Hypergraphs and homogeneous Lotka-Volterra systems with linear Darboux polynomials

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Abstract

We associate homogeneous n-component Lotka-Volterra systems which admit k additional linear Darboux polynomials, with admissible hypergraphs of order n and size k. We study the equivalence relation on admissible hypergraphs induced by linear transformations of the associated LV-systems, for $n \leq 5$. We present a new 13-parameter 5-component superintegrable Lotka-Volterra system, i.e. one that is not equivalent to a so-called tree-system. We conjecture that tree-systems associated with nonisomorphic trees are not equivalent, which we verified for $n < 9$.

1 Introduction

In [5, 6] a one-to-one correspondence was established between trees on n vertices and homogeneous n-component Lotka-Volterra systems with $n-1$ linear Darboux polynomials. These so-called tree-systems contain $3n-2$ parameters and were shown to be maximally superintegrable, cf. [7] where tree-systems were shown to be measure-preserving with a rational measure that is also preserved by their Kahan discretisation.

For tree-systems, the linear Darboux polynomials depend on only two variables, and they correspond to the edges of the associated tree. If we allow Darboux polynomials to depends on more than $m = 2$ variables, the corresponding edges become hyperedges (connecting m vertices), and hence the associated graphs become hypergraphs. An n-component Lotka-Volterra system with k linear Darboux polynomials can be identified with a hypergraph on n-vertices with k hyperedges. However, not all hypergraphs correspond to a class of LV-systems. We will call the ones that do correspond to a class of LV-systems *admissible hypergraphs*. The hypergraph of an LV-system A is not necessarily isomorphic to the hypergraph of an LV-system B obtained from A by a linear transformation of the variables. This is due to the fact that although Darboux polynomials are invariant under linear transformations, the number of variables they depend on is not. Thus, linear transformations induce an equivalence relation on the set of admissible hypergraphs with a fixed number of hyperedges, and it is this equivalence relation that we are interested in.

The paper is organised as follows. In the next section, we start with some preliminaries on Darboux polynomials, and how these give rise to integrals for LV-systems. We provide the conditions on the matrix $\mathbf{A}$ and the vector $\mathbf{b}$ such that the linear combination, with nonzero coefficients

$$P_{i_1,\dots,i_m} = \alpha_1 x_{i_1} + \alpha_2 x_{i_2} + \cdots + \alpha_m x_{i_m}, \tag{1}$$

is a Darboux polynomial for the Lotka-Volterra system

$$\frac{dx_i}{dt} = x_i \left(b_i + \sum_{j=1}^{n} A_{ij} x_j \right), \quad i = 1, \ldots, n. \tag{2}$$

In section 3, we introduce the reader to hypergraphs. We give some examples of admissible and nonadmissible hypergraphs, as well as equivalent and nonequivalent ones (with respect to linear transformation of their associated LV-systems). In section 4, we provide all equivalent classes of admissible hypergraphs on n vertices, for $2 \leq n \leq 5$, and for each of them we provide a representative LV-system. We have found one 5-component maximally superintegrable LV-system that is not equivalent to a tree-system. We expect there exist higher component maximally superintegrable LV-systems that are not equivalent to tree-systems. We have verified, for $n < 9$, that n-component LV-systems with trees that are not isomorphic are not linearly equivalent. We conjecture the statement is true for all n.

2 Linear Darboux polynomials for LV-systems

A polynomial $P(x)$ is a Darboux Polynomial (DP) for a ODE $\frac{dx}{dt} = f(x)$ (where f is a polynomial), if there exists another polynomial $C(x)$, called the cofactor, such that $\frac{dP}{dt} = CP$. One observes that each n-component LV system, Eq. (2), has at least n DPs, namely the variables x_i, $i = 1, \ldots, n$.

An integral is a DP with cofactor 0. As a product of powers of DPs is a DP whose cofactor is a linear combination of the cofactors of the DPs, constructing integrals becomes a linear algebra problem once one has sufficiently many DPs, cf. [1]. Thus one can say that LV systems are on the verge of having an integral; one more linear DP suffices. An elementary method to determine k integrals from k additional linear DPs for a homogeneous LV-system (Eq. (2) with $\mathbf{b} = \mathbf{0}$), given in [5, Section 2], is the following. Let $\mathbf{B}$ be the $k \times n$ matrix whose ith row contains the coefficients of the cofactor of the ith DP, P_i, i.e. we have, for $i = 1, \ldots, k$, $\dot{P}_i = P_i \sum_j B_{ij} x_j$. Then k integrals are given by, with $\mathbf{Z} = -\mathbf{B}\mathbf{A}^{-1}$,

$$P_i \prod_{j=1}^{n} x_j^{Z_{i,j}}, \quad i = 1, \ldots, m. \tag{3}$$

Notation 1. *Let $N = \{1, 2, \ldots, n\}$. For all $I = \{i_1, \ldots, i_m\} \subset N$ we denote $I^{\mathrm{c}} = N \setminus I$. A DP of the form* (1)*, i.e. linear in x_i for all $i \in I$, with $\alpha_i \neq 0$, will be variously called an I-DP, or an m-DP.*

In [5], it was also shown that the expression $P_{i,k} = \alpha x_i + \beta x_k$ with $\alpha\beta \neq 0$, is a DP of Eq. (2) if and only if, for some constant b and all $j \in \{i, k\}^c$, we have

$$A_{i,j} = A_{k,j}, \quad b_i = b_k \ = \ b \quad \alpha(A_{k,k} - A_{i,k}) = \beta(A_{k,i} - A_{i,i}),$$

and $(A_{k,k} - A_{i,k})(A_{k,i} - A_{i,i}) \neq 0$.[1] Such a DP is a 2-DP and is associated to an edge in a graph. In [6], it was shown that one can choose $n-1$ of these 2-DPs so that the associated graph is a tree. There are then $3n - 2$ free parameters left in the matrix $\mathbf{A}$, and, for generic values of these parameters, the corresponding $n-1$ integrals are functionally independent [6]. We will now consider m-DPs with $m > 2$.

Lemma 2. *The expression* (1) *is an m-DP for* (2)*, with*

$$(\mathbf{A}_{i_j,i_j} - \mathbf{A}_{i_1,i_j})(\mathbf{A}_{i_1,i_1} - \mathbf{A}_{i_j,i_1}) \neq 0 \quad \forall j = 2, 3, \ldots, m,$$

[1] The statement is true when $\det(A) \neq 0$, otherwise we also have that $P_{i,k}$ is a DP for all α, β if $A_{k,k} - A_{i,k} = A_{k,i} - A_{i,i}) = 0$.

iff, for some constant b,

$$\begin{aligned} b_i &= b \quad \forall i \in I && (4)\\ \mathbf{A}_{i,j} &= \mathbf{A}_{k,j} \quad \forall i,k \in I, j \in I^{\mathrm{c}} && (5)\\ \alpha_1(\mathbf{A}_{i_j,i_j} - \mathbf{A}_{i_1,i_j}) &= \alpha_j(\mathbf{A}_{i_j,i_1} - \mathbf{A}_{i_1,i_1}) \quad \forall j = 2,3,\ldots,m, && (6)\end{aligned}$$

and the following $(m-1)(m-2)/2$ *additional conditions on the entries* $\mathbf{A}_{i,j}$ *with* $i,j \in I$ *hold:* $C_{j,k} = 0$*, for* $j < k \in \{2,3,\ldots,m\}$*, where, cf. [3, Eq. (1.5)],*

$$\begin{aligned} C_{j,k} := {} & (\mathbf{A}_{i_k,i_1} - \mathbf{A}_{i_1,i_1})(\mathbf{A}_{i_j,i_j} - \mathbf{A}_{i_1,i_j})(\mathbf{A}_{i_j,i_k} - \mathbf{A}_{i_k,i_k}) \\ & + (\mathbf{A}_{i_j,i_1} - \mathbf{A}_{i_1,i_1})(\mathbf{A}_{i_k,i_j} - \mathbf{A}_{i_j,i_j})(\mathbf{A}_{i_k,i_k} - \mathbf{A}_{i_1,i_k}). \end{aligned}$$

