Some rigidity results related to the Obata type equation

Yiwei Liu and Yi-Hu Yang*

November 28, 2024

Abstract

Let (Ω^{n+1}, g) be an $(n+1)$-dimensional smooth compact connected Riemannian manifold with smooth boundary $\partial\Omega = \Sigma$ and f a smooth function on Ω which satisfies the Obata type equation $\nabla^2 f - fg = 0$ with Robin boundary condition $f_\nu = cf$, where $c = \coth\theta > 1$. In this paper, we consider the rigidity of Ω. First, we derive the warped product structure of Ω by the properties of the Obata type equation and the set $\Omega_0 = \{x \in \Omega \mid f(x) = 0\}$. Second, we determine the specific structure of Ω_0 under appropriate curvature assumptions and provide the structure of Ω accordingly.

1 Introduction

Let (Ω^{n+1}, g) be an $(n+1)$-dimensional $(n \geq 2)$ smooth compact connected Riemannian manifold with smooth boundary $\partial\Omega = \Sigma$. In this paper, we concern the Obata type equation $\nabla^2 f - fg = 0$ with Robin boundary condition $f_\nu = cf$ and related rigidity of Ω.

For a compact Riemannian manifold (M, g) with boundary or without boundary, one can consider the following Obata type equation

$$\nabla^2 f + kfg = 0,$$

where $k \in \{1, 0, -1\}$.

For the case $k = 1$ and manifolds without boundary, the above equation is related to many rigidity results. In [11], Lichnerowicz proved that the first eigenvalue λ_1 of a closed manifold of dimension n whose Ricci curvature has lower bound $n-1$ is at least n, that is, $\lambda_1 \geq n$. Accordingly, Obata [13] proved that if the equality holds, then the manifold must be isometric to a standard sphere by using the following equation

$$\nabla^2 f + fg = 0.$$

*Partially supported by NSF of China (No. 12071283)

For a compact manifold (Ω^{n+1}, g) with boundary $\partial\Omega = \Sigma$, one also can consider the Obata type equation with various boundary conditions. In [16], Reilly obtained the Lichnerowicz-type lower bound for the first Dirichlet eigenvalue μ_1 on Ω with $\mathrm{Ric}^{\Omega} \geq n$ and the boundary Σ being of non-negative mean curvature, that is, $\mu_1 \geq n+1$. In particular, he also proved that $\mu_1 = n+1$ if and only if Ω is isometric to the standard hemisphere. Here, the proof of the rigidity result depends again on the following Obata type equation with Dirichlet boundary condition

$$\begin{cases} \nabla^2 f + fg = 0, & \text{in } \Omega, \\ f = 0, & \text{on } \Sigma. \end{cases}$$

Later, Escobar [4] and Xia [20] independently proved that the first Neumann eigenvalue η_1 on Ω satisfies $\eta_1 \geq n+1$ if $\mathrm{Ric}^{\Omega} \geq n$ and the boundary is convex. Moreover, $\eta_1 = n+1$ if and only if Ω is isometric to the standard hemisphere and the proof of this rigidity result also depends on the Obata type equation with Neumann boundary condition

$$\begin{cases} \nabla^2 f + fg = 0, & \text{in } \Omega, \\ f_\nu = 0, & \text{on } \Sigma, \end{cases}$$

where ν is the outward unit normal. Recently, Chen, Lai and Wang [2] studied the Obata type equation with Robin boundary condition

$$\begin{cases} \nabla^2 f + fg = 0, & \text{in } \Omega, \\ f_\nu - cf = 0, & \text{on } \Sigma, \end{cases}$$

and also obtained many rigidity results.

