 function $f: X \to \mathbb{R}$ is plurisubharmonic (resp. strictly plurisubharmonic) if the (1, 1)-form $i\partial \bar{\partial} f$ is nonnegative (resp. positive). A continuous function $f: X \to \mathbb{R}$ is plurisubharmonic if for every chart $\phi: V \subset X \to W \subset \mathbb{C}^n$, every $a \in W$ and every $\xi \in \mathbb{C}^n$ such that $|\xi| < d(a, {}^cW)$, we have

$$f \circ \phi^{-1}(a) \le \frac{1}{2\pi} \int_0^{2\pi} f \circ \phi^{-1}(a + e^{i\theta}\xi) d\theta.$$

It is strictly plurisubharmonic if for every $x \in X$ there are holomorphic coordinates (z_1, \ldots, z_n) defined on some neighborhood of x and a constant c > 0 such that $z \mapsto f(z) - c ||z||^2$ is plurisubharmonic.

A complex manifold X is said to be holomorphically convex if the holomorphic hull \widehat{K} of any compact subset K of X, which is defined by

$$\widehat{K} := \{ x \in X \mid \forall f \in \mathcal{O}(X), |f(x)| \le \sup_{V} |f| \}$$

is compact. The manifold X is *Stein* if it is holomorphically convex and if, in addition, for any pair of distinct points x, y of X, there is a holomorphic function $f : X \longrightarrow \mathbb{C}$ such that $f(x) \neq f(y)$. Grauert's theorem asserts that a manifold is Stein if and only if it admits a strictly plurisubharmonic exhaustion function, see [Gra58]. Alternatively, a manifold is Stein if and only if it is holomorphically convex and does not contain any compact subvariety of positive dimension. This follows from the existence of the Remmert reduction of a holomorphically convex manifold [Pet94, Theorem 2.1]. We will also use the following result, that we subsequently refer to as Grauert's theorem, since it derives from it.

Theorem ([Gra58], [Pet94, Corollary 2.4]). Let X be a complex manifold admitting an exhaustion function which is smooth and strictly plurisubharmonic outside a compact set. Then X is holomorphically convex.

We will also use the following version of Docquier-Grauert's theorem.

Theorem ([DG60], [Siu78, Theorem 5.2]). The union of an increasing 1-parameter family of Stein manifolds is Stein

1.2 Convexity, Busemann functions and Patterson-Sullivan measures

For the remainder of the section, (X, ω) denotes a simply connected complete Kähler manifold with complex structure denoted by J and sectional curvature bounded above by -1, and Γ is a group acting freely and properly discontinuously by holomorphic isometries on X. Let d be the Riemannian distance associated with ω . We recall that if $\phi : X \to \mathbb{R}$ is a function of class \mathcal{C}^2 , then the form $i\partial\overline{\partial}\phi$ is related to the Riemannian Hessian $D^2(\phi)$ of ϕ by

$$2i\partial\overline{\partial}\phi(v,Jv) = D^2(\phi)(v,v) + D^2(\phi)(Jv,Jv), \qquad (2)$$

see [GW73].

Fix a point $o \in X$. For all $\xi \in \partial X$, the Busemann function at ξ is the function defined by

$$\forall x \in X, B_{\xi}(x) := B_{\xi}(x, o) := \lim_{z \to \xi} (d(x, z) - d(o, z)).$$

The Busemann function at ξ depends on $o \in X$ only up to an additive constant. This fonction is of class C^2 , see [HH77, Proposition 3.1], and it depends continuously on $\xi \in \partial X$. It is moreover strictly plurisubharmonic and more precisely we have

$$i\partial\overline{\partial}B_{\xi} \ge \omega. \tag{3}$$

This inequality is a consequence of [GW79, Proposition 2.28], see also [SY82; Che13]. We also call $(\xi, x, y) \in \partial X \times X^2 \longmapsto B_{\xi}(x, y) \in \mathbb{R}$ the Busemann function on X.

In this article, we will also use Patterson-Sullivan theory. It is used in the proofs of Proposition 6, Corollary 8, and Theorem 4. We now recall the definition and some basic properties of

Patterson-Sullivan measures, and we refer the reader to [Rob03; Pat76; Sul79; Nic89] for more details. Let X and Γ be as above, and δ be the critical exponent of Γ , whose definition was recalled in the introduction. We also assume that Γ is non-elementary, which means that Γ does not stabilize a geodesic of X, nor a point of ∂X . A Patterson-Sullivan measure is a Γ -conformal density of dimension δ , which means that it is a family of measures $(\mu_x)_{x \in X}$ on ∂X such that $\gamma_* \mu_x = \mu_{\gamma x}$ for all $x \in X$ and $\gamma \in \Gamma$, and such that

$$\forall x, y \in X, \frac{d\mu_x}{d\mu_y} = e^{-\delta B_{\bullet}(x,y)}.$$
(4)

We now recall a construction of a Patterson-Sullivan measure when Γ has divergent type, i.e. when the series

$$\sum_{\gamma \in \Gamma} e^{-\delta d(o, \gamma o)}$$

diverges. The general case is explained for instance in [Nic89, Section 3.1]. Fix an arbitrary point $o \in X$ and for $s > \delta$, set

$$\Phi(s) := \sum_{\gamma \in \Gamma} e^{-sd(o,\gamma o)}.$$

The space of probability measures on $\overline{X} := X \cup \partial X$ being compact for the weak topology, define μ_o as an accumulation point as $s > \delta$ tends towards δ of the probability measures

$$\frac{1}{\Phi(s)} \sum_{\gamma \in \Gamma} e^{-sd(o,\gamma o)} \delta_{\gamma o},$$

where δ_{γ_o} denotes the Dirac measure at γ_o . It can be verified that this measure is supported on ∂X . The other measures μ_x for $x \in X$ are defined from μ_o by means of Formula (4). All these measures are finite and we denote by $\|\mu_x\| := \mu_x(\partial X)$ the total mass of the measure μ_x . It is not difficult to see that these measures are supported on the limit set $\Lambda(\Gamma)$ of Γ . Since $\Lambda(\Gamma)$ is the smallest non-empty closed and Γ -invariant subset of ∂X , we deduce that the support of the Patterson-Sullivan measures is exactly $\Lambda(\Gamma)$.

1.3 **Proof of Proposition 5**

Proposition 5 is proven by an application of the next lemma to the square of the distance function to $C/\Gamma \subset \mathbb{H}^n_{\mathbb{C}}/\Gamma$. In this lemma, a continuous function ϕ from a complete Riemannian manifold M to \mathbb{R} is called *convex* if for all geodesic $\eta : \mathbb{R} \to M$, the function $\phi \circ \eta$ is convex, and it is called *strictly convex* if for any compact subset $K \subset M$, there exists a constant $\alpha > 0$ such that, for any unit-speed geodesic $\eta : [0, 1] \to K$, the function $t \in [0, 1] \mapsto \phi \circ \eta(t) - \alpha t^2$ is convex.

Lemma 9. Let (M, ω) be a Kähler manifold. Assume that there exists a continuous function $\phi : M \to \mathbb{R}$ which is convex on M, and strictly convex on $M \setminus \phi^{-1}(0)$. Then the compact connected subvarieties of positive dimension of M are included in the level set $\phi^{-1}(0)$. If moreover ϕ is an exhaustion, then M is holomorphically convex.

Proof. We are going to use that on a Kähler manifold (M, ω) , every continuous convex function is plurisubharmonic [GW73, Theorem 3], from which it follows that a strictly convex function $f: M \to \mathbb{R}$ is strictly plurisubharmonic. Indeed, let x be a point in M and (z_1, \ldots, z_n) be holomorphic coordinates defined in a neighborhood Ω of x. Then for any open subset V with $\overline{V} \subset \Omega$, there exists a constant c > 0 such that $z \mapsto f(z) - c ||z||^2$ is convex, and therefore plurisubharmonic, on V. Thus f is strictly plurisubharmonic. When f is of class C^2 , these statements are a simple consequence of Formula (2).

Let $\phi: M \to \mathbb{R}$ be a continuous function which is convex on M, and strictly convex on $M \setminus \phi^{-1}(0)$. Then ϕ is plurisubharmonic on M and strictly plurisubharmonic on $M \setminus \phi^{-1}(0)$. Moreover for any connected compact subvariety A of M, the function $\phi|_A$ is constant by the maximum principle, so A is included in $\phi^{-1}(0)$ or in $M \setminus \phi^{-1}(0)$. Notice that A cannot be contained in $M \setminus \phi^{-1}(0)$ because in that case $\phi|_A$ would be constant and strictly plurisubharmonic. Thus $A \subset \phi^{-1}(0)$. If moreover ϕ is an exhaustion, then by Richberg's theorem [Dem, Theorem I.5.21], there exists a continuous plurisubharmonic function $\tilde{\phi}: M \to \mathbb{R}$ which is smooth and strictly plurisubharmonic on $M \setminus \phi^{-1}(0)$ and such that $\tilde{\phi} \ge \phi$ on $M \setminus \phi^{-1}(0)$. In particular $\tilde{\phi}$ is an exhaustion, and by Grauert's theorem, M is holomorphically convex.