Proof. The expression $P = P_{i_1,\ldots,i_m}$, as given by Eq. (1), is a DP for LV-system (2) iff there is an affine function

$$C = \beta_0 + \sum_{i=1}^{n} \beta_i x_i$$

such that

$$\dot{P} - CP = \sum_{i=1}^{n} d_i x_i + \sum_{1 \le i \le j \le n} c_{i,j} x_i x_j = 0.$$

Hence, P is a DP iff $d_i = 0$ for all $i \in N$ and $c_{i,j} = 0$ for all $i,j \in N$ such that $i \le j$. We have

$$d_i = \begin{cases} \alpha_i(nb_i - \beta_0) & i \in I, \\ 0 & i \in I^{\mathrm{c}}. \end{cases}$$

Defining $b = \beta_0/n$, we have $b_i = b$ for all $i \in I$, (4). For $i_k \in I$ and $j \in I^{\mathrm{c}}$ we have $c_{i_k,j} = \alpha_k(\mathbf{A}_{i_k,j} - \beta_j)$, the vanishing of which is equivalent to condition (5), and it yields (our choice of parametrisation) $\beta_j = \mathbf{A}_{i_1,j}$. For $i_j \in I$, we find $c_{i_j,i_j} = \alpha_j(\mathbf{A}_{i_j,i_j} - \beta_{i_j})$ from which we find the values of the coefficients of the cofactor,

$$\beta_j = \mathbf{A}_{j,j} \tag{7}$$

when $j \in I$. For $i,j \in I^{\mathrm{c}}$ the coefficients $c_{i,j}$ and $c_{i,i}$ are identically zero. For $i_k, i_j \in I$ and $i_j \neq i_k$, we have

$$c_{i_k,i_j} = (\mathbf{A}_{i_k,i_j} - \beta_{i_j})\alpha_k + (\mathbf{A}_{i_j,i_k} - \beta_{i_k})\alpha_j. \tag{8}$$

Taking $k = 1$ and $j \in \{2,3,\ldots,m\}$ in (8), we find, using (7) the condition (6). The equations (8) comprise $\binom{m}{2}$ conditions, and we have used $m-1$ of them. Thus there are $\binom{m}{2} - m + 1 = (m-1)(m-2)/2$ additional conditions. Consider $k, j \neq k \in \{2,3,\ldots,m\}$, i.e. precisely $\binom{m-1}{2} = (m-1)(m-2)/2$ cases. Setting

$$\alpha_j = \alpha_1 \frac{\mathbf{A}_{i_j,i_j} - \mathbf{A}_{i_1,i_j}}{\mathbf{A}_{i_j,i_1} - \mathbf{A}_{i_1,i_1}}, \quad \alpha_k = \alpha_1 \frac{\mathbf{A}_{i_k,i_k} - \mathbf{A}_{i_1,i_k}}{\mathbf{A}_{i_k,i_1} - \mathbf{A}_{i_1,i_1}}$$

and using (7), the condition $c_{i_k,i_j} = 0$ yields

$$(\mathbf{A}_{i_k,i_j} - \mathbf{A}_{i_j,i_j}) \frac{\mathbf{A}_{i_k,i_k} - \mathbf{A}_{i_1,i_k}}{\mathbf{A}_{i_k,i_1} - \mathbf{A}_{i_1,i_1}} + (\mathbf{A}_{i_j,i_k} - \mathbf{A}_{i_k,i_k}) \frac{\mathbf{A}_{i_j,i_j} - \mathbf{A}_{i_1,i_j}}{\mathbf{A}_{i_j,i_1} - \mathbf{A}_{i_1,i_1}} = 0,$$

which is equivalent to the vanishing of $C_{k,j}$. □

Remark 3. *Note that in the above lemma (and its proof) we have singled out the first coefficient of the DP,* α_1*, in which to express the others. We could have made a different choice. By taking* $k = h$ *and* $j \in \{1,2,\ldots,m\} \setminus \{h\}$ *in (8), one would find*

$$\alpha_h(\mathbf{A}_{i_j,i_j} - \mathbf{A}_{i_h,i_j}) = \alpha_j(\mathbf{A}_{i_j,i_h} - \mathbf{A}_{i_h,i_h}) \neq 0,$$

and the remaining conditions would be

$$(\mathbf{A}_{i_k,i_h} - \mathbf{A}_{i_h,i_h})(\mathbf{A}_{i_j,i_j} - \mathbf{A}_{i_h,i_j})(\mathbf{A}_{i_j,i_k} - \mathbf{A}_{i_k,i_k}) + (\mathbf{A}_{i_j,i_h} - \mathbf{A}_{i_h,i_h})(\mathbf{A}_{i_k,i_j} - \mathbf{A}_{i_j,i_j})(\mathbf{A}_{i_k,i_k} - \mathbf{A}_{i_h,i_k}) = 0, \tag{9}$$

where $j,k \in \{1,2,\ldots,m\} \setminus \{h\}$.

3 Admissible hypergraphs, equivalence

By a hypergraph on n vertices (undirected) we mean a subset of the powerset of $N = \{1, 2, \ldots, n\}$. The number of vertices is the order and the number of (hyper)edges is the size of the hypergraph. An example of a hypergraph of order 3 and size 6 is $\{\{1\}, \{2\}, \{3\}, \{1, 3\}, \{1, 2, 3\}\}$. This would be the hypergraph associated to the 3-component LV-system with two additional Darboux polynomials

$$x_1 + 2x_3, \quad 3x_1 + 4x_2 + 6x_3. \tag{10}$$

Two hypergraphs are isomorphic if they can be obtained from each other by a permutation of N, e.g. the above hypergraph is isomorphic to $\{\{1\}, \{2\}, \{3\}, \{2, 3\}, \{1, 2, 3\}\}$. Because we can relabel the variables of the LV-system, we are only interested in nonisomorphic hypergraphs. Moreover, because we restrict ourselves to the study of LV-systems, and their hypergraphs always contain the full set of singletons $\{\{1\}, \ldots, \{n\}\}$, we may as well omit them, i.e. in the sequel we assume our hypergraphs do not contain loops (and also no multiple edges). Thus, to the above mentioned LV-system we then associate the hypergraph of size 2, $\{\{1, 3\}, \{1, 2, 3\}\}$, reflecting only the additional DPs. The number of nonisomorphic hypergraphs of order n grows quite rapidly [4, A000612]

$$1, 2, 6, 40, 1992, 18666624, 12813206169137152, 33758171486592987164087845043830784, \ldots.$$

The triangular array counting the number of hypergraphs of order n and size k is given by [4, A371830]. Excluding singletons, the number of hypergraphs of order n is [4, A317794]

$$1, 1, 2, 8, 180, 612032, 200253854316544, 263735716028826427534807159537664, \ldots.$$

We have generated all hypergraphs of order n and size k, where the degree of each hyperedge is at least 2, for $n \leq 5$ and, when $k = 5$, $k \leq 7$. Their number is given in Table 1, cf. [2].