For the case $k = 0$, the following boundary value problem

$$\begin{cases} \nabla^2 f = 0, & \text{in } \Omega, \\ f_\nu - cf = 0, & \text{on } \Sigma \end{cases}$$

has been further studied, where c is a positive constant. In [15], Raulot and Savo proved that Ω is isometric to a Euclidean ball with radius $\frac{1}{c}$ if Ω has non-negative sectional curvature, the principal curvatures of the boundary Σ are bounded from below by c, and there exists a non-constant smooth function f satisfying the above boundary value problem. Later, Xia and Xiong [19] extend this result to a compact manifold Ω whose Ricci curvature satisfies $\mathrm{Ric}^{\Omega} \geq 0$ and the mean curvature H of $\Sigma \geq c$. It is worth noting that their result is used to prove the rigidity part in Escobar's conjecture for manifolds with non-negative sectional curvature.

As for the case $k = -1$, Kanai [8] proved that a complete manifold (M^n, g) is isometric to the standard hyperbolic space $\mathbb{H}^n$ if and only if there exists a non-constant function f on M with a critical point and satisfying $\nabla^2 f - fg = 0$. At the same time, he also discussed the case that f does not have critical points.

In [5], Galloway and Jang focused on the following Dirichlet boundary problem

$$\begin{cases} \nabla^2 f - fg = 0, & \text{in } \Omega, \\ f = a, & \text{on } \Sigma, \end{cases}$$

and proved some rigidity results. Very recently, Lai and Zhou [9] studied the obata type equation $\nabla^2 f - fg = 0$ with various boundary conditions and f being of interior critical points, and gave a series of rigidity results in the standard hyperbolic space $\mathbb{H}^{n+1}$. For more information about the Obata type equations, interested readers also can refer to [1] and [18] .

The aim of the present paper is to concentrate on the equation $\nabla^2 f - fg = 0$ with Robin boundary condition $f_\nu = cf$, where $c = \cosh\theta > 1$ and $\theta > 0$ (so that f has no critical points, cf. Proposition 2.1), and prove some rigidity results. Our first result is the following

Theorem 1.1. *Let (Ω^{n+1}, g) be an $(n+1)$-dimensional ($n \geq 2$) smooth compact connected Riemannian manifold with smooth boundary Σ. Assume that there exists a non-constant function $f \in C^\infty(\Omega)$ and a constant $c > 1$ such that*

$$\begin{cases} \nabla^2 f - fg = 0, & in\, \Omega, \\ f_\nu - cf = 0, & on\, \Sigma; \end{cases} \tag{1}$$

and set $\Omega_0 = \{p \in \Omega \,|\, f(p) = 0\}$ and $c = \coth\theta$ ($\theta > 0$). Then, we have

(1) If f is constant on some boundary component, then Ω is isometirc to the warped product space

$$\Omega_0 \times [-\theta, \theta]_t, \;\; g = dt^2 + (\cosh t)^2 g_{|\Omega_0}.$$

(2) If f is non-constant on any boundary component, then Σ is connected and Ω is isometirc to a Z_2-symmetric domain in the warped product space

$$\widehat{\Omega} = \Omega_0 \times (-\infty, \infty)_t, \;\; g = dt^2 + (\cosh t)^2 g_{|\Omega_0},$$

which is bounded by the graph functions $\pm\phi$, where $\phi \in C^\infty(\Omega_0^\circ) \cap C(\Omega_0)$ satisfies

$$\frac{\cosh\phi}{\sqrt{1 + (\cosh\phi)^{-2} |\nabla_{\Omega_0}\phi|^2_{g_{|\Omega_0}}}} = c\sinh\phi, \;\; in\ \Omega_0^\circ,$$

$$\phi > 0, \;\; in\ \Omega_0^\circ,$$

and

$$\phi = 0, \;\; on\ \partial\Omega_0.$$

Remark. (1) Theorem 1.1 is similar to Theorem 1.3 in [2]. We will use a similar method to prove this theorem.

(2) The two cases in Theorem 1.1 are completely discrete. In other words, if f is non-constant on any boundary component, it won't be a disturbance of a certain constant.

(3) The structure of Ω_0 is unknown. We need additional conditions to determine Ω_0.