Proof of Proposition 5. Let $d_C^2 : X \to \mathbb{R}_+$ be the square of the distance function to C. This function is convex on X, and it is strictly convex on $X \setminus C$. This is proved in [BH23, Lemma 4.5], and for the sake of completeness we now outline a proof of the strict convexity of d_C^2 on $X \setminus C$. Fix some $\epsilon > 0$ and let $\gamma : [0, L] \to X$ be a unit-speed geodesic such that $d(\gamma(0), C) \ge \epsilon$ and $d(\gamma(L), C) \ge \epsilon$. Denote by x and y the projections of $\gamma(0)$ and $\gamma(L)$ in C and let $\eta : [0, L'] \to X$ be the unit-speed geodesic joining x and y. Using [KS93, Corollary 2.1.3 - Formula 2.1(iv)], we get :

$$2d_C^2\left(\gamma\left(\frac{L}{2}\right)\right) - d_C^2(\gamma(0)) - d_C^2(\gamma(L)) \le -\frac{1}{2}(L - L')^2.$$

Finally, since X has sectional curvature bounded above by -1, there is a positive constant a depending only on ϵ such that $(L - L')^2 \ge aL^2$. Using [GW76, Lemma 1], we conclude that d_C^2 is strictly convex on $X \setminus C$.

To conclude, notice that the function d_C^2 is Γ -invariant, so it defines a convex function ϕ : $X/\Gamma \to \mathbb{R}$, which is strictly convex outside C/Γ . The proposition is obtained by applying Lemma 9 with the function ϕ .

Remark. Proposition 5 provides an alternative proof of [Kap22, Proposition 5]: suppose that X has negatively pinched curvature and that there is a surjective holomorphic map $f: X/\Gamma \to B$ with compact fibers on a complex manifold B with $\dim(B) < \dim(X)$. Then $\Lambda(\Gamma) = \partial X$. In particular X/Γ cannot have convex ends.

1.4 **Proof of Proposition 6**

The next lemma, used in the proof of Proposition 6, asserts that a certain function defined in [CMW23] is strictly plurisubharmonic when the critical exponent of Γ is less than 2. This function and the flow it defines also play an important role in Section 4. An alternative proof of the first point of Proposition 6, which uses comparison arguments from [GW79] is outlined below.

Lemma 10. Let (X, ω) be a simply connected complete Kähler manifold with sectional curvature bounded above by -1, and Γ be a non-elementary group acting freely and properly discontinuously by holomorphic isometries on X. Denote by δ the critical exponent of Γ , by $(\mu_x)_{x \in X}$ a Patterson-Sullivan measure associated with Γ and by $\|\mu_x\|$ the total mass of the measure μ_x for every $x \in X$. Then the Γ -invariant function on X defined by $f(x) := -\ln\|\mu_x\|$ satisfies

$$i\partial\overline{\partial}f \ge \delta(1-\frac{\delta}{2})\omega.$$

Proof. For every $x \in X$, let $\overline{\mu_x}$ be the normalized probability measure $\overline{\mu_x} = \frac{\mu_x}{\|\mu_x\|}$. Fixing a point $p \in X$, denote by $B_{\theta} := B_{\theta}(\cdot, p)$ the Busemann function at a point $\theta \in \partial X$ which vanishes at p. From dominated convergence together with the \mathcal{C}^2 -regularity of Busemann functions and Formula (4), it follows that f is of class \mathcal{C}^2 . Moreover, a computation using Formula (4) shows that the Hessian $D^2(f)$ at a point $x \in X$ is given by

$$D^{2}(f)(v,v) = \delta \int_{\partial X} D^{2}B_{\theta}(v,v)d\overline{\mu_{x}}(\theta) + \delta^{2} \left(\left(\int_{\partial X} dB_{\theta}(v)d\overline{\mu_{x}}(\theta) \right)^{2} - \int_{\partial X} dB_{\theta}(v)^{2}d\overline{\mu_{x}}(\theta) \right).$$

Using Identity (2), we get

$$i\partial\partial\overline{f}(v,Jv) \ge \delta \int_{\partial X} i\partial\overline{\partial}B_{\theta}(v,Jv)d\overline{\mu_x}(\theta) - \frac{\delta^2}{2} \int_{\partial X} (dB_{\theta}(v)^2 + dB_{\theta}(Jv)^2)d\overline{\mu_x}(\theta).$$

It is easily verified that for all tangent vector v,

$$dB_{\theta}(v)^2 + dB_{\theta}(Jv)^2 \le ||v||^2,$$

where $||v||^2 := \omega(v, Jv)$. Using this minoration together with Inequality (3), we deduce that

$$i\partial\overline{\partial}f \ge \delta(1-\frac{\delta}{2})\omega.$$

Proof of Proposition 6. If Γ is elementary, the result is already known, see [Che13, Theorem 1.1 and Proposition 1.3]. Assume that Γ is non-elementary and that $\delta < 2$. Then the function fdefined in Lemma 10 is strictly plurisubharmonic. Suppose moreover that there is a constant C > 0 such that

$$\operatorname{Ricci}(\omega) \ge -C\omega(\cdot, J\cdot)$$

Then there is a constant C' > 0 such that

$$\partial \overline{\partial} (C'f) + \operatorname{Ricci}(\omega)(\cdot, J \cdot) \ge 0.$$

Using [Che13, Proposition 4.1], we obtain that the holomorphic functions on X/Γ separate points and give local coordinate systems.

Remark. A strictly plurisubharmonic function on X/Γ can also be constructed following Dey and Kapovich's ideas, providing an alternative proof for the first point of Proposition 6. Here is an outline of the argument. Let ϕ be the function defined on X by $\phi(x) := \tanh(d(o, x))^2$ for some basepoint $o \in X$. An application of the comparison result [GW79, Theorem A] together with Formula (2) gives that ϕ is strictly plurisubharmonic on X: this is obtained by comparing the Hessian of ϕ with the Hessian of the function $\tilde{\phi}$ defined on $\mathbb{H}^{2n}_{\mathbb{R}}$, the real hyperbolic space of dimension 2n, by $\tilde{\phi}(x) := \tanh(d_{\text{hyp}}(\tilde{o}, x))^2$ for some basepoint \tilde{o} of $\mathbb{H}^{2n}_{\mathbb{R}}$. Then because

$$0 \le 1 - \phi \le 4e^{-2d(o,\cdot)},$$

we deduce that the convergence of the series

$$\sum_{\gamma\in\Gamma}e^{-2d(o,\gamma o)}$$

implies that

$$\sum_{\gamma \in \Gamma} (\phi(\gamma \cdot x) - 1)$$

converges uniformly on compact subsets of X to a strictly plurisubharmonic and Γ -invariant function ψ . When $X = \mathbb{H}^n_{\mathbb{C}}$, ϕ is the squared euclidean norm on the unit ball, and the above series is the one constructed in [DK20].

1.5 Proofs of Corollaries 7 and 8

Proof of Corollary 7. This follows directly from Propositions 5 and 6, using that a manifold is Stein if and only if it is holomorphically convex and does not contain any compact subvariety of positive dimension. Alternatively, each one of the three strictly plurisubharmonic functions $\phi + e^f$, $\phi + e^{\psi}$ and f is an exhaustion on X/Γ , with ϕ as in the proof of Proposition 5, f as in the proof of Proposition 6 and ψ as in the remark above. For the function f, this can be shown using that $(X \cup \Omega(\Gamma))/\Gamma$ is compact, where $\Omega(\Gamma) \subset \partial X$ is the discontinuity subset of Γ .

Proof of Corollary 8. Let f be the Γ -invariant function defined in Lemma 10 and o be a point of X/Γ . By [Tam10] or [Cho+10, §4 of Ch. 26], there is a constant C > 0 and a smooth function $g: X/\Gamma \to \mathbb{R}$ such that $g \ge d(\cdot, o)$ and $D^2(g) \ge -C\omega(\cdot, J \cdot)$. Using Identity (2), we deduce that $i\partial \overline{\partial}g \ge -C\omega$. Now define

$$\phi = g + C'f,$$

where C' is a positive constant such that

$$-C+C'\delta(1-\frac{\delta}{2})>0.$$

Then ϕ is strictly plurisubharmonic by Lemma 10. Since $(\mu_x)_{x \in X}$ has subexponential growth, there is a constant C'' > 0 such that

$$f \ge -\frac{1}{2C'}d(\cdot, o) - \ln C''.$$

We deduce that

$$\phi \ge \frac{1}{2}d(\cdot, o) - \ln C'',$$

which implies that ϕ is proper. Thus X/Γ is a Stein manifold.

Remark. The hypothesis of bounded sectional curvature can be weakened to bounded Ricci curvature together with an assumption on the ball volumes, see [Hua19].

1.6 Compact subvarieties of positive dimension of X/Γ

We now summarize known criteria for asserting that X/Γ does not admit a compact subvariety of positive dimension.

Proposition 11. Let (X, ω) be a simply connected complete Kähler manifold with sectional curvature bounded above by -1, and Γ a group acting freely and properly discontinuously by holomorphic isometries on X. Denote by d the Riemannian distance associated with ω . Then each one of the following condition is sufficient to assert that X/Γ does not admit a compact subvariety of positive dimension.