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11
2	1	1										
3	1	2	2	2	1							
4	1	3	7	16	28	35	35	28	16	7	3	1
5	1	4	15	62	243	841	2544	6672				

Table 1: The number of hypergraphs of order n and size $k > 1$, where the degree of each hyperedge is at least 2.

For $n = 3$, the set of nonisomorphic hypergraphs is given in Fig. 1.

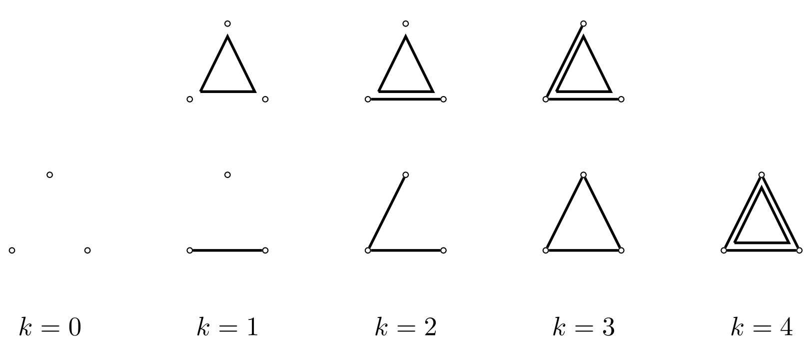

Figure 1: Nonisomorphic hypergraphs on 3 vertices. Edges represent 2-DPs, and triangles represent 3-DPs (if the hypergraph is admissible).

As we now know, cf. [5, 6], for every tree there is a class of LV-systems, whose DPs/independent integrals correspond to the edges of the tree. The question arises whether for every hypergraph there is a corresponding class of LV-systems. This is not the case, as is illustrated by the following examples. Hypergraphs with a corresponding class of LV-systems will be called LV-admissible, or admissible for short. For admissible hypergraphs, we would be interested to know the number of free parameters.

Example 4. *Consider the hypergraph on three vertices with $k = 4$ hyperedges, see Fig. 1. The condition on the entries of*

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$$

for the LV-system to admit a 3-DP are

$$(a_2 - a_5)(a_1 - a_7)(a_6 - a_9) - (a_3 - a_9)(a_1 - a_4)(a_5 - a_8) = 0 \tag{11}$$

as well as

$$(a_5 - a_2)(a_1 - a_4)(a_9 - a_3)(a_1 - a_7) \neq 0. \tag{12}$$

The latter is to ensure that none of the coefficients of the 3-DP

$$(a_1 - a_4)(a_1 - a_7)x_1 + (a_1 - a_7)(a_2 - a_5)x_2 + (a_1 - a_4)(a_3 - a_9)x_3$$

vanishes. For the LV-system to admit three 2-DPs the requirements are

$$a_8 = a_2, a_7 = a_4, a_6 = a_3. \tag{13}$$

Substituting (13) *into* (11) *yields*

$$2(a_2 - a_5)(a_1 - a_4)(a_3 - a_9) = 0,$$

which violates (12)*. Hence, the hypergraph on three vertices with four hyperedges is not admissible.*

In the next two examples, the conditions can be satisfied but there are consequences.

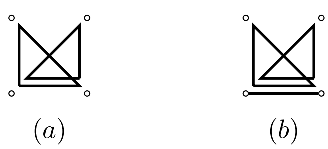

Figure 2: Hypergraphs on 4 vertices with 2 resp. 3 hyperedges.

Example 5. *Consider the hypergraph on four vertices with two hyperedges as in Fig. 2(a). The LV-system with interaction matrix*

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{pmatrix}$$

has an $\{1,2,3\}$-DP if $C_{2,3} = 0$ and $a_4 = a_8 = a_{12}$. It has a $\{1,2,4\}$-DP if $C_{2,4} = 0$ and $a_3 = a_7 = a_{15}$. So, as a consequence, we have $a_4 = a_8$ and $a_3 = a_7$, which implies that the LV-system also admits a $\{1,2\}$-DP. Hence, the hypergraph in Fig. 2(a) is not admissible. On the other hand, the hypergraph in Fig. 2(b) is admissible (with 10 free parameters).

Example 6. *Any admissible hypergraph with edges of degree 2 that form a loop contains the edges of degree 2 between all pairs of vertices of the loop, cf. [7, Proposition 4.1].*

Another issue we'd like to address is equivalence. LV-systems that are related to each other by a linear transformation, may correspond to different hypergraphs, which are not isomorphic. We will call such hypergraphs LV-equivalent, or equivalent for short.

Example 7. *Consider the LV-system with interaction matrix*

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 5 & 9 \\ 2 & 1 & 3 \end{pmatrix}.$$

It admits the two DPs (10)*, and these give rise to the following integrals*

$$\frac{(x_1 + 2x_3)^3 \, x_1^3}{x_2 x_3}, \quad \frac{(3x_1 + 4x_2 + 6x_3) \, x_3^3}{x_1^3 x_2}.$$

By introducing the DPs

$$y_1 = x_1, \quad y_2 = x_1 + 2x_3, \quad y_3 = 3x_1 + 4x_2 + 6x_3,$$

as new variables, the equations for $\mathbf{y}$ *take the form of an LV-system with interaction matrix*

$$\frac{1}{4} \begin{pmatrix} 2 & -1 & 1 \\ -2 & 3 & 1 \\ -2 & -9 & 5 \end{pmatrix}$$

and, the $\mathbf{y}$*-system admits the additional DPs* $2x_3 = y_2 - y_1$ *and* $4x_2 = y_3 - 3y_2$*, i.e. it is a tree-system. Indeed, the two nonisomorphic hypergraphs in Fig. 1 with* $k = 2$ *are LV-equivalent (see section 4.2).*

4 Classification of n-component homogeneous LV-systems with linear DPs, for $n \leq 5$

For each $n \leq 5$, we consider the set of nonisomorphic hypergraphs (with $k \leq 7$ hyperedges if $n = 5$) and determine which ones are admissible. Then, for each associated LV-system we consider the finite set of linear transformations where the new variables are Darboux polynomials, and hence yield LV-systems. These systems may be associated to hypergraphs that are nonisomorphic but LV-equivalent. For $n < 5$ we present the equivalence classes of admissible hypergraphs, for $n = 5$ we provide non-equivalent ones.

4.1 2-component LV systems

Not considering the empty graph, there is only one graph on two vertices: the tree ∘——∘. The associated LV-system

$$\begin{aligned} \dot{x_1} &= x_1 \left(a_1 x_1 + b_1 x_2 \right) \\ \dot{x_2} &= x_2 \left(c_1 x_1 + a_2 x_2 \right). \end{aligned} \tag{14}$$

has 4 parameters, and admits the additional Darboux polynomial $P = (c_1 - a_1)x_1 + (a_2 - b_1)x_2$ and corresponding integral $x_1^{a_4(a_3-a_1)} x_2^{a_1(a_2-a_4)} P^{a_1 a_4 - a_2 a_3}$.

4.2 3-component LV-systems

There are 7 nonisomorphic hypergraphs of order 3 and positive size, cf. Fig. 1.