In the second part of the paper, we discuss the structure of Ω_0 (and then the structure of Ω) under some appropriate curvature assumptions. For the first case in Theorem 1.1, we know that Ω_0 is a closed manifold which is conformal to the boundary component(s) and the second fundamental form h on Σ satisfies $h = \frac{1}{c} g_{|\Sigma}$. To this end, we first make the following curvature assumption

(K) Let S be some boundary component and $k \in \{2, \cdots, n\}$. For any orthonormal vectors $\{e_1, e_2, \cdots, e_k\}$ in TS, we assume that

$$-\sum_{j=2}^{k} R^{\Omega}(e_1, e_j, e_1, e_j) \geq (k-1)(1 - \frac{2}{c^2}),$$

where the R^{Ω} is the curvature tensor of Ω.

Under the above assumption and a lower bound condition for the diameter of boundary components, we then have the following

Theorem 1.2. *Let (Ω^{n+1}, g) and f be as in Theorem 1.1, where f is constant on some boundary component. Assume that there exists a boundary component which satisfies the assumption (K) and the diameter d of this component satisfies $d \geq \frac{c}{\sqrt{c^2-1}}\pi$. Then Ω_0 is isometric to the standard sphere $\mathbb{S}^n$ and Ω is isometirc to the warped product space*

$$\mathbb{S}^n \times [-\theta, \theta]_t, \ \ g = dt^2 + (\cosh t)^2 g_{\mathbb{S}^n}.$$

For the second case, we know that Ω_0 is a compact manifold with boundary. By the idea in Proposition 4.3 in [19], we can prove the following theorem which can be seen as an extension of Corollary 4.5 in [12].

Theorem 1.3. *Let (Ω^{n+1}, g) and f be as in Theorem 1.1. If the Ricci curvature of Ω satisfies $Ric^{\Omega} \geq -n$ and the mean curvature of Σ is bounded from below by c, then Ω is isometric to a geodesic ball of radius $\tanh^{-1}(\frac{1}{c})$ in the hyperbolic space $\mathbb{H}^{n+1}$.*

It is worth noting that the curvature assumptions in Theorem 1.3 implies that the boundary Σ is connected (see Lemma 2.1 in [10]). Therefore, Theorem 1.3 corresponds to the second case in Theorem 1.1 and f is non-constant on boundary Σ.

Similarly, we can also extend the above result to standard sphere $\mathbb{S}^{n+1}$; accordingly, we need to use the Obata type equation $\nabla^2 f + fg = 0$.

Theorem 1.4. *Let (Ω^{n+1}, g) be an $(n+1)$-dimensional $(n \geq 2)$ smooth compact connected Riemannian manifold with smooth boundary Σ. Assume that there exist a non-constant function $f \in C^\infty(\Omega)$ and a constant $c > 0$ such that*

$$\begin{cases} \nabla^2 f + fg = 0, & in\, \Omega, \\ f_\nu - cf = 0, & on\, \Sigma. \end{cases} \tag{2}$$

If the Ricci curvature of Ω satisfies $Ric^\Omega \geq n$ and the mean curvature of Σ is bounded from below by c, then Ω is isometric to a geodesic ball of radius $\tan^{-1}\left(\frac{1}{c}\right)$ in the standard sphere $\mathbb{S}^{n+1}$.

We also point out that Theorem 1.4 is an extension of Corollary 4.4 in [12].

The paper is organized as follows. Section 2 gives some necessary preliminaries, including some basic definitions and some known results which are needed later. Section 3 concentrates on the Obata type equation (1) and the proof of Theorem 1.1. In Section 4, we consider the structure of Ω_0 and prove Theorems 1.2, 1.3 and 1.4.

2 Preliminaries

This section mainly introduces some basic definitions and some known results which are needed in the later proofs.