- (a) The group Γ is parabolic.
- (b) The critical exponent of Γ satisfies $\delta(\Gamma) < 2$.
- (c) There exists a Γ -invariant geodesically convex subset C of X which is included in a totally real submanifold M of X.
- (d) The Kähler form ω is exact on X/Γ .
- (e) The cohomology group $H^2(\Gamma, \mathbb{R})$ vanishes.

- (f) There is a complete vector field on X/Γ whose flow contracts complex subspaces.
- *Proof.* (a) If Γ is parabolic, there is a Busemann function at some point $\xi \in \partial X$ which is invariant under the action of Γ , see [EO73, Proposition 7.8]. As a consequence X/Γ admits a strictly plurisubharmonic function.
- (b) This follows from [DK20] for the complex hyperbolic space and from Proposition 6 in the general case.
- (c) This is a consequence of Proposition 5. Indeed, let A be a compact connected subvariety of positive dimension included in X/Γ . Then $A \subset C/\Gamma$, and taking a smooth point x of the lift \tilde{A} of A in X, we obtain that $T_x \tilde{A} \cap J(T_x \tilde{A}) \subset T_x M$, which contradicts the hypothesis that M is totally real. Alternatively, it can be shown that in this case the distance squared function to the convex core is strictly plurisubharmonic, see [Che13].
- (d) This is a well known fact about Kähler geometry.
- (e) The manifold X/Γ is a $K(\Gamma, 1)$, and in particular its cohomology identifies with that of Γ . Therefore, if $H^2(\Gamma, \mathbb{R}) = 0$, the Kähler form ω is exact on X/Γ . This argument appears in a dual version in [MO18, Lemma 4.2], and also in [Kap22, page 27].
- (f) See for instance [CMW23, Proof of Theorem 1.5].

- *Examples.* 1. When $X = \mathbb{H}^n_{\mathbb{C}}$ and Γ is a discrete and torsion-free subgroup of $\operatorname{PO}(n, 1)$, embedded in $\operatorname{PU}(n, 1)$ so as to stabilize a copy of $\mathbb{H}^n_{\mathbb{R}}$, the quotient $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ does not contain a compact subvariety of positive dimension by the third point. In particular, as proven in [Che13], the quotient of X by the group generated by a hyperbolic element is a Stein manifold.
 - 2. Let $\Gamma = \pi_1(\Sigma_g)$ be a surface group and $\rho : \Gamma \to PU(n,1)$ be a discrete and faithful representation of Γ in PU(n,1). The *Toledo invariant* τ of ρ is the real number

$$\tau := \frac{1}{2\pi} [\phi^* \omega] \in H^2(\Sigma_g, \mathbb{R}) \simeq \mathbb{R},$$

where $\phi: \Sigma_g \to X/\rho(\Gamma)$ is any homotopy equivalence between Σ_g and $X/\rho(\Gamma)$. Then ω is exact on $X/\rho(\Gamma)$ if and only if $\tau = 0$. This example appears in [Kap22, page 27].

3. If Γ is a free group, then $H^2(\Gamma, \mathbb{R}) = 0$ so X/Γ does not contain any compact subvariety of positive dimension.

To conclude this section, let us note that the question of finding sufficient conditions on the group Γ for the quotient X/Γ to be Stein admits a natural generalization to the case where X is a higher rank Hermitian symmetric space and Γ is a group acting freely and properly discontinuously by holomorphic isometries on X. Here is a family of examples of quotients of the bidisk $\mathbb{D} \times \mathbb{D}$ that are easily proven to be Stein manifolds.

Example. Let $\Gamma = \pi_1(S_g)$ be a cocompact lattice in $PSL_2(\mathbb{R})$. Define the following action of Γ on the bidisk $\mathbb{D} \times \mathbb{D}$

$$\forall \gamma \in \Gamma, \forall (z, w) \in \mathbb{D} \times \mathbb{D}, \ \gamma \cdot (z, w) := (\gamma \cdot z, \overline{\gamma \cdot w}).$$

Then $\Delta := \{(z, \overline{z}) \mid z \in \mathbb{D}\}$ is a totally real geodesically convex subset of $\mathbb{D} \times \mathbb{D}$, on which Γ acts cocompactly. Using [Che13, Proposition 3.2], we get that $(\mathbb{D} \times \mathbb{D})/\Gamma$ is a Stein manifold.

2 Discrete parabolic subgroups of PU(n, 1)

This section is organised as follows. We first recall the definition of the complex hyperbolic distance on the ball, and describe the stabilizer of a point at infinity. Then we state and prove Theorem 13 which characterises the discrete and torsion-free parabolic subgroups of PU(n, 1) for which $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold, and which implies Theorem 2-(b). We then show that if Γ is a discrete parabolic subgroup which satisfies $\delta(\Gamma) < 2$ or preserves a totally real geodesic submanifold of $\mathbb{H}^n_{\mathbb{C}}$, then Γ is virtually Abelian, thus completing the proof of Theorem 2. Afterwards we give an example of a discrete parabolic subgroup with $\delta(\Gamma) = 2$ and for which $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is not Stein. We also construct a complex hyperbolic bundle of punctured disks over a non-compact Cousin manifold. This complex hyperbolic bundle is not holomorphically convex, but holomorphic functions separate points by [Mie24, Theorem 1.1]. Notice that a parabolic quotient $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is Stein if and only if it is holomorphically convex, as follows from Proposition 11-(a).

2.1 The parabolic biholomorphisms of the ball

Let h be the Hermitian form on \mathbb{C}^{n+1} associated with the quadratic form

$$q(z_1, \dots, z_{n+1}) := -|z_1|^2 + \sum_{i=2}^{n+1} |z_i|^2,$$

and $[\cdot] : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ denote the projection onto the complex projective space \mathbb{CP}^n . The open subset of \mathbb{CP}^n defined by

$$\mathbb{H}^n_{\mathbb{C}} := \{ [v] \in \mathbb{CP}^n \mid q(v) < 0 \}$$

is biholomorphic to the unit ball of \mathbb{C}^n . It can be endowed with a complete Kähler metric of negative sectional curvature pinched between -4 and -1, for which the distance between two points $x, y \in \mathbb{H}^n_{\mathbb{C}}$ is given by the formula

$$\cosh^2 d(x,y) = \frac{h(\widetilde{x},\widetilde{y})h(\widetilde{y},\widetilde{x})}{h(\widetilde{x},\widetilde{x})h(\widetilde{y},\widetilde{y})},$$

where $\tilde{x}, \tilde{y} \in \mathbb{C}^{n+1}$ denote lifts of x, y. Moreover, every biholomorphism of the ball is an isometry for this metric, and the group of biholomorphic isometries of $\mathbb{H}^n_{\mathbb{C}}$ is isomorphic to $\mathrm{PU}(n, 1)$. The action of $\mathrm{PU}(n, 1)$ on $\mathbb{H}^n_{\mathbb{C}}$ extends to an action by homeomorphisms on the closed ball $\mathbb{H}^n_{\mathbb{C}} \cup \partial \mathbb{H}^n_{\mathbb{C}}$, where

$$\partial \mathbb{H}^n_{\mathbb{C}} := \{ [v] \in \mathbb{CP}^n \mid q(v) = 0 \}.$$

Let us fix a point $\xi \in \partial \mathbb{H}^n_{\mathbb{C}}$. There exists a basis $f_{\xi} = (f_1, f_2, e_1, \dots, e_{n-1})$ of \mathbb{C}^{n+1} such that $\xi = [f_1]$ and in which the quadratic form q has the following expression

$$q\left(af_1 + bf_2 + \sum_{i=1}^{n-1} u_i e_i\right) = 2\Re(a\overline{b}) + \sum_{i=1}^{n-1} |u_i|^2.$$

The biholomorphism

$$\begin{cases} \{(a,u) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid 2\Re(a) + \|u\|^2 < 0\} & \longrightarrow & \mathbb{H}^n_{\mathbb{C}} \\ (a,u) & \longmapsto & [af_1 + f_2 + u] \end{cases}$$

defines a global chart of $\mathbb{H}^n_{\mathbb{C}}$, in which Busemann functions at $[f_1]$ are the translates of the function b defined by

$$e^{2b(a,u)} = \frac{-2}{2\Re(a) + \|u\|^2}$$

A horoball at $[f_1]$ is a sublevel set of b.