1. The two LV-systems related to the hypergraphs of size 1 are not LV-equivalent. Each of them has 8 free parameters. Their LV-matrices are respectively

$$\begin{pmatrix} a_1 & a_2 & a_5 \\ a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 \end{pmatrix}, \qquad \begin{pmatrix} a_1 & a_4 + \frac{(a_2-a_8)(a_1-a_3)(a_4-a_7)}{(a_5-a_8)(a_1-a_6)} & a_2 \\ a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 \end{pmatrix}.$$

2. The bottom hypergraph of size 2 corresponds to the tree-system, with $3n-2=7$ parameters (here we have taken the parametrisation for tree-systems, as in [6]),

$$\begin{aligned} \dot{x_1} &= x_1\,(a_1x_1 + b_1x_2 + b_2x_3) \\ \dot{x_2} &= x_2\,(c_1x_1 + a_2x_2 + b_2x_3) \\ \dot{x_3} &= x_3\,(c_1x_1 + c_2x_2 + a_3x_3)\,. \end{aligned} \tag{15}$$

   The edges are $e_1=(1,2)$, $e_2=(2,3)$, related to the additional DPs

$$P_1 = (c_1-a_1)\,x_1 + (a_2-b_1)\,x_2, \quad P_2 = (c_2-a_2)\,x_2 + (a_3-b_2)\,x_3.$$

   Performing the linear transformation $\mathbf{x} \mapsto \mathbf{y} = (x_1, P_1, P_2)$ we obtain the LV system

$$\frac{dy_i}{dt} = y_i \left(\sum_{j=1}^{3} A_{ij} y_j \right), \quad i = 1, \ldots, 3, \tag{16a}$$

   where

$$\mathbf{A} = \begin{pmatrix} \frac{a_1a_2a_3 - a_1b_2c_2 - a_2b_2c_1 - a_3b_1c_1 + b_1b_2c_1 + b_2c_1c_2}{(a_3-b_2)(a_2-b_1)} & \frac{a_2b_2 + a_3b_1 - b_1b_2 - b_2c_2}{(a_3-b_2)(a_2-b_1)} & \frac{b_2}{a_3-b_2} \\ \frac{2a_1a_2a_3 - a_1a_2b_2 - a_1a_3b_1 + a_1b_1b_2 - a_1b_2c_2 - a_2a_3c_1 + b_2c_1c_2}{(a_3-b_2)(a_2-b_1)} & \frac{a_2a_3 - b_2c_2}{(a_3-b_2)(a_2-b_1)} & \frac{b_2}{a_3-b_2} \\ \frac{2a_1a_2a_3 - a_1a_2b_2 - a_1a_3c_2 - a_2a_3c_1 - a_3b_1c_1 + a_3c_1c_2 + b_1b_2c_1}{(a_3-b_2)(a_2-b_1)} & \frac{2a_2a_3 - a_2b_2 - a_3c_2}{(a_3-b_2)(a_2-b_1)} & \frac{a_3}{a_3-b_2} \end{pmatrix}, \tag{16b}$$

   which satisfies the condition $C_{2,3}=0$. The LV system (16) admits the DPs

$$(a_1-c_1)y_1 + y_2, \quad (a_1-c_1)\,(a_2-c_2)\,y_1 + (a_2-c_2)\,y_2 + (a_2-b_1)\,y_3,$$

   and corresponds to the other hypergraph of size 2 in Fig. 1. This hypergraph is therefore LV-equivalent to the tree on 3 vertices.

3. The two nonisomorphic hypergraphs of size 3 are also LV-equivalent. Their LV-systems are subsystems of the $k=2$ ones (eqs. 15 and 16). In each case the matrix $\mathbf{A}$ satisfies one extra condition, they contain 6 parameters. We note that although the LV-system has three DPs, there are only two functionally independent integrals.

4. As we saw in Example 4, the hypergraph with $k=4$ is not admissible.

The number of hypergraphs of order 3 and size $k>0$, as well as how many of are admissible and nonequivalent is given in Table 2. There are four nonequivalent 3-component LV-systems with additional DPs.

k	1	2	3	4	total
hypergraphs	2	2	2	1	7
admissible	2	2	2	0	6
nonequivalent	2	1	1	0	4

Table 2: The number of order 3 hypergraphs, admissible hypergraphs, and equivalence classes of admissible hypergraphs, with $k>0$ hyperedges.

4.3 4-component LV systems

The number of hypergraphs of order 4 and size $k > 0$, as well as how many of them are admissible and nonequivalent is given in Table 3.

k	1	2	3	4	5	6	7	8	9	10	11	total
hypergraphs	3	7	16	28	35	35	28	16	7	3	1	179
admissible	3	6	9	9	0	3	0	0	0	0	0	30
nonequivalent	3	3	3	1	0	1	0	0	0	0	0	11

Table 3: The number of order 4 hypergraphs, admissible hypergraphs, and equivalence classes of admissible hypergraphs, with $k > 0$ hyperedges.

1. Nonisomorphic hypergraphs of size 1 are not LV-equivalent. For $n = 4$, the size 1 hypergraphs are given in Fig. 3.

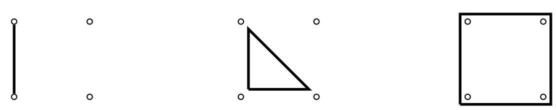

Figure 3: Nonequivalent admissible hypergraphs of order 4 and size 1.

Their LV-matrices, with respectively 14,13 and 13 parameters, are given by

$$\begin{pmatrix} a_1 & a_2 & a_5 & a_6 \\ a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} \end{pmatrix}, \quad \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_4 \\ a_8 & a_6 - \frac{(a_2-a_6)(a_1-a_8)(a_7-a_9)}{(a_1-a_5)(a_3-a_9)} & a_9 & a_4 \\ a_{10} & a_{11} & a_{12} & a_{13} \end{pmatrix},$$

and

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_9 + \frac{(a_3-a_9)(a_1-a_5)(a_6-a_8)}{(a_2-a_6)(a_1-a_7)} & a_{13} + \frac{(a_4-a_{13})(a_1-a_5)(a_6-a_{12})}{(a_2-a_6)(a_1-a_{11})} \\ a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_9 - \frac{(a_3-a_9)(a_1-a_{11})(a_{10}-a_{13})}{(a_1-a_7)(a_4-a_{13})} & a_{13} \end{pmatrix}.$$

2. The size 2 LV-admissible hypergraphs are depicted in Fig. 4. There are 3 equivalence classes which contain respectively 1,2 and 3 nonisomorphic hypergraphs.

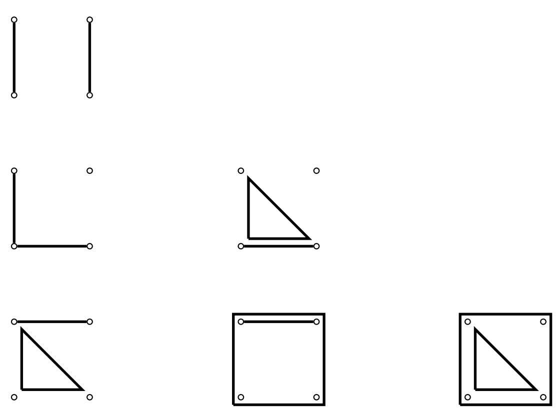

Figure 4: Admissible hypergraphs of order 4 and size 2. Hypergraphs next to each other are LV-equivalent.

We take the hypergraphs whose edges are of lowest degree as representatives (these are the ones on the left in Fig. 4). Their LV-matrices, with respectively 12, 12 and 11 parameters, are

$$\begin{pmatrix} a_1 & a_2 & a_5 & a_6 \\ a_3 & a_4 & a_5 & a_6 \\ a_9 & a_{10} & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{pmatrix}, \qquad \begin{pmatrix} a_1 & a_6 & a_4 & a_8 \\ a_2 & a_3 & a_4 & a_8 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{pmatrix}$$

and

$$\begin{pmatrix} a_1 & a_7 & a_3 & a_4 \\ a_5 & a_6 & a_3 & a_4 \\ a_9 & a_7 & a_{10} & a_{11} + \frac{(a_4-a_{11})(a_1-a_9)(a_{10}-a_8)}{(a_3-a_{10})(a_1-a_2)} \\ a_2 & a_7 & a_8 & a_{11} \end{pmatrix}.$$

3. The size 3 LV-admissible hypergraphs are depicted in Fig. 5. There are 3 equivalence classes which contain respectively 2,2 and 5 nonisomorphic hypergraphs.