Let (Ω^{n+1}, g) be an $(n+1)$-dimensional smooth compact connected Riemannian manifold with boundary $\partial\Omega = \Sigma$ and $g|_\Sigma$ the restricted metric on Σ; denote by $\langle \cdot, \cdot \rangle$ the inner product on Ω as well as Σ. Denote by ∇^Ω, ∇, Δ, and ∇^2 the connection, the gradient, the Laplacian, and the Hessian on Ω respectively, while by ∇_Σ and Δ_Σ the gradient and the Laplacian on Σ respectively. Let ν be the unit outward normal of Σ; denote by h, A^ν, and H the second fundamental form, the Weingarten transformation, and the mean curvature of Σ with respect to ν respectively, here

$$h(X, Y) = -\langle \nabla^\Omega_X Y, \nu \rangle, \quad \langle A^\nu(X), Y \rangle = h(X, Y),$$

and

$$H = \frac{\mathrm{tr}_g h}{n}.$$

The principal curvatures of Σ are defined to be the eigenvalues of h and A^ν. Let R^Ω be the curvature tensor of Ω, i.e., for tangent vectors X, Y, Z, W,

$$R^\Omega(X, Y, Z, W) = \langle \nabla^\Omega_X \nabla^\Omega_Y Z - \nabla^\Omega_Y \nabla^\Omega_X Z - \nabla^\Omega_{[X,Y]} Z, W \rangle;$$

Ric^Ω be the Ricci curvature tensor of Ω. Let dV and dv be the canonical volume element of Ω and Σ respectively.

Let $f \in C^\infty(\Omega)$ satisfy the equation (1), which is non-constant and

$$\Omega_a = \{x \in \Omega \mid f(x) = a\}.$$

Now, we give some basic facts related to the Obata type equations.

Proposition 2.1. *Let $f \in C^\infty(\Omega)$ satisfy the equation (1) which is non-constant, then there exists a constant $A > 0$ such that*

$$|\nabla f|^2 - f^2 = A.$$

Proof. A direct calculation shows that, for any tangent vector (field) X,

$$X(|\nabla f|^2 - f^2) = 2\nabla^2 f(X, \nabla f) - 2f\langle X, \nabla f\rangle = 0.$$

Therefore, $|\nabla f|^2 - f^2$ is constant. Since f is non-constant, $f_\nu = cf$, and $c > 1$, we have $|\nabla f|^2 - f^2 > 0$. Then the conclusion follows. □

Proposition 2.1 clearly implies that f has no critical points in Ω. Without lose of generality, we always assume that $A = 1$ in the following. Here, we also record the following fact, for its proof one can refer to [2].

Proposition 2.2. *Let $f \in C^\infty(\Omega)$ be a non-constant function which satisfies equation (1), then the integral curves of $\frac{\nabla f}{|\nabla f|}$ are geodesics.*

For later convenience, we here introduce the warped product structure of the space forms and the corresponding equations of geodesic spheres. These facts will be used in the proofs of Theorems 1.3 and 1.4. For sake of clarity, we first see the Euclidean case.

Let $\mathbb{R}^{n+1} = \{(x^1, x^2, \cdots, x^{n+1}) \mid x^i \in \mathbb{R}\}$ and $g_{\mathbb{R}^{n+1}} = (dx^1)^2 + (dx^2)^2 + \cdots + (dx^{n+1})^2$, and identify $\mathbb{R}^n$ with $\{(x^1, x^2, \cdots, x^{n+1}) \in \mathbb{R}^{n+1} \mid x^1 = 0\}$. Obviously, the Euclidean space $\mathbb{R}^{n+1}$ is isometric to the warped product space

$$\mathbb{R}^n \times (-\infty, \infty)_t, \;\; g = dt^2 + g_{\mathbb{R}^n}.$$

Let $p \in \mathbb{R}^n \subset \mathbb{R}^{n+1}$ and $B_p^{n+1}(r)$ be the (closed) ball of radius r centered at p in $\mathbb{R}^{n+1}$. Then, $B_p^n(r) = B_p^{n+1}(r) \cap \mathbb{R}^n$ is the (closed) ball of radius r centered at p in $\mathbb{R}^n$. Now, we consider $B_p^{n+1}(r)$ as a bounded domain in the above warped product space $\mathbb{R}^n \times (-\infty, \infty)_t$. A direct calculation shows that there exists a unique non-negative function $\phi \in C^\infty((B_p^n(r))^\circ) \cap C(B_p^n(r))$ such that

$$\partial B_p^{n+1}(r) = \{(x, \pm\phi(x)) \mid x \in B_p^n(r)\}.$$

In particular, ϕ satisfies the following equation

$$\phi^2(x) + d^2(x, p) = r^2,$$

where d is the distance function in $\mathbb{R}^n$.