In the basis f_{ξ} , let us define three subgroups M, A and N of PU(n, 1) by the associated groups of matrices

$$\mathcal{M} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & T \end{pmatrix} \mid T \in \mathrm{U}(n-1) \right\},\$$
$$\mathcal{A} = \left\{ \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & \mathrm{I}_{n-2} \end{pmatrix} \mid t \in \mathbb{R} \right\},\$$
$$\mathcal{N} = \left\{ \begin{pmatrix} 1 & a & -^t \overline{b} \\ 0 & 1 & 0 \\ 0 & b & \mathrm{I}_{n-2} \end{pmatrix} \mid b \in \mathbb{C}^{n-1}, a \in \mathbb{C}, \|b\|^2 = -2\Re(a) \right\}.$$

Then the stabilizer of a point $\xi \in \partial \mathbb{H}^n_{\mathbb{C}}$ in $\mathrm{PU}(n, 1)$ decomposes as

$$\operatorname{Stab}_{\xi}(\mathbb{H}^n_{\mathbb{C}}) = MAN.$$

For $T \in U(n-1)$, $b \in \mathbb{C}^{n-1}$ and $c \in \mathbb{R}$, let (T, b, c) denote the element of the group MN defined in the basis f_{ξ} by the matrix

$$\left(\begin{array}{ccc} 1 & -\frac{\|b\|^2}{2} + ic & -\langle T \cdot, b \rangle \\ 0 & 1 & 0 \\ 0 & b & T \end{array}\right)$$

The group law on MN is given by

$$(T, b, c) \cdot (T', b', c') = (TT', b + Tb', c + c' + \Im\langle b, Tb' \rangle),$$

where $\langle \cdot, \cdot \rangle$ is the standard Hermitian product on \mathbb{C}^{n-1} . This group identifies with a semi-direct product $\mathrm{U}(n-1) \ltimes N$, and N is isomorphic to the Heisenberg group of real dimension 2n-1. The center Z(N) of N is the set of elements of the form $(\mathrm{Id}, 0, c)$, with $c \in \mathbb{R}$. We denote by π the projection of $\mathrm{U}(n-1) \ltimes N$ onto $\mathrm{U}(n-1)$, and by Π the morphism $\mathrm{U}(n-1) \ltimes N \to \mathrm{U}(n-1) \ltimes \mathbb{C}^{n-1}$ which to $(T, b, c) \in \mathrm{U}(n-1) \ltimes N$ associates the holomorphic isometry $z \mapsto Tz + b$ of \mathbb{C}^{n-1} .

A subgroup Γ of $\operatorname{PU}(n, 1)$ acts freely and properly discontinuously on $\mathbb{H}^n_{\mathbb{C}}$ if and only if it is torsion-free and discrete in $\operatorname{PU}(n, 1)$. It is said to be *parabolic* if it fixes a point ξ in $\partial \mathbb{H}^n_{\mathbb{C}}$ and if all the eigenvalues of its elements are of modulus 1, which amounts to saying that, in the model described above, Γ is a subgroup of $\operatorname{U}(n-1) \ltimes N$, or equivalently that Γ preserves horoballs at ξ . We write for all $\gamma \in \Gamma$

$$\gamma = (\pi(\gamma), b(\gamma), c(\gamma)), \text{ with } \pi(\gamma) \in \mathcal{U}(n-1), b(\gamma) \in \mathbb{C}^{n-1}, c(\gamma) \in \mathbb{R}, \text{ and}$$

 $\Pi(\Gamma) = (\pi(\gamma), b(\gamma)).$

The parabolic group Γ is said to be *unipotent* if $\pi(\Gamma)$ is trivial.

2.2 A characterization of Stein parabolic quotients of the ball

As explained in Subsection 2.1, we identify a parabolic subgroup of PU(n, 1) with a subgroup of $U(n-1) \ltimes N$, and we denote by π , respectively II, the projection of $U(n-1) \ltimes N$ onto U(n-1), respectively $U(n-1) \ltimes \mathbb{C}^{n-1}$. The following lemma is probably classical, and we give its proof, after the statement of Theorem 13, for the reader's convenience.

Lemma 12. Let Γ be a discrete and torsion-free parabolic subgroup of PU(n, 1). Then there exists a finite-index subgroup Γ_1 of Γ such that $\Pi(\Gamma_1)$ is Abelian.

Let Γ and Γ_1 be as in this lemma. Set

$$V_1 := \bigcap_{\gamma \in \Gamma_1} \ker(\mathrm{Id} - \pi(\gamma)),$$

and let $p: \mathbb{C}^{n-1} \to V_1$ be the orthogonal projection onto V_1 . Finally, define

$$W_1 := \operatorname{Span}(\{p(b(\gamma)) \mid \gamma \in \Gamma_1\}),$$

where $b(\gamma) = \Pi(\gamma) \cdot 0 \in \mathbb{C}^{n-1}$. In the following statement, a linear subspace W of \mathbb{C}^{n-1} is said to be totally real if $W \cap iW = \{0\}$.

Theorem 13. Let Γ be a discrete and torsion-free parabolic subgroup of PU(n, 1), and let Γ_1, p, V_1 and W_1 be as above. Then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold if and only if W_1 is totally real.

Proof of Lemma 12. As in Subsection 2.1, let us fix a basis of \mathbb{C}^{n+1} which induces an identification between Γ and a discrete subgroup of $U(n-1) \ltimes N$. By Margulis Lemma, Γ is virtually nilpotent. The existence of Γ_1 follows from the classical fact that a nilpotent subgroup of $U(n-1) \ltimes \mathbb{C}^{n-1}$ is virtually Abelian. To show this fact, the first observation, that we will not prove here, is that a nilpotent subgroup of U(n-1) is virtually Abelian. Let Γ_1 be a finite-index nilpotent subgroup of Γ such that $\pi(\Gamma_1)$ is Abelian. Seeking a contradiction, let us assume that the nilpotent group $\Pi(\Gamma_1)$ is not Abelian. There is a non-trivial element z in the center of $\Pi(\Gamma_1)$ which can be written as a product of commutators $z = [x_1, y_1] \dots [x_k, y_k]$, with $x_1, \dots, x_k, y_1, \dots, y_k \in \Pi(\Gamma_1)$. Let us write

$$\begin{aligned} x_i &= (\pi(x_i), b(x_i)), \\ y_i &= (\pi(y_i), b(y_i)), \\ z &= (\mathrm{Id}, b(z)), \end{aligned}$$

with $\pi(x_i), \pi(y_i) \in U(n-1)$ which commute, and $b(x_i), b(y_i), b(z) \in \mathbb{C}^{n-1}$. We will now show that b(z) = 0, which means that z is trivial, a contradiction. Given $\pi_1, \pi_2 \in U(n-1)$ which commute and $b_1, b_2 \in \mathbb{C}^{n-1}$, we compute that

$$\begin{split} [(\pi_1, b_1), (\pi_2, b_2)] &= (\pi_1, b_1) \ (\pi_2, b_2) \ (\pi_1^{-1}, -\pi_1^{-1}b_1) \ (\pi_2^{-1}, -\pi_2^{-1}b_2) \\ &= (\pi_1 \pi_2, b_1 + \pi_1 b_2) (\pi_1^{-1} \pi_2^{-1}, -\pi_1^{-1}b_1 - \pi_1^{-1} \pi_2^{-1}b_2) \\ &= (\mathrm{Id}, b_1 + \pi_1 b_2 + (\pi_1 \pi_2) (-\pi_1^{-1}b_1 - \pi_1^{-1} \pi_2^{-1}b_2)) \\ &= (\mathrm{Id}, (\mathrm{Id} - \pi_2) b_1 - (\mathrm{Id} - \pi_1) b_2). \end{split}$$

Therefore

$$b(z) = \sum_{i=1}^{k} (\mathrm{Id} - \pi(y_i))b(x_i) - (\mathrm{Id} - \pi(x_i))b(y_i).$$
(5)

Moreover, z commutes with all elements of $E := \{x_1, \ldots, x_k, y_1, \ldots, y_k\}$, which implies that

$$\forall \gamma \in E, \quad \pi(\gamma)b(z) = b(z). \tag{6}$$

Choose a basis $e = (e_1, \ldots, e_{n-1})$ that diagonalizes all elements of $\pi(E)$ and express $\pi(\gamma)$ in this basis as $\text{Diag}(a_1(\gamma), \ldots, a_{n-1}(\gamma))$ for $\gamma \in E$. For all $j \in \{1, \ldots, n-1\}$, if there exists $\gamma \in E$ such that $a_j(\gamma) \neq 1$, then the j^{th} coordinate $b_j(z)$ of b(z) in the basis e must vanish according to Formula (6), and if $a_j(\gamma) = 1$ for all $\gamma \in E$, then $b_j(z) = 0$ according to Formula (5). Thus b(z) = 0, which gives the contradiction we were looking for and proves that $\Pi(\Gamma_1)$ is Abelian. \square

Proof of Theorem 13. Let Γ_1 be as in Lemma 12. Since $\mathbb{H}^n_{\mathbb{C}}/\Gamma_1$ is a finite covering of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$, one of these two manifolds is Stein if and only if the other one is. To simplify the notation, we can thus assume without loss of generality that $\Gamma_1 = \Gamma$.

Step 1. Separation of the elliptic and unipotent parts of Γ .