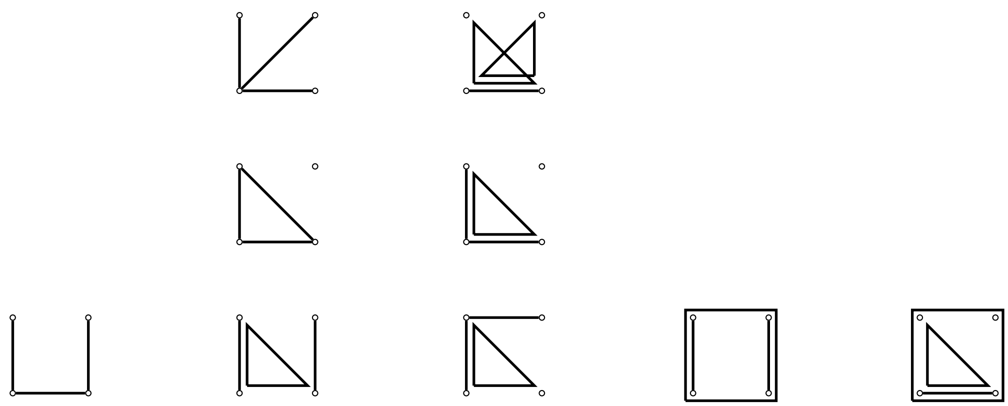

Figure 5: Admissible hypergraphs on 4 vertices with 3 hyperedges. Hypergraphs next to each other are LV-equivalent.

Representative LV-matrices, with respectively 10, 11 and 10 parameters, are

$$\begin{pmatrix} a_1 & a_8 & a_9 & a_6 \\ a_2 & a_3 & a_9 & a_6 \\ a_4 & a_8 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} \end{pmatrix}, \quad \begin{pmatrix} a_1 & a_5 & a_3 & a_7 \\ a_4 & a_2 & a_3 & a_7 \\ a_4 & a_5 & a_6 & a_7 \\ a_8 & a_9 & a_{10} & a_{11} \end{pmatrix}, \quad \begin{pmatrix} a_1 & a_4 & a_9 & a_6 \\ a_7 & a_2 & a_9 & a_6 \\ a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} \end{pmatrix} \tag{17}$$

For the second LV-system, there are only 2 functionally independent integrals. The other LV-systems are tree-systems.

4. The size 4 LV-admissible hypergraphs are depicted in Fig. 6. There is 1 equivalence class which contains 9 nonisomorphic hypergraphs.

   The simplest representative LV-matrix,

$$\begin{pmatrix} a_1 & a_7 & a_8 & a_5 \\ a_3 & a_2 & a_8 & a_5 \\ a_3 & a_7 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 \end{pmatrix}$$

   has 9 parameters. It is (equivalent to) a subcase of each of the LV-systems with 3 DPs, cf. Eq. 17.

5. There are no size 5 LV-admissible hypergraphs of order 4.

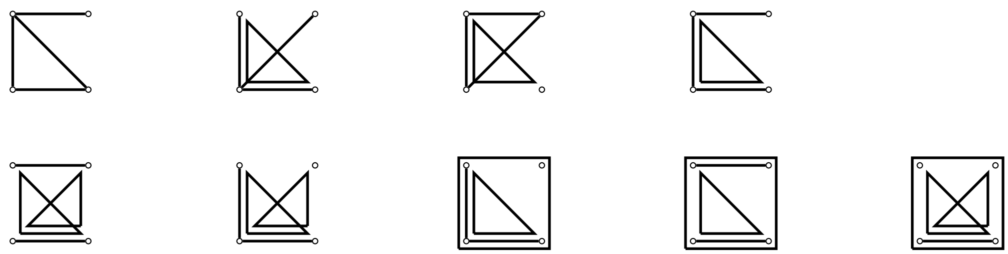

Figure 6: LV-equivalent admissible hypergraphs on 4 vertices with 4 hyperedges.

6. There are 3 nonisomorphic LV-equivalent admissible hypergraphs of order 4 and size 6, cf. Fig. 7.

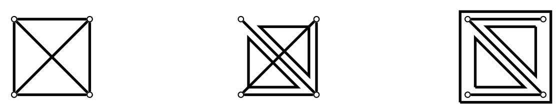

Figure 7: LV-equivalent admissible hypergraphs on 4 vertices with 6 hyperedges.

A representative LV-matrix, associated to the complete graph on 4 vertices, is

$$\begin{pmatrix} a_1 & a_6 & a_7 & a_4 \\ a_5 & a_2 & a_7 & a_4 \\ a_5 & a_6 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{pmatrix},$$

which has 8 parameters.

4.4 5-component LV systems

The number of hypergraphs of order 5 and size $0 < k \leq 7$, as well as how many of are admissible and nonequivalent is given in Table 4.

k	1	2	3	4	5	6	7
hypergraphs	4	15	62	243	841	2544	6672
admissible	4	12	27	45	60	18	21
nonequivalent	4	6	7	7	4	2	1

Table 4: The number of order 5 hypergraphs, admissible hypergraphs, and equivalence classes of admissible hypergraphs, with $0 < k \leq 7$ hyperedges.

1. Nonisomorphic hypergraphs of size 1 are not LV-equivalent. For $n = 5$, the size 1 hypergraphs are given in Fig. 8.

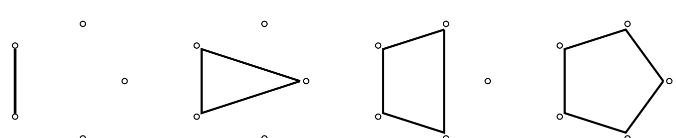

Figure 8: Nonequivalent admissible hypergraphs on 5 vertices with 1 hyperedge.

with respectively 22,20,19 and 19 parameters, are given by

$$\begin{pmatrix} a_1 & a_2 & a_5 & a_6 & a_7 \\ a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\ a_{18} & a_{19} & a_{20} & a_{21} & a_{22} \end{pmatrix}, \quad \begin{pmatrix} a_1 & a_2 & a_3 & a_9 & a_{10} \\ a_4 & a_5 & a_6 & a_9 & a_{10} \\ a_1 - \frac{(a_5-a_7)(a_3-a_8)(a_1-a_4)}{(a_2-a_5)(a_6-a_8)} & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \end{pmatrix},$$

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_{14} \\ a_5 & a_6 & a_9 + \frac{(a_6-a_8)(a_3-a_9)(a_1-a_5)}{(a_2-a_6)(a_1-a_7)} & a_{13} + \frac{(a_1-a_5)(a_4-a_{13})(a_6-a_{11})}{(a_2-a_6)(a_1-a_{10})} & a_{14} \\ a_7 & a_8 & a_9 & a_{13} + \frac{(a_1-a_7)(a_4-a_{13})(a_9-a_{12})}{(a_1-a_{10})(a_3-a_9)} & a_{14} \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{15} & a_{16} & a_{17} & a_{18} & a_{19} \end{pmatrix},$$