Now, we consider the hyperbolic space $\mathbb{H}^{n+1}$. We use the model of the upper half-space, i.e.

$$\mathbb{H}^{n+1} = \{(x^1, x^2, \cdots, x^{n+1}) \mid x^{n+1} > 0\}$$

and

$$g_{\mathbb{H}^{n+1}} = \frac{(dx^1)^2 + (dx^2)^2 + \cdots + (dx^{n+1})^2}{(x^{n+1})^2};$$

identify $\mathbb{H}^n$ with $\{(x^1, x^2, \cdots, x^{n+1}) \in \mathbb{H}^{n+1} \mid x^1 = 0\}$. For any point $p = (0, x^2, \cdots, x^{n+1}) \in \mathbb{H}^n$, a direct calculation shows that

$$\gamma_p(t) = (\frac{\sinh t}{\cosh t} x^{n+1}, x^2, \cdots, x^n, \frac{x^{n+1}}{\cosh t})$$

is the (arc length parameter) minimizing geodesic satisfying $\gamma_p(0) = p$ and $\gamma_p'(0) \perp T_p\mathbb{H}^n$. Then, one has the following smooth map

$$\Psi : \mathbb{H}^n \times (-\infty, \infty)_t \to \mathbb{H}^{n+1}$$

by setting $\Psi(p,t) = \gamma_p(t)$. It is easy to see that Ψ is a diffeomorphism and the pull-back metric $\Psi^* g_{\mathbb{H}^{n+1}}$ on $\mathbb{H}^n \times (-\infty, \infty)$ can be written as

$$g = dt^2 + (\cosh t)^2 g_{\mathbb{H}^n}.$$

Thus, the hyperbolic space $\mathbb{H}^{n+1}$ is actually isometric to the following warped product space

$$\mathbb{H}^n \times (-\infty, \infty)_t, \;\; g = dt^2 + (\cosh t)^2 g_{\mathbb{H}^n};$$

this is the warped product structure of $\mathbb{H}^{n+1}$.

Let $p \in \mathbb{H}^n \subset \mathbb{H}^{n+1}$ and $B_p^{n+1}(r)$ be the closed geodesic ball of radius r centered at p in $\mathbb{H}^{n+1}$. Let $B_p^n(r) = B_p^{n+1}(r) \cap \mathbb{H}^n$; obviously $B_p^n(r)$ is the closed geodesic ball of radius r centered at p in $\mathbb{H}^n$. By the above discussion, we can also consider $B_p^{n+1}(r)$ as a bound domain in the product space $\mathbb{H}^n \times (-\infty, \infty)_t$. Set

$$\partial B_{p+}^{n+1}(r) = \partial B_p^{n+1}(r) \cap (\mathbb{H}^n \times [0, \infty)_t).$$

Then, it is easy to see that there exists a non-negative function $\phi \in C^\infty((B_p^n(r))^\circ) \cap C(B_p^n(r))$ so that $\partial B_{p+}^{n+1}(r)$ can be written as

$$\{(x, \phi(x)) \in \mathbb{H}^n \times [0, \infty)_t \mid x \in B_p^n(r)\}.$$

For a point $x \in B_p^n(r)$ together with the points p and $(x, \phi(x))$, one has the corresponding geodesic triangle in $\mathbb{H}^{n+1}$; by the law of cosine in $\mathbb{H}^{n+1}$, we have

$$\cosh r = \cosh \phi(x) \cosh d(x,p),$$

that is,

$$\frac{\tanh^2 d(x,p)}{\tanh^2 r} + \frac{\sinh^2 \phi(x)}{\sinh^2 r} = 1,$$

where d is the distance function in $\mathbb{H}^n$. Similarly, one can consider the lower half part

$$\partial B_{p-}^{n+1}(r) = \partial B_p^{n+1}(r) \cap (\mathbb{H}^n \times (-\infty, 0]_t).$$