The group $\pi(\Gamma)$ is Abelian, hence there is an orthonormal basis $e = (e_1, \ldots, e_{n-1})$ of \mathbb{C}^{n-1} as well as morphisms a_1, \ldots, a_{n-1} from Γ to the unit circle in \mathbb{C} such that, in the basis e

$$\pi(\gamma) = \operatorname{Diag}(a_1(\gamma), \dots, a_{n-1}(\gamma)).$$

For $\gamma \in \Gamma$, let $(b_1(\gamma), \ldots, b_{n-1}(\gamma))$ be the coordinates of $b(\gamma)$ in the basis *e*. Up to permuting the elements of *e*, we can assume that $(e_{k+1}, \ldots, e_{n-1})$ forms a basis of V_1 for some integer $k \in \{0, \ldots, n-1\}$. For all $i \in \{1, \ldots, k\}$, there is an element $\gamma_i \in \Gamma$ such that $a_i(\gamma_i) \neq 1$. Set

$$\lambda_i := \frac{b_i(\gamma_i)}{1 - a_i(\gamma_i)}.$$

As $\Pi(\Gamma)$ is Abelian, we have

$$\forall \gamma \in \Gamma, \forall i \le k, b_i(\gamma) = \lambda_i (1 - a_i(\gamma)).$$

We set $\Lambda := {}^t(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0) \in \mathbb{C}^{n-1}$, so that

$$\forall \gamma \in \Gamma, (\pi(\gamma) - \mathrm{Id})\Lambda = {}^{t}(-b_{1}(\gamma), \dots, -b_{k}(\gamma), 0, \dots 0).$$

Let T_{Λ} be the element (Id, $\Lambda, 0$) of U(n - 1) $\ltimes N$ and, for all $\gamma \in \Gamma$, define $\phi(\gamma) := T_{\Lambda}^{-1} \gamma T_{\Lambda}$. A computation shows that for all $\gamma \in \Gamma$, we have

$$\phi(\gamma) = (\pi(\gamma), b(\phi(\gamma)), c(\phi(\gamma))), \text{ where } b(\phi(\gamma)) = b(\gamma) + (\pi(\gamma) - \mathrm{Id})\Lambda \in V_1 \text{ and } c(\phi(\gamma)) \in \mathbb{R}.$$

Set

$$\phi(\gamma)_e := (\pi(\gamma), 0, 0) \text{ and } \phi(\gamma)_u := (\mathrm{Id}, b(\phi(\gamma)), c(\phi(\gamma)))$$

Using that $b(\phi(\gamma)) \in V_1$ for all $\gamma \in \Gamma$, we get

$$\forall \gamma, \gamma' \in \Gamma, [\phi(\gamma)_e, \phi(\gamma')_u] = \mathrm{Id}.$$

It is also easily verified that $\phi(\gamma) = \phi(\gamma)_e \phi(\gamma)_u$ for all $\gamma \in \Gamma$. More generally, U(k), seen as a subgroup of U(n-1) fixing V_1 pointwise, commutes with $\phi(\gamma)_u$ for $\gamma \in \Gamma$. In particular, $\phi(\Gamma)_E := \{\phi(\gamma)_e \mid \gamma \in \Gamma\}$ and $\phi(\Gamma)_U := \{\phi(\gamma)_u \mid \gamma \in \Gamma\}$ are groups and $\phi(\Gamma)_U < N$. Moreover, $\phi(\Gamma)_U$ is discrete in N. Indeed, let $(\gamma_k)_{k \in \mathbb{N}}$ be a sequence in Γ such that $\phi(\gamma_k)_u \xrightarrow[k \to +\infty]{} Id$. After passing to a subgroup, we can assume that the sequence $\phi(\gamma_k)_e$ converges to a limit $M \in U(n-1)$. Thus $\phi(\gamma_k)$ converges to M, and since $\phi(\Gamma)$ is discrete, this sequence has to be stationary. Hence $\phi(\Gamma)_U$ is discrete in N. Step 2 Characterization of Stein quotients. We can rewrite W_1 as

$$W_1 = \operatorname{Span}_{\mathbb{R}}(\{b(\phi(\gamma)) \mid \gamma \in \Gamma\}).$$

From [Mie24, Theorem 1.4], we obtain that the quotient $\mathbb{H}^n_{\mathbb{C}}/\phi(\Gamma)_U$ is Stein if and only if W_1 is totally real.

Assume that W_1 is totally real. Then $\mathbb{H}^n_{\mathbb{C}}/\phi(\Gamma)_U$ is a Stein manifold, so it has a strictly plurisubharmonic exhaustion function $\psi_U : \mathbb{H}^n_{\mathbb{C}}/\phi(\Gamma)_U \to \mathbb{R}_+$. Moreover, the holomorphic action of U(k) on $\mathbb{H}^n_{\mathbb{C}}$ descends to the quotient $\mathbb{H}^n_{\mathbb{C}}/\phi(\Gamma)_U$, and by averaging ψ_U over the orbits of U(k), we can assume that ψ_U is U(k)-invariant. Then ψ_U lifts to a strictly plurisubharmonic function $\widetilde{\psi_U} : \mathbb{H}^n_{\mathbb{C}} \to \mathbb{R}_+$ which is invariant by $\phi(\Gamma)_U$ and U(k). This function descends to a strictly plurisubharmonic function $\psi : \mathbb{H}^n_{\mathbb{C}}/\phi(\Gamma) \to \mathbb{R}_+$, which is an exhaustion. Thus, $\mathbb{H}^n_{\mathbb{C}}/\phi(\Gamma)$, and therefore $\mathbb{H}^n_{\mathbb{C}}/\Gamma$, are Stein.

Conversely, if $\mathbb{H}^n_{\mathbb{C}}/\Gamma$, hence $\mathbb{H}^n_{\mathbb{C}}/\phi(\Gamma)$, is a Stein manifold, then $\mathbb{H}^n_{\mathbb{C}}/\phi(\Gamma)$ admits a strictly plurisubharmonic exhaustion function. This implies that $\mathbb{H}^n_{\mathbb{C}}/\phi(\Gamma)_U$ has a strictly plurisubharmonic exhaustion function, thus $\mathbb{H}^n_{\mathbb{C}}/\phi(\Gamma)_U$ is a Stein manifold. Therefore, W_1 is totally real.

Using Theorem 13, we obtain as a corollary the second point of Theorem 2.

Corollary 14. Let Γ be a discrete and torsion-free parabolic subgroup of PU(n, 1). If Γ is virtually Abelian, then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold.

Proof. As in the proof of Theorem 13, we assume without loss of generality that Γ is Abelian and we decompose $\phi(\Gamma)$ into an elliptic part $\phi(\Gamma)_E$ and a unipotent part $\phi(\Gamma)_U$. Then $\phi(\Gamma)_U$ is a discrete and Abelian parabolic subgroup of PU(n, 1). It is known that the quotient of the complex hyperbolic space by such a subgroup is a Stein manifold, see [Che13]. This is also a particular case of [Mie24, Theorem 1.4], because with the notations of Step 1 above, it can be verified that for all $\gamma, \gamma' \in \Gamma$, the identity $[\phi(\gamma), \phi(\gamma')] = Id$ implies that

$$\Im \langle b(\phi(\gamma)), b(\phi(\gamma')) \rangle = 0,$$

and this implies that W_1 is totally real, so that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold.

2.3 Proof of Theorem 2

We first recall a formula for the critical exponent of a discrete and torsion-free parabolic subgroup Γ of PU(n, 1), for which we refer to [CI99] or [DOP00, §3]. Let Γ_1 be a finite-index subgroup of Γ such that $\Pi(\Gamma_1)$ is Abelian. Define $l \in \{0, 1\}$ as the dimension of the real subspace spanned by $Z(N) \cap \Gamma_1$, where $Z(N) \simeq \mathbb{R}$ is the center of \mathbb{R} , and $k \in \{0, \ldots 2n - 2\}$ as the dimension of the subspace of \mathbb{C}^{n-1} spanned by $\Pi(\Gamma_1)$. Then

$$\delta(\Gamma) = \frac{2l+k}{2}.$$
(7)

Proof of Theorem 2. The second point of the theorem is given by Corollary 14. For the first point, let Γ be a discrete and torsion-free parabolic subgroup of PU(n, 1) which is not virtually Abelian. We will show that $\delta(\Gamma) \geq 2$ by finding two elements $x, y \in \Gamma$ that generate a group of critical exponent equal to 2. Let us fix, as in Subsection 2.1, a basis f of \mathbb{C}^{n+1} which induces an identification between Γ and a subgroup of $U(n-1) \ltimes N$. Let Γ_1 be a finite-index subgroup of Γ such that $\Pi(\Gamma_1)$ is Abelian, given by Lemma 12. Since the set of commutators of elements of Γ_1 is included in the kernel of Π , which coincides with the center Z(N) of N, and Γ_1 is not Abelian, we deduce that Γ contains two elements x and y such that $\Pi(x)$ and $\Pi(y)$ commute, but x and y do not. Then according to Formula (7), the critical exponent of the group generated by x and y is $\frac{2l+k}{2}$, where $l \in \{0,1\}$ is the dimension of the \mathbb{R} -span of the elements $c(\gamma)$ for $\gamma \in \Gamma$ and $k \in \{0,1,2\}$ is the dimension of the \mathbb{R} -span of $\pi(x)$ and $\pi(y)$. Since x and y do not commute, we see that l = 1 and k = 2. Thus $\delta(\langle x, y \rangle) = 2$.

Corollary 15. Let Γ be a discrete and torsion-free parabolic subgroup of PU(n, 1). If Γ preserves a totally real geodesic submanifold of $\mathbb{H}^n_{\mathbb{C}}$, then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is Stein.