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_{10} + \frac{(a_7-a_9)(a_1-a_6)(a_3-a_{10})}{(a_2-a_7)(a_1-a_8)} & a_{14} + \frac{(a_1-a_6)(a_4-a_{14})(a_7-a_{12})}{(a_2-a_7)(-a_{11}+a_1)} & a_{19} + \frac{(a_1-a_6)(a_5-a_{19})(a_7-a_{16})}{(a_2-a_7)(a_1-a_{15})} \\ a_8 & a_9 & a_{10} & a_{14} + \frac{(a_1-a_8)(a_4-a_{14})(a_{10}-a_{13})}{(a_1-a_{11})(a_3-a_{10})} & a_{19} + \frac{(a_1-a_8)(a_5-a_{19})(a_{10}-a_{17})}{(a_1-a_{15})(a_3-a_{10})} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{19} + \frac{(a_5-a_{19})(a_{14}-a_{18})(a_1-a_{11})}{(a_1-a_{15})(a_4-a_{14})} \\ a_{15} & a_{16} & a_{17} & a_{18} & a_{19} \end{pmatrix}.$$

2. Representatives of the size 2 admissible hypergraph LV-equivalence classes are depicted in Fig. 9.

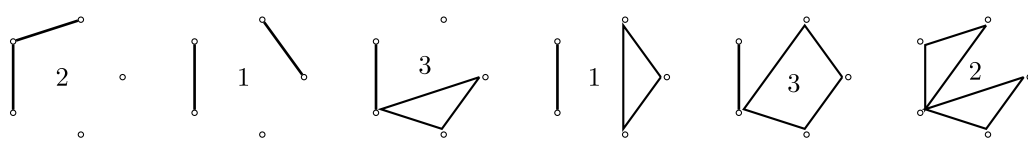

Figure 9: Nonequivalent admissible hypergraphs on 5 vertices with 2 hyperedges. We have added the number of nonisomorphic hypergraphs that are LV-equivalent.

Corresponding LV-matrices, where the number of parameters is the index of the parameter in the bottom right corner, are

$$\begin{pmatrix} a_1 & a_6 & a_4 & a_8 & a_9 \\ a_2 & a_3 & a_4 & a_8 & a_9 \\ a_5 & a_6 & a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{15} & a_{16} & a_{17} & a_{18} & a_{19} \end{pmatrix}, \quad \begin{pmatrix} a_1 & a_2 & a_5 & a_6 & a_7 \\ a_3 & a_4 & a_5 & a_6 & a_7 \\ a_{10} & a_{11} & a_8 & a_9 & a_{14} \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{15} & a_{16} & a_{17} & a_{18} & a_{19} \end{pmatrix},$$

$$\begin{pmatrix} a_1 & a_9 & a_2 & a_3 & a_{12} \\ a_4 & a_5 & a_2 & a_3 & a_{12} \\ a_1 - \frac{(a_2-a_6)(a_1-a_8)(a_7-a_{11})}{(a_3-a_{11})(a_6-a_{10})} & a_9 & a_6 & a_7 & a_{12} \\ a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \end{pmatrix},$$

$$\begin{pmatrix} a_1 & a_2 & a_5 & a_6 & a_7 \\ a_3 & a_4 & a_5 & a_6 & a_7 \\ a_{13} & a_{14} & a_8 & a_{11} + \frac{(a_9-a_{17})(a_{11}-a_{16})(a_8-a_{10})}{(a_{12}-a_{17})(a_8-a_{15})} & a_9 \\ a_{13} & a_{14} & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \end{pmatrix},$$

$$\begin{pmatrix} a_1 & a_{13} & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_2 & a_3 & a_4 \\ a_7 & a_{13} & a_8 & a_{11} + \frac{(a_1-a_7)(a_3-a_{11})(a_8-a_{10})}{(a_1-a_9)(a_2-a_8)} & a_{16} + \frac{(a_1-a_7)(a_4-a_{16})(a_8-a_{14})}{(a_1-a_{12})(a_2-a_8)} \\ a_9 & a_{13} & a_{10} & a_{11} & a_{16} + \frac{(a_1-a_9)(a_4-a_{16})(a_{11}-a_{15})}{(a_1-a_{12})(a_3-a_{11})} \\ a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \end{pmatrix},$$

$$\begin{pmatrix} a_1 & a_2 & a_{10} + \frac{(a_2-a_6)(a_1-a_8)(a_7-a_{10})}{(a_6-a_9)(a_1-a_5)} & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_3 & a_4 \\ a_8 & a_9 & a_{10} & a_3 & a_4 \\ a_{11} & a_2 & a_{10} + \frac{(a_2-a_6)(a_1-a_8)(a_7-a_{10})}{(a_6-a_9)(a_1-a_5)} & a_{12} & a_{15} + \frac{(a_4-a_{15})(a_1-a_{11})(a_{12}-a_{14})}{(a_1-a_{13})(a_3-a_{12})} \\ a_{13} & a_2 & a_{10} + \frac{(a_2-a_6)(a_1-a_8)(a_7-a_{10})}{(a_6-a_9)(a_1-a_5)} & a_{14} & a_{15} \end{pmatrix}.$$

3. Representatives of the size 3 admissible hypergraph LV-equivalence classes are depicted in Fig. 10.

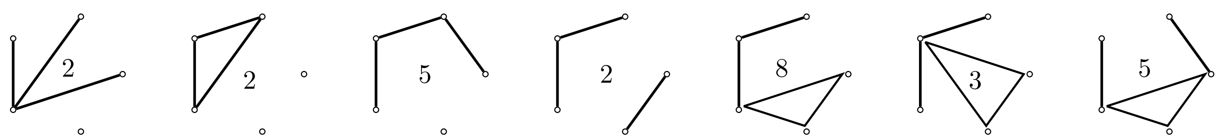

Figure 10: Nonequivalent admissible hypergraphs on 5 vertices with 3 hyperedges.

Associated LV-matrices are:

$$\begin{pmatrix} a_1 & a_8 & a_9 & a_6 & a_{11} \\ a_2 & a_3 & a_9 & a_6 & a_{11} \\ a_4 & a_8 & a_5 & a_6 & a_{11} \\ a_7 & a_8 & a_9 & a_{10} & a_{11} \\ a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \end{pmatrix}, \quad \begin{pmatrix} a_1 & a_5 & a_3 & a_7 & a_8 \\ a_4 & a_2 & a_3 & a_7 & a_8 \\ a_4 & a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} & a_{13} \\ a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \end{pmatrix}$$

(which only has two independent integrals),

$$\begin{pmatrix} a_1 & a_4 & a_9 & a_6 & a_{11} \\ a_7 & a_2 & a_9 & a_6 & a_{11} \\ a_3 & a_4 & a_5 & a_6 & a_{11} \\ a_7 & a_8 & a_9 & a_{10} & a_{11} \\ a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \end{pmatrix}, \quad \begin{pmatrix} a_1 & a_6 & a_4 & a_8 & a_9 \\ a_2 & a_3 & a_4 & a_8 & a_9 \\ a_5 & a_6 & a_7 & a_8 & a_9 \\ a_{12} & a_{13} & a_{14} & a_{10} & a_{11} \\ a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \end{pmatrix}$$

$$\begin{pmatrix} a_1 & a_5 & a_{12} & a_3 & a_7 \\ a_{10} & a_2 & a_{12} & a_3 & a_7 \\ a_4 & a_5 & a_6 & a_3 & a_7 \\ a_{10} & a_2 - \frac{(a_3-a_8)(a_9-a_{14})(-a_{11}+a_2)}{(a_7-a_{14})(a_8-a_{13})} & a_{12} & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \end{pmatrix},$$

$$\begin{pmatrix} a_1 & a_{11} & a_{12} & a_2 & a_3 \\ a_4 & a_5 & a_{12} & a_2 & a_3 \\ a_6 & a_{11} & a_7 & a_2 & a_3 \\ a_1 - \frac{(a_2-a_8)(a_9-a_{14})(-a_{10}+a_1)}{(a_3-a_{14})(a_8-a_{13})} & a_{11} & a_{12} & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \end{pmatrix},$$

$$\begin{pmatrix} a_1 & a_{11} & a_2 & a_{13} & a_3 \\ a_4 & a_5 & a_2 & a_{13} & a_3 \\ a_{10} & a_{11} & a_6 & a_{13} & a_7 \\ a_{10} & a_{11} & a_8 & a_9 & a_7 \\ a_1 - \frac{(a_3-a_{14})(-a_{12}+a_6)(-a_{10}+a_1)}{(-a_6+a_2)(a_7-a_{14})} & a_{11} & a_{12} & a_{13} & a_{14} \end{pmatrix}.$$