Thus, we have the following

Proposition 2.3. *There exists some* $\phi \in C^\infty((B_p^n(r))^\circ) \cap C(B_p^n(r))$ *which is non-negative and satisfies*

$$\frac{\tanh^2 d(x,p)}{\tanh^2 r} + \frac{\sinh^2 \phi(x)}{\sinh^2 r} = 1$$

(where d *is the distance function in* $\mathbb{H}^n$*), so that the geodesic sphere* $\partial B_p^{n+1}(r)$ *can be written as*

$$\{(x, \pm\phi(x)) \mid x \in B_p^n(r)\}$$

in the product space $\mathbb{H}^n \times (-\infty, \infty)_t$*.*

By a similar argument, the standard sphere $\mathbb{S}^{n+1}$ can be isometrically written as the warped product space

$$\mathbb{S}^n \times (-\frac{\pi}{2}, \frac{\pi}{2})_t, \;\; g = dt^2 + (\cos t)^2 g_{\mathbb{S}^n}.$$

Let $p \in \mathbb{S}^n \subset \mathbb{S}^{n+1}$ and $B_p^{n+1}(r)$ be the closed geodesic ball of radius $r < \frac{\pi}{2}$ centered at p in $\mathbb{S}^{n+1}$. Let $B_p^n(r) = B_p^{n+1}(r) \cap \mathbb{S}^n$. We then have the following

Proposition 2.4. *For* $r < \frac{\pi}{2}$*, there exist some* $\phi \in C^\infty((B_p^n(r))^\circ) \cap C(B_p^n(r))$ *which is non-negative and satisfies*

$$\frac{\tan^2 d(x,p)}{\tan^2 r} + \frac{\sin^2 \phi(x)}{\sin^2 r} = 1$$

(where d *is the distance function in* $\mathbb{S}^n$*), so that the geodesic sphere* $\partial B_p^{n+1}(r)$ *can be written as*

$$\{(x, \pm\phi(x)) \mid x \in B_p^n(r)\}$$

in the product space $\mathbb{S}^n \times (-\frac{\pi}{2}, \frac{\pi}{2})_t$*.*

3 The warped product structure of Ω and proof of Theorem 1.1

Now let us concentrate on Theorem 1.1. We first consider the case that f is constant on some boundary component.

Proposition 3.1. *Let* (Ω^{n+1}, g) *and* f *be as in Theorem 1.1. If* f *is constant on some boundary component, then* Σ *is not connected. Moreover, we also have*

$$\max_\Omega f = \max_\Sigma f = \frac{1}{\sqrt{c^2 - 1}} = \sinh\theta$$

and

$$\min_\Omega f = \min_\Sigma f = \frac{-1}{\sqrt{c^2 - 1}} = -\sinh\theta.$$

Proof. Since $|\nabla f|^2 - f^2 = 1$ and $f_\nu = cf$, we have

$$|\nabla_\Sigma f|^2 + (c^2 - 1)f^2 = 1, \text{ on } \Sigma.$$

Let $S \subset \Sigma$ be the boundary component such that $f|_S$ is constant. Without loss of generality, we assume that

$$f|_S = \frac{-1}{\sqrt{c^2 - 1}} = -\sinh\theta.$$

If Σ is connected, we then know that there exists a point $q \in \Omega^\circ$ such that

$$f(q) = \max_\Omega f$$

and

$$\nabla f(q) = 0,$$

which is a contradiction, i.e. Σ is not connected. In addition, we also conclude that

$$\max_\Omega f = \max_\Sigma f$$

and

$$\min_\Omega f = \min_\Sigma f.$$

Since f is non-constant on Ω and $|\nabla_\Sigma f|^2 + (c^2 - 1)f^2 = 1$, we then have

$$\max_\Omega f = \frac{1}{\sqrt{c^2 - 1}} = \sinh\theta$$

and

$$\min_\Omega f = \frac{-1}{\sqrt{c^2 - 1}} = -\sinh\theta.$$

This finishes the proof. ∎

Proposition 3.2. *Let $S \subset \Sigma$ be the boundary component such that $f|_S = -\sinh\theta$ as in Proposition 3.1. Then Ω is isometirc to the warped product space*

$$S \times [0, 2\theta]_t, \;\; g = dt^2 + \frac{(\cosh(t - \theta))^2}{(\cosh\theta)^2} g|_S.$$