Proof. We can realize Γ as a discrete and virtually nilpotent subgroup of

$$\mathbb{P}((O(k-1) \ltimes \mathbb{R}^{k-1}) \times U(n-k))$$

for some integer $k \in \{1, ..., n\}$. Consequently, Γ is virtually Abelian. We deduce from Theorem 13 that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold.

Corollary 16. Let Γ be a discrete and torsion-free parabolic subgroup of PU(2,1). Then $\mathbb{H}^2_{\mathbb{C}}/\Gamma$ is Stein if and only if Γ is virtually Abelian.

Proof. If Γ is not virtually Abelian, choose two elements $x, y \in \Gamma$ as in the proof of Theorem 2. Since x and y do not commute, we get that $\Im\langle b(x), b(y) \rangle \neq 0$, and thus $W := \operatorname{Span}_{\mathbb{R}}(b(x), b(y)) \subset \mathbb{C}$ is equal to \mathbb{C} . In particular W is not totally real and using Theorem 13, we deduce that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ admits a covering $\mathbb{H}^n_{\mathbb{C}}/\langle x_0, y_0 \rangle$ which is not Stein. As any covering of a Stein manifold is Stein, see [Ste56; Siu78], this implies that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is not Stein.

2.4 Examples of parabolic quotients of the ball

In the following two examples, we fix, as in Subsection 2.1, a basis $f = (f_1, f_2, e_1, \ldots, e_{n-1})$ of \mathbb{C}^{n+1} which induces an identification between parabolic subgroups of PU(n, 1) fixing $[f_1]$ and subgroups of $U(n-1) \ltimes N$.

Example. Here is an example of a discrete unipotent subgroup Γ of PU(n, 1) with $\delta(\Gamma) = 2$, for which $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is not holomorphically convex. The group Γ generated by $\gamma_1 := (\mathrm{Id}, e_1, 0)$ and $\gamma_2 := (\mathrm{Id}, ie_1, 0)$ is the set of all elements of the form

$$(\mathrm{Id}, (k_1 + ik_2)e_1, 2\ell - k_1k_2),$$

where $(k_1, k_2, \ell) \in \mathbb{Z}^3$. In particular, Γ is discrete, and Formula (7) shows that $\delta(\Gamma) = 2$. Finally, the quotient $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ naturally identifies with a bundle of punctured disks over the base $B := \mathbb{C}/(\mathbb{Z}+i\mathbb{Z}) \times \mathbb{C}^{n-2}$. If $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ were holomorphically convex, it would be Stein by Proposition 11-(a) and we would deduce that B is a Stein manifold by [CD97, Lemma 1.6], which is not the case. Therefore, $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is not holomorphically convex.

Example. Here is an example of a complex hyperbolic bundle of punctured disks over a Cousin group. We work in dimension n = 3, but this example generalizes to any dimension $n \ge 3$. Using the identification introduced before the previous example, define three vectors in $\mathbb{C}^2 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ by $v_1 = e_1, v_2 = e_2$, and $v_3 = ae_1 + be_2$ for $(a, b) \in \mathbb{C}^2$ two complex numbers such that

$$\begin{cases} \lambda := \Im(a) = \Im(b) \neq 0, \\ \Re(a) - \Re(b) \notin \mathbb{Q}. \end{cases}$$

The fact that $\Im(a) \neq 0$ and $\Im(b) \neq 0$ implies that v_1, v_2 and v_3 are \mathbb{R} -linearly independent, and both conditions together imply that 1, *a* and *b* are \mathbb{Z} -linearly independent. We deduce that the subgroup Γ_0 of \mathbb{C}^2 generated by v_1, v_2 and v_3 is discrete, and that the quotient \mathbb{C}^2/Γ_0 has no compact factor and does not admit any non-constant holomorphic function (see for example [Nap90, pages 451-452]). Let Γ be the subgroup of $U(n-1) \ltimes N$ generated by the three elements $\gamma_i = (\text{Id}, v_i, 0)$ for i = 1, 2 and 3. The equality $\Im(a) = \Im(b) = \lambda$ implies that

$$[\gamma_3, \gamma_1] = [\gamma_3, \gamma_2] = (\mathrm{Id}, 0, 2\lambda).$$

Any element of Γ is of the form $\gamma_1^{k_2} \gamma_2^{k_2} \gamma_3^{k_3} [\gamma_3, \gamma_1]^{\ell}$, with $(k_1, k_2, k_3, \ell) \in \mathbb{Z}^4$, and we deduce that Γ is the set of all elements of the form

 $(\mathrm{Id}, k_1v_2 + k_2v_2 + k_3v_3, ((k_1 + k_2)k_3 + 2\ell)\lambda).$

with $(k_1, k_2, k_3, \ell) \in \mathbb{Z}^4$. Consequently, Γ is discrete, and $\mathbb{H}^3_{\mathbb{C}}/\Gamma$ is biholomorphic to a bundle of punctured disks over \mathbb{C}^2/Γ_0 . Since \mathbb{C}^2/Γ_0 is not Stein, we deduce as in the previous example that $\mathbb{H}^3_{\mathbb{C}}/\Gamma$ is not holomorphically convex. Additionally, Formula (7) shows that $\delta(\Gamma) = \frac{5}{2}$.

3 Holomorphic convexity and geometrically finite subgroups

We begin by reviewing the structure of a geometrically finite quotient of the ball, then we prove Theorems 3 and 1.

3.1 A description of geometrically finite quotients of the ball

We first recall some general concepts about discrete subgroups of $\operatorname{PU}(n, 1)$. Let Γ be a discrete subgroup of $\operatorname{PU}(n, 1)$. Its *limit set* $\Lambda(\Gamma)$ is the closed subset of $\partial \mathbb{H}^n_{\mathbb{C}}$ defined as the accumulation set of an orbit Γo , for some point $o \in \mathbb{H}^n_{\mathbb{C}}$, and it does not depend on the choice of the point $o \in \mathbb{H}^n_{\mathbb{C}}$. The *domain of discontinuity* $\Omega(\Gamma)$ of Γ is an open subset of $\partial \mathbb{H}^n_{\mathbb{C}}$ which can be defined as the complement of the limit set. These sets are invariant under the action of Γ on $\partial \mathbb{H}^n_{\mathbb{C}}$, and in particular, the closed geodesic convex hull of the limit set forms a Γ -invariant closed subset of $\mathbb{H}^n_{\mathbb{C}}$. The quotient C_{Γ} of this convex hull by Γ is a closed subset of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$, called *convex core* of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$.

We now pass to geometrically finite groups, for the definition of which we refer to [Bow95]. Let Γ be a geometrically finite and torsion-free subgroup of PU(n, 1). We are going to use the following description of $X_{\Gamma} := \mathbb{H}^n_{\mathbb{C}}/\Gamma$. This manifold has a boundary $\partial X_{\Gamma} := \Omega(\Gamma)/\Gamma$ and it decomposes as

$$X_{\Gamma} =: Q \cup \bigcup_{i=1}^{k} E_i,$$

where Q is relatively compact in $X_{\Gamma} \cup \partial X_{\Gamma}$, k is an integer, and for $i \in \{1, \ldots, k\}$, each E_i is an open subset of X_{Γ} biholomorphic to the quotient of a horoball $b_i^{-1}((-\infty, 0))$ by a maximal parabolic subgroup P_i of Γ , for some Busemann function b_i . Moreoever $C_{\Gamma} \cap Q$ is compact.

3.2 Proof of Theorem 3

For the proof of Theorem 3, we will need the following lemma about parabolic quotients of the ball.

Lemma 17. Let P be a discrete and torsion-free parabolic subgroup of PU(n, 1), and $\xi \in \partial \mathbb{H}^n_{\mathbb{C}}$ a point fixed by P. The following statements are equivalent:

1. $\mathbb{H}^n_{\mathbb{C}}/P$ is a Stein manifold.

- 2. For any horoball $H \subset \mathbb{H}^n_{\mathbb{C}}$ at ξ , the quotient H/P is a Stein manifold.
- 3. There exists a horoball $H \subset \mathbb{H}^n_{\mathbb{C}}$ at ξ for which H/P is a Stein manifold.

Proof. $(1 \implies 2)$ If $\mathbb{H}^n_{\mathbb{C}}/P$ is a Stein manifold, it admits a strictly plurisubharmonic exhaustion function $\psi : \mathbb{H}^n_{\mathbb{C}}/P \to \mathbb{R}_+$. The Busemann function $b : \mathbb{H}^n_{\mathbb{C}} \to \mathbb{R}$ at ξ is invariant under the action of P. Let $H_{\lambda} := b^{-1}((-\infty, \lambda))$ be a horoball at ξ . The function $\psi + \frac{1}{\lambda - b}$ defined on H_{λ}/P is a strictly plurisubharmonic exhaustion function on H_{λ}/P , which shows that H_{λ}/P is a Stein manifold.