4. Representatives of the size 4 admissible hypergraph LV-equivalence classes are depicted in Fig. 11.

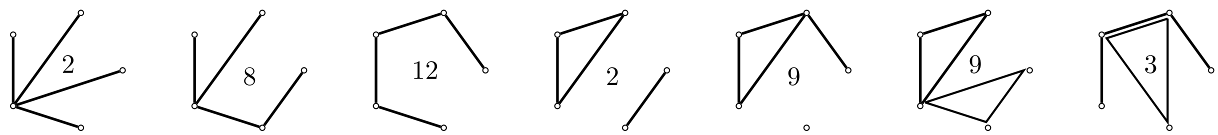

Figure 11: Nonequivalent admissible hypergraphs on 5 vertices with 4 hyperedges.

Associated LV-matrices are:

$$
\begin{pmatrix}
a_1 & a_{10} & a_{11} & a_{12} & a_8 \\
a_2 & a_3 & a_{11} & a_{12} & a_8 \\
a_4 & a_{10} & a_5 & a_{12} & a_8 \\
a_6 & a_{10} & a_{11} & a_7 & a_8 \\
a_9 & a_{10} & a_{11} & a_{12} & a_{13}
\end{pmatrix}, \quad
\begin{pmatrix}
a_1 & a_6 & a_{11} & a_{12} & a_8 \\
a_9 & a_2 & a_{11} & a_{12} & a_8 \\
a_3 & a_6 & a_4 & a_{12} & a_8 \\
a_5 & a_6 & a_{11} & a_7 & a_8 \\
a_9 & a_{10} & a_{11} & a_{12} & a_{13}
\end{pmatrix},
$$

$$
\begin{pmatrix}
a_1 & a_{10} & a_6 & a_{12} & a_8 \\
a_4 & a_2 & a_6 & a_{12} & a_8 \\
a_9 & a_{10} & a_3 & a_{12} & a_8 \\
a_4 & a_5 & a_6 & a_7 & a_8 \\
a_9 & a_{10} & a_{11} & a_{12} & a_{13}
\end{pmatrix}, \quad
\begin{pmatrix}
a_1 & a_5 & a_3 & a_7 & a_8 \\
a_4 & a_2 & a_3 & a_7 & a_8 \\
a_4 & a_5 & a_6 & a_7 & a_8 \\
a_{11} & a_{12} & a_{13} & a_9 & a_{10} \\
a_{11} & a_{12} & a_{13} & a_{14} & a_{15}
\end{pmatrix}, \tag{18}
$$

$$
\begin{pmatrix}
a_1 & a_7 & a_8 & a_5 & a_{10} \\
a_3 & a_2 & a_8 & a_5 & a_{10} \\
a_3 & a_7 & a_4 & a_5 & a_{10} \\
a_6 & a_7 & a_8 & a_9 & a_{10} \\
a_{11} & a_{12} & a_{13} & a_{14} & a_{15}
\end{pmatrix}, \quad
\begin{pmatrix}
a_1 & a_{10} & a_{11} & a_2 & a_3 \\
a_5 & a_4 & a_{11} & a_2 & a_3 \\
a_5 & a_{10} & a_6 & a_2 & a_3 \\
a_1 - \frac{(-a_7+a_2)(-a_9+a_1)(a_8-a_{13})}{(a_3-a_{13})(-a_{12}+a_7)} & a_{10} & a_{11} & a_7 & a_8 \\
a_9 & a_{10} & a_{11} & a_{12} & a_{13}
\end{pmatrix} \tag{19}
$$

$$
\begin{pmatrix}
a_1 & a_2 & a_{11} & a_{12} & a_3 \\
a_4 & a_5 & a_{11} & a_{12} & a_3 \\
a_6 & a_2 & a_7 & a_{12} & a_3 \\
a_4 & a_8 & a_{11} & a_9 & a_3 \\
a_1 - \frac{(a_1-a_4)(a_5-a_{10})}{-a_5+a_2} & a_{10} & a_{11} & a_{12} & a_{13}
\end{pmatrix}. \tag{20}
$$

We note that the LV-system related to the second matrix of (18) and the LV-systems related to the matrices (19) have 3 independent integrals only. The LV-system related to (20) is a new superintegrable LV-system that is not equivalent to a tree-system. It has the same number of parameters as the 5-component tree-systems. The two nonisomorphic hypergraphs that are LV-equivalent are depicted in Fig. 12.

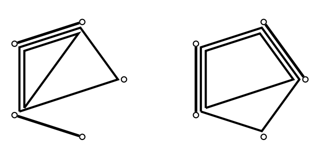

Figure 12: The two hypergraphs that are LV-equivalent to the last hypergraph in Fig. 11.

5. Representatives of the size 5 admissible hypergraph LV-equivalence classes are depicted in Fig. 13.

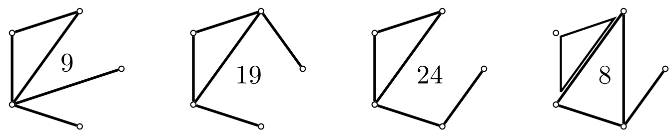

Figure 13: Nonequivalent admissible hypergraphs on 5 vertices with 5 hyperedges.

As the LV-systems are subsystems of the LV-systems with 4 DPS, we do not give their matrices explicitly. In each case, the number of parameters is 12.

6. Representatives of size 6,7 and 10 admissible hypergraph LV-equivalence classes are depicted in Fig. 14.

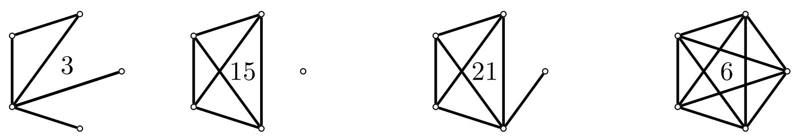

Figure 14: Nonequivalent admissible hypergraphs on 5 vertices with 6,7 or 10 hyperedges.

The number of parameters in these LV-systems is respectively 14, 11, 11, and 10. Any LV-system with $k > 7$ DPs also has 7 DPs. There are 9 nonisomorphic hypergraphs of size 8 which have the representative admissible hypergraph of size 7 as subgraph. None of these are admissible, but two of them give rise to a hypergraph of size 10 (cf. Examples 5 and 6). There are 6 nonisomorphic hypergraphs of size 10 that are LV-equivalent to the complete graph of order 5.

5 Concluding remarks

We have classified homogeneous n-component LV-systems with k additional linear Darboux polynomials, up to linear transformations, for $n \leq 5$. The number of them, for each n and k, is given in Table 5.

$n \setminus k$	1	2	3	4	5	6	7	8	9	10
2	1									
3	2	1	1	0						
4	3	3	3	1	0	1	0	0	0	0
5	4	6	7	7	4	2	1	0	0	1

Table 5: The number of n-component LV-systems with k additional linear DPs.