Proof. Since $f|_S = -\sinh\theta$ and $f_\nu = cf$, we have

$$f_\nu|_S = -\cosh\theta.$$

$\forall\, p \in S$, we consider the integral curve $\gamma_p(t)$ of $\frac{\nabla f}{|\nabla f|}$ starting from p. By Proposition 2.2, we know that γ_p is a geodesic. Then a direct calculation shows

$$f(\gamma_p(t)) = \sinh(t - \theta).$$

Since Ω is compact and $\nabla f \neq 0$, we know that $\gamma_p(t)$ will meet $\Sigma \setminus S$ at a time t_p. Let $t_{p_0} = \min_{p\in S} t_p$ and γ_{p_0} be the corresponding geodesic starting from p_0 such that $\gamma_{p_0}(t_{p_0}) \in \Sigma \setminus S$. We then know that $\gamma'_{p_0}(t_{p_0}) \perp T_{\gamma_{p_0}(t_{p_0})}\Sigma$. Therefore, we have

$$\frac{d}{dt} f(\gamma_{p_0}(t))|_{t=t_{p_0}} = cf(\gamma_{p_0}(t_{p_0})),$$

that is,

$$\cosh(t_{p_0} - \theta) = c\sinh(t_{p_0} - \theta),$$

which shows

$$t_{p_0} = 2\theta$$

and

$$f(\gamma_{p_0}(t_{p_0})) = \sinh\theta = \max_{\Omega} f.$$

Then for any $p \in S$, we have

$$t_p = 2\theta.$$

Let $\gamma_{p_0}(2\theta)$ belongs to some boundary component $S' \neq S$. For any $p \in S$, there exists a smooth curve $\phi(s)$ in S from p_0 to p. Then $\gamma_{\phi(s)}(2\theta) = \exp_{\phi(s)}(2\theta\frac{\nabla f}{|\nabla f|}(\phi(s)))$ is a smooth curve in Σ from $\gamma_{p_0}(2\theta)$ to $\gamma_p(2\theta)$, which shows that $\gamma_p(2\theta) \in S'$, $\forall\, p \in S$.

Now, by Morse theory, we know that Ω is isometric to the warped product space $S \times [0, 2\theta]_t$ with mertic

$$g = dt^2 + g(t),$$

where $g(t)$ is a family of metrics on S and $g(0) = g|_S$. Moreover,

$$f = \sinh(t - \theta).$$

Since $\nabla^2 f - fg = 0$, a direct calculation shows that

$$\frac{1}{2} f'(t)g'(t) - f(t)g(t) = 0,$$

that is,

$$\frac{1}{2}\cosh(t-\theta)g'(t) = \sinh(t-\theta)g(t),$$

which shows that

$$g(t) = \frac{(\cosh(t-\theta))^2}{(\cosh\theta)^2} g|_S.$$

We then finish the proof. □

Proof of Theorem 1.1 (1): The first case in Theorem 1.1 follows from Propositions 3.1 and 3.2. □

Next, we concentrate on the second case in Theorem 1.1. Since f is non-constant on any boundary component and $|\nabla f| \neq 0$, we conclude that Ω_0 is a compact manifold of dimension n with boundary $\partial\Omega_0$. Moreover, we also have $\Omega_0^\circ = \Omega^\circ \cap \Omega_0$ and $\partial\Omega_0 = \Sigma \cap \Omega_0$.