The implication $2 \implies 3$ is immediate. We now show that $3 \implies 2$. We fix, as in Subsection 2.1, a basis $f = (f_1, f_2, e_3, \ldots, e_{n+1})$ of \mathbb{C}^{n+1} which induces an identification between parabolic elements of PU(n, 1) fixing $\xi = [f_1]$ and elements of $U(n-1) \ltimes N$. The elements of P, seen as biholomorphisms of \mathbb{CP}^n , commute with the biholomorphisms $L_t : \mathbb{CP}^n \to \mathbb{CP}^n$ defined for all real numbers t in the basis f by the matrices

$$\begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathbf{I}_{n-1} \end{pmatrix}.$$

With the notations of Subsection 2.1, it is easily checked that for any pair (λ, μ) of real numbers, the map L_t with $t = e^{-2\lambda} - e^{-2\mu}$ sends the horoball $H_{\lambda} := b^{-1}((-\infty, \lambda))$ to the horoball $H_{\mu} := b^{-1}((-\infty, \mu))$. We deduce that the quotients of horoballs at ξ by P are all biholomorphic. $(2 \implies 1)$ The manifold $\mathbb{H}^n_{\mathbb{C}}/P$ is the union of a one-parameter family of Stein manifolds, so it is Stein by the theorem of Docquier-Grauert recalled in Subsection 1.1.

We now come to the proof of Theorem 3.

Proof of Theorem 3. $(1 \implies 2)$ Suppose that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ admits a plurisubharmonic exhaustion function $\phi : \mathbb{H}^n_{\mathbb{C}}/\Gamma \to \mathbb{R}$. Let P be a maximal parabolic subgroup of Γ . There exists a Busemann function b, invariant under P, such that the set $C := b^{-1}((-\infty, 0))/P$ is biholomorphic to an open subset of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$. The function $\phi|_C + \frac{-1}{b}$ is a strictly plurisubharmonic exhaustion function of C, and therefore C is a Stein manifold. Using Lemma 17, we deduce that $\mathbb{H}^n_{\mathbb{C}}/P$ is a Stein manifold. If now P is any parabolic subgroup of Γ , it is contained in a maximal parabolic subgroup P_0 of Γ . The manifold $\mathbb{H}^n_{\mathbb{C}}/P$ is a covering of $\mathbb{H}^n_{\mathbb{C}}/P_0$, which is Stein, and therefore $\mathbb{H}^n_{\mathbb{C}}/P$ is Stein.

 $(2 \implies 3)$ This proof is inspired by [Che13, Proof of Theorem 1.4]. Suppose that, for every maximal parabolic subgroup $P < \Gamma$, the quotient $\mathbb{H}^n_{\mathbb{C}}/P$ is a Stein manifold. As explained in Subsection 3.1, the manifold $X_{\Gamma} := \mathbb{H}^n_{\mathbb{C}}/\Gamma$ decomposes as

$$X_{\Gamma} =: Q \cup \bigcup_{i=1}^{k} E_i,$$

where Q is relatively compact in $X_{\Gamma} \cup \partial X_{\Gamma}$, k is an integer, and for $i \in \{1, \ldots, k\}$, each E_i is an open subset in X_{Γ} biholomorphic to the quotient of a horoball $b_i^{-1}((-\infty, 0))$ by a maximal parabolic subgroup P_i of Γ , for some Busemann function b_i . We also define $E'_i \subset E_i$ as the quotient of the horoball $b_i^{-1}((-\infty, -1))$ by P_i . Moreover, the convex core C_{Γ} of X_{Γ} has compact intersection with Q. By the arguments given in the proofs of Proposition 6 and Lemma 9, we get that the squared distance function to the convex hull $C(\Gamma)$ of the limit set descends to a convex function ϕ on X_{Γ} , which is strictly convex outside C_{Γ} . By Richberg's theorem, there exists a continuous plurisubharmonic function ϕ which is smooth and strictly plurisubharmonic outside C_{Γ} , and such that

$$\phi \le \widetilde{\phi} \le \phi + \frac{1}{2},$$

see [Dem, Theorem I.5.21]. Moreover, Lemma 17 implies that for any $i \in \{1, \ldots, k\}$, the open subset E_i of X_{Γ} is a Stein manifold, and admits a strictly plurisubharmonic exhaustion function. Let Ψ_i be a smooth non-negative function that coincides with this function on E'_i and vanishes outside E_i . For any integer $j \in \mathbb{N}$, let T^i_j be the compact subset of X_{Γ} defined by

$$T_j^i := \{ x \in \overline{E_i} \setminus E_i' \mid j \le \widetilde{\phi}(x) \le j+1 \}.$$

Then, as soon as $j \geq 1$, the function ϕ is strictly plurisubharmonic on T_i^j , so there exists a constant $\beta_j^i > 0$ such that $i\partial \bar{\partial} \Psi_i \geq -\beta_j^i i \partial \bar{\partial} \phi$ on T_j^i . It follows that there exists a strictly increasing convex function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lambda(t) \xrightarrow[t \to +\infty]{} +\infty$ and such that

$$N:=\lambda\circ\widetilde{\phi}+\sum_{i=1}^k\Psi_i$$

is strictly plurisubharmonic on the set

$$\bigcup_{i=1}^k \bigcup_{j\ge 1} T_j^i.$$

On Q, this function N coincides with $\lambda \circ \tilde{\phi}$ and it is strictly plurisubharmonic on $Q \cap X_{\Gamma} \setminus C_{\Gamma}$. On each E'_i , since Ψ_i is strictly plurisubharmonic and $\tilde{\phi}$ is plurisubharmonic, N is strictly plurisubharmonic outside the compact set

$$(C_{\Gamma} \cap Q) \cup \bigcup_{i=1}^{k} T_0^i.$$

Moreover, N is an exhaustion function. Indeed, if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X_{Γ} without accumulation point, then, after extracting a subsequence, it converges to the boundary ∂X_{Γ} or has values in one of the open sets E'_i . In the first case where $x_n \longrightarrow x_{\infty} \in \Omega(\Gamma)/\Gamma$, we claim that $\phi(x_n) \longrightarrow +\infty$. Assuming the contrary, we obtain a sequence $(\widetilde{x_n})_{n\in\mathbb{N}}$ in $\mathbb{H}^n_{\mathbb{C}}$ converging to an element $\widetilde{x_{\infty}} \in \Omega(\Gamma)$, which remains at bounded distance from $C(\Gamma)$, and thus another sequence $(c_n)_{n\in\mathbb{N}}$ in $C(\Gamma)$ converging to $\widetilde{x_{\infty}}$. Thus, $\widetilde{x_{\infty}} \in \Omega(\Gamma) \cap \partial C(\Gamma)$. This is a contradiction, because $\partial C(\Gamma) = \Lambda(\Gamma)$, see [And83]. In the case where the sequence lies in E'_i , it does not accumulate and therefore, after after passing to a subsequence, we have $\Psi(x_n) \to +\infty$. Thus N is an exhaustion function. Finally, a second application of Richberg's theorem shows that there exists a continuous exhaustion function $\widetilde{N} : X_{\Gamma} \to \mathbb{R}$ which is smooth and strictly plurisubharmonic outside a compact set. Therefore, X is holomorphically convex.

The implication $3 \implies 1$ is classical, see for example [Dem, Theorem I.6.14].

Remark. If we replace $\mathbb{H}^n_{\mathbb{C}}$ by a simply connected complete Kähler manifold (X, ω) with negatively pinched sectional curvature, and assume that Γ is a group acting freely and geometrically finitely by holomorphic isometries on X, I do not know if Lemma 17 remains true (the proof uses the holomorphic maps L_t whose existence is specific to the complex hyperbolic case). In Theorem 3, it remains true that $1 \iff 3$. To show that $1 \implies 3$, one argues as in the proof above,

noticing that if X/Γ admits a plurisubharmonic exhaustion ϕ , then the open sets E_i appearing in the decomposition

$$X/\Gamma = Q \cup \bigcup_{i=1}^{\kappa} E_i$$

are Stein manifolds, with a strictly plurisubharmonic exhaustion given by $\phi + \frac{-1}{b_i}$, where b_i is a Busemann function on X associated to a parabolic point corresponding to the cusp E_i .

3.3 Proof of Theorem 1

For the proof of Theorem 1-(a), we will need the following lemma, which is presumably classical, and the proof of which we include for completeness.

Lemma 18. Let X be a complete simply connected Riemannian manifold with negatively pinched curvature, and P a discrete and torsion-free parabolic subgroup of isometries of X. Then P is cyclic or contains a copy of \mathbb{Z}^2 .

Proof of Lemma 18. By Margulis' lemma, P contains a finite-index nilpotent subgroup P'. Moreover, P' is finitely generated according to [Bow93]. We distinguish two cases:

- If P' is Abelian, then P' is cyclic or contains a copy of Z². Since a virtually cyclic torsion-free group is cyclic, we deduce that P is cyclic or contains a copy of Z².
- Otherwise, let g be a non-trivial element in the center of P', and h an element of P' which does not belong to the center of P'. Then g and h generate a subgroup isomorphic to \mathbb{Z} or \mathbb{Z}^2 . Suppose, by contradiction, that this group is cyclic. Then g and h are powers of an element $k \in P'$. In a torsion-free and finitely generated nilpotent group, the centralizers of an element and its powers coincide, and consequently g and h have the same centralizer in P'. This yields a contradiction, and consequently P contains a copy of \mathbb{Z}^2 .