Each DP gives rise to an integral, however, they are not necessarily functionally independent. The maximum number of independent integrals for an n-component system is $n - 1$. For an admissible hypergraph of size k, which contains a subgraph of order o and size $l > o - 1$ the number of independent integrals for the corresponding LV-system is at most $k - l + o - 1$. The number of LV-systems with functionally integrals of the form (3) is given in Table 6. (Note that LV-systems with more linear DPs than independent integrals are subsystems of a larger system, i.e. with more pararameters. These are not counted.)

Of special interest are the n-component LV-systems which have $n-1$ functionally independent integrals, the so-called superintegrable ones. A large class of such systems are associated to tree-systems, cf. [5, 6]. For $n \leq 4$ all superintegrable hypergraph-systems are tree-systems. For $n = 5$

$n \setminus k$	1	2	3	4
2	1			
3	2	1		
4	3	3	2	
5	4	6	6	4

Table 6: The number of n-component LV-systems with k functionally independent integrals of the form (3).

this is not true. The LV-system with matrix (20) is a hypergraph-system which is not a tree-system. For $n > 5$, we expect there exist more superintegrable hypergraph-systems that are not equivalent to tree-systems.

The reader should be able, for each of the matrices we have provided, to figure out what the labeling is on the corresponding hypergraph, by inspecting which of the DP-conditions are satisfied, cf. Lemma 2. For the matrix (20), which corresponds to the right-most hypergraph in Fig. 11, it is easy to see that the 2-edges are $\{1,2\},\{1,3\},\{2,4\}$ and the 3-edge is $\{1,2,5\}$. The DPs are

$$(a_4 - a_1)\,x_1 + (a_5 - a_2)\,x_2, \quad (a_6 - a_1)\,x_1 + (a_7 - a_{11})\,x_3, \quad (a_8 - a_5)\,x_2 + (a_9 - a_{12})\,x_4,$$
$$(a_1 - a_4)\,(a_5 - a_{10})\,x_1 + (a_2 - a_5)\,(a_5 - a_{10})\,x_2 + (a_2 - a_5)\,(a_3 - a_{13})\,x_5,$$

and 4 functionally independent integrals can be obtained from the DPs by using equation (3).

We have determined the sets of hypergraphs that are LV-equivalent to trees of order $n < 9$. Their sizes are given in Table 7, as well as in the appendix.

n	trees	LV-equivalent hypergraphs
2	1	1
3	1	2
4	2	2, 5
5	3	2, 8, 12
6	6	2, 8, 5, 18, 21, 30
7	11	2, 8, 8, 18, 21, 30, 22, 79, 12, 55, 76
8	23	2, 8, 8, 18, 21, 5, 30, 30, 32, 79, 21, 55, 18, 48, 60, 79, 126, 79, 112, 207, 30, 144, 195

Table 7: The number of trees of order $n < 9$ and the numbers of LV-equivalent hypergraphs.

The following statement is true for trees of order smaller than nine.

Conjecture 8. *Nonisomorphic trees are not LV-equivalent.*

A Trees of orders $6, 7, 8$ and the number of LV-equivalent hypergraphs

In Figs. 15, 16 and 17, we provide for each tree of order 6,7 and 8 the number of LV-equivalent hypergraphs.

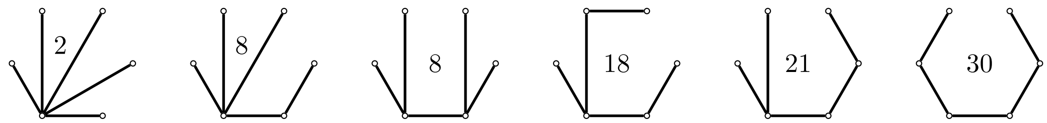

Figure 15: Trees of order 6 and the number of LV-equivalent nonisomorphic hypergraphs.

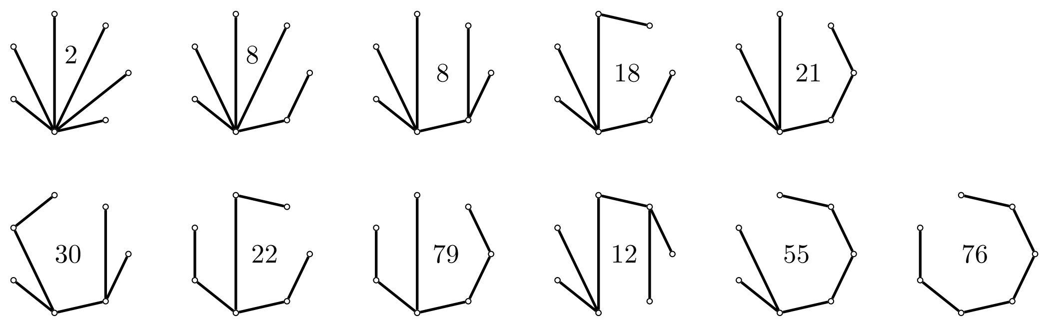

Figure 16: Trees of order 7 and the number of LV-equivalent nonisomorphic hypergraphs.

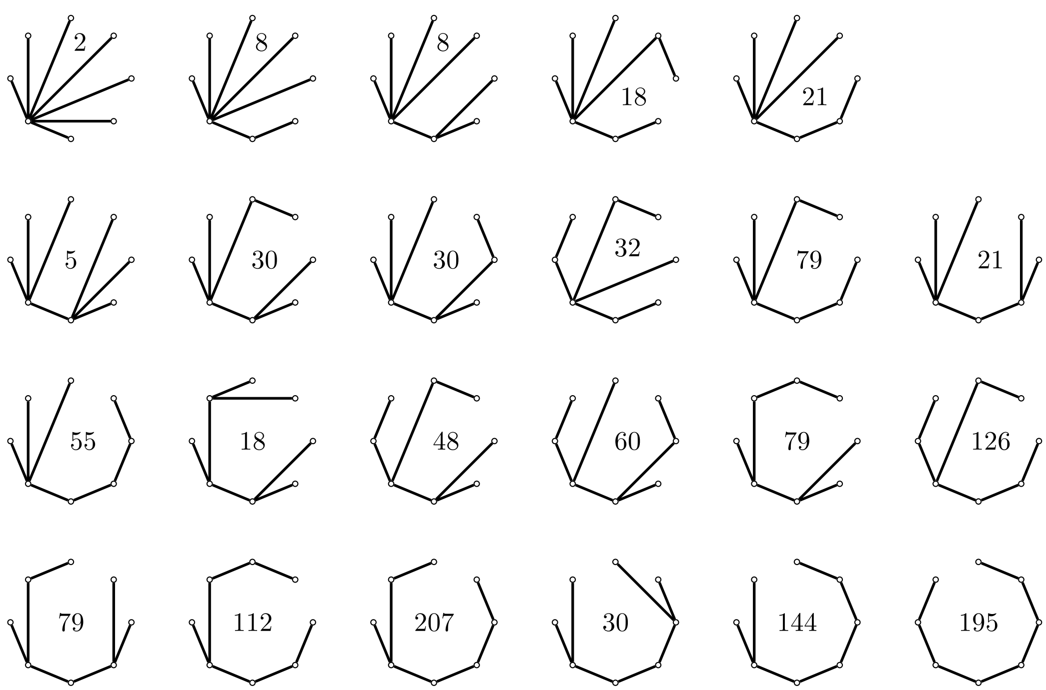

Figure 17: Trees of order 8 and the number of LV-equivalent nonisomorphic hypergraphs.

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