Proposition 3.3. *(1) For any $p \in \Omega$ with $f(p) > 0$, the integral curve of $-\frac{\nabla f}{|\nabla f|}$ starting at p will meet Ω_0° before it reaches Σ. For any $p \in \Omega$ with $f(p) < 0$, the integral curve of $\frac{\nabla f}{|\nabla f|}$ starting at p will meet Ω_0° before it reaches Σ.*

(2) For $0 < a_1 < a_2$, the integral curve of $-\frac{\nabla f}{|\nabla f|}$ defines an injective map from Ω_{a_2} to Ω_{a_1}.For $0 > a_1 > a_2$, the integral curve of $\frac{\nabla f}{|\nabla f|}$ defines an injective map from Ω_{a_2} to Ω_{a_1}.

Proof. Take a point p with $f(p) \neq 0$. By a similar proof of Lemma 4.3 in [2], we know that the corresponding integral curve (geodesic) starting at p will meet Ω_0 before it reaches Σ.

Now, we prove that the integral curve meets Ω_0° before it reaches Σ. In fact, we only need to prove that for any point $q \in \partial\Omega_0$, the geodesic $\gamma_q(t)$ which satisfies $\gamma_q(0) = q$ and $\gamma_q'(0) = \frac{\nabla f}{|\nabla f|}(q)$ or $-\frac{\nabla f}{|\nabla f|}(q)$ is not contained in Ω° when t is close to 0. Since $f(q) = 0$ and $f_\nu = cf$, we know that $\frac{\nabla f}{|\nabla f|}(q)$ is tangent to $T_q\Sigma$. Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal frame near q in $T\Sigma$ such that $e_1(q) = \frac{\nabla f}{|\nabla f|}(q)$, then at the point q, we have

$$c(e_1 f) = e_1(\nu f) = \nabla^2 f(e_1, \nu) + \nabla^\Omega_{e_1}\nu f = \nabla^\Omega_{e_1}\nu f,$$

which shows

$$h(e_1, e_1) = c > 0.$$

We then know that the geodesic $\gamma_q(t)$ discussed above is not contained in Ω° when t is close to 0.

Obviously, (2) follows from (1). ∎

For any $a \in (-\sinh\theta, \sinh\theta)$, by Proposition 3.3, we know that the integral curve discussed above defines an injective map

$$\Psi_a : \Omega_a \to \Omega_0.$$

Therefore, Ω can be considered as a bounded domain of the warped product space $(\widehat{\Omega}, g)$, where

$$\widehat{\Omega} = \Omega_0 \times (-\infty, \infty)_t,\ \ g = dt^2 + (\cosh t)^2 g|_{\Omega_0}$$

and

$$f = \sinh t.$$

Since Ω is connetced and $\Omega_0 \subset \Omega$, we also know that Ω_0 is connected.

As for the boundary Σ, we have the following

Proposition 3.4. *Let $\Sigma_+ = \{p \in \Sigma \mid f(p) \geq 0\}$. Then*

$$\Sigma_+ = \{(x, \phi(x)) \in \widehat{\Omega} \mid x \in \Omega_0\},$$

where $\phi \in C^\infty(\Omega_0^\circ) \cap C(\Omega_0)$ is non-negative and satisfies

$$\frac{\cosh\phi}{\sqrt{1 + (\cosh\phi)^{-2}|\nabla_{\Omega_0}\phi|^2_{g|_{\Omega_0}}}} = c\sinh\phi, \ \ in\ \Omega_0^\circ,$$

$$\phi > 0, \quad in\ \Omega_0^\circ,$$

and

$$\phi = 0, \quad on\ \partial\Omega_0.$$

Proof. Let $S = \{p \in \Sigma | f(p) > 0\}$. Obviously, $\Sigma_+ = S \cup \partial\Omega_0$. For any $p \in S$, by Proposition 3.3, the integral curve of $-\frac{\nabla f}{|\nabla f|}$ starting at p will meet Ω_0° before it reaches Σ. In addition, for any $x \in \Omega_0^\circ$, the integral curve of $\frac{\nabla f}{|\nabla f|}$ starting at x will meet S at some point p. Let $\gamma : [0, t_x] \to