Proof of Theorem 1. Let Γ be a geometrically finite and torsion-free subgroup of PU(n, 1). Assume first that Γ is Gromov-hyperbolic. Then Γ does not contain a copy of \mathbb{Z}^2 , see [BH99, Corollary III. Γ .3.10], so according to Lemma 18, the non-trivial parabolic subgroups of Γ are cyclic. The quotient of the complex hyperbolic space by the action of a cyclic parabolic group is a Stein manifold, as follows, for example, from Theorem 2, see also [dFI01] or [Mie10]. Theorem 3 implies that $\mathbb{H}^n_{\Gamma}/\Gamma$ is holomorphically convex.

In particular, if Γ is free, then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is holomorphically convex. Using Proposition 11-(e), we deduce that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is Stein.

Suppose now that $\delta(\Gamma) < 2$. For any parabolic subgroup $P < \Gamma$, we have $\delta(P) \leq \delta(\Gamma) < 2$, so $\mathbb{H}^n_{\mathbb{C}}/P$ is Stein according to Theorem 2. Using Theorem 3, we deduce that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is holomorphically convex. Since $\delta(\Gamma) < 2$, this manifold does not contain any compact analytic subvariety of positive dimension according to [DK20, Theorem 15] or Proposition 6. We deduce that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is Stein.

Finally, suppose that Γ preserves a totally real and totally geodesic submanifold $\mathbb{H}^k_{\mathbb{R}} \subset \mathbb{H}^n_{\mathbb{C}}$. Then according to Corollary 15, for any parabolic subgroup P of Γ , the quotient $\mathbb{H}^n_{\mathbb{C}}/P$ is a Stein manifold. Using Theorem 3, we deduce that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is holomorphically convex. This manifold does not contain any compact analytic subvariety of positive dimension, according for example to Proposition 11-(c). Thus $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is Stein.

We conclude this section by a remark which follows from Lemma 18.

Remark. Let X be a complete simply connected Riemannian manifold with negatively pinched curvature, and Γ a group containing no copy of \mathbb{Z}^2 . If Γ acts faithfully, discretely, and geometrically finitely by isometries on X, then Γ is Gromov-hyperbolic. Indeed, a geometrically finite group is hyperbolic relative to its parabolic subgroups. Under the assumption that Γ contains no copies of \mathbb{Z}^2 , the parabolic subgroups of Γ are cyclic, and in particular Gromov-hyperbolic. This implies that Γ itself is Gromov-hyperbolic, see [Osi06]. We can use this fact to exhibit finitely generated groups which admit a discrete and faithful representation in PU(n, 1) but no discrete, faithful, and geometrically finite representation of N in PU(n, 1). To do this, let first Γ_0 be a cocompact arithmetic lattice of the simplest type of PU(n, 1), for the definition of which we refer to [BW00, §VIII.5]. Then there exists a finite-index torsion-free subgroup $\Gamma < \Gamma_0$ and a morphism $\phi : \Gamma \to \mathbb{Z}$ such that $N := \ker(\phi)$ is finitely generated but not hyperbolic, see [LP24], and also [IMM23] for related results. As a subgroup of Γ , the group N cannot contain a copy of \mathbb{Z}^2 . Thus, there exists by construction a discrete and faithful representation of N in PU(n, 1), but there is no discrete, faithful, and geometrically finite representation of N in PU(n, 1).

4 Discrete subgroups with critical exponent equal to 2

In this section we give two proofs of Theorem 4, using the techniques developped in [CMW23]. The first one uses the function f defined in Lemma 10. The second one involves the *natural flow*, which is defined by the complete vector field $\mathfrak{X} = \nabla f$.

Proof of Theorem 4. Let Γ be a discrete and torsion-free subgroup of $\operatorname{PU}(n,1)$ with critical exponent $\delta = 2$ and assume that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ contains a compact subvariety A of positive dimension. First, we remark that Γ is non-elementary, as a consequence of Proposition 11-(a) and (c). Thus Γ admits a Patterson-Sullivan measure $(\mu_x)_{x \in \mathbb{H}^n_{\mathbb{C}}}$. Fix a point $p \in \mathbb{H}^n_{\mathbb{C}}$ and, for all $\theta \in \partial \mathbb{H}^n_{\mathbb{C}}$, denote by $B_{\theta} := B_{\theta}(\cdot, p)$ the Busemann function at θ which vanishes at p. Let f be the Γ -invariant function defined on $\mathbb{H}^n_{\mathbb{C}}$ by $f(x) := -\ln \|\mu_x\|$. By Lemma 10, for all tangent vector v at a point $x \in \mathbb{H}^n_{\mathbb{C}}$, we have

$$i\partial\overline{\partial}f(v,Jv) \ge 0.$$

Let $\widetilde{A} \subset \mathbb{H}^n_{\mathbb{C}}$ be the lift of A, x be a regular point of \widetilde{A} and v be a non-zero vector in $T_x \widetilde{A}$. Then the plurisubharmonic function $f|_{\widetilde{A}}$ is constant and consequently $i\partial\overline{\partial}f(v, Jv) = 0$. The inequality given by Lemma 10 is thus an equality for this vector v. Using that Patterson-Sullivan measures are supported on $\Lambda(\Gamma)$, one sees that this equality can only happen when

$$\forall \theta \in \Lambda(\Gamma), dB_{\theta}(v)^2 + dB_{\theta}(Jv)^2 = ||v||^2.$$

This is possible if and only if $v \in \mathbb{C}v_{x\theta}$ for all $\theta \in \Lambda(\Gamma)$ and all $v \in T_x \widetilde{A}$, where $v_{x\theta} \in T_x \mathbb{H}^n_{\mathbb{C}}$ is the unit vector at x pointing in the direction of θ . We deduce that A has dimension 1 and that

$$\forall \theta \in \Lambda(\Gamma), \ v_{x\theta} \in T_x \widetilde{A}.$$

Let D be the unique complex geodesic containing x for which $T_x D = T_x \widetilde{A}$. Then $\Lambda(\Gamma) \subset \partial D$, and hence the convex hull of $\Lambda(\Gamma)$ is contained in D. We deduce that Γ preserves D. Moreover $A \subset D/\Gamma$ by Proposition 5. To conclude, notice that D/Γ is a Riemann surface containing a compact subvariety of positive dimension, so D/Γ is compact and Γ is a complex Fuchsian group.

Remark. We now outline a second proof of Theorem 4, inspired by [CMW23, Theorem 1.5]. According to [CMW23, Lemma 2.2], the vector field \mathfrak{X} defined by ∇f on $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is complete, and defines a smooth flow $(\phi_t)_{t \in \mathbb{R}}$. If $x \in \mathbb{H}^n_{\mathbb{C}}$ and (Y_1, \ldots, Y_k) is a k-frame of $T_x \mathbb{H}^n_{\mathbb{C}}$ which spans a subspace V, then the infinitesimal contraction rate of this frame by $(\phi_t)_{t \in \mathbb{R}}$ is given by the real number $\operatorname{tr}(\nabla df(x)|_V)$ (see [CMW23, Lemma 2.5] for a precise statement). For $x \in \mathbb{H}^n_{\mathbb{C}}$ and $\theta \in \partial \mathbb{H}^n_{\mathbb{C}}$, there is a real basis $(e_1, e_2, \ldots, e_{2n})$ of $T_x \mathbb{H}^n_{\mathbb{C}}$ with $e_1 = v_{x\theta}$ the unit vector pointing in the direction of θ and $e_2 = Je_1$, in which the matrix of $L_{\theta} + J^*L_{\theta}$ is $\operatorname{Diag}(2 - \delta, 2 - \delta, 2, \ldots, 2)$. In particular, if $\delta(\Gamma) = 2$ and $V \subset T_x \mathbb{H}^n_{\mathbb{C}}$ is a complex subspace, then

$$\operatorname{tr}(L_{\theta|V}) = \frac{1}{2} \operatorname{tr}\left((L_{\theta} + J^* L_{\theta})|_V\right) \ge 0,$$

with equality if and only if V has complex dimension 1 and $V = \mathbb{C}v_{x\theta}$. Let \widetilde{A} be the lift in $\mathbb{H}^n_{\mathbb{C}}$ of a compact subvariety of positive dimension $A \subset \mathbb{H}^n_{\mathbb{C}}/\Gamma$. Then we have for all regular point xof \widetilde{A}

$$\mathrm{tr}(\nabla df|_{T_x\widetilde{A}}) \geq \int_{\partial \mathbb{H}^n_{\mathbb{C}}} \mathrm{tr}(L_{\theta|T_x\widetilde{A}}) d\mu_x(\theta) \geq 0.$$

If, for some regular point x of A, the above inequality was strict, then ϕ_{-t} would contract A for sufficiently small t > 0, which would contradict the fact that A is a volume minimizer in its homology class. Fixing from now on a regular point x of A, we deduce that A has dimension 1 and, since Patterson-Sullivan measures of Γ are supported on the limit set $\Lambda(\Gamma)$ of Γ , we get

$$\forall \theta \in \Lambda(\Gamma), v_{x\theta} \in T_x A.$$

We conclude as in the first proof.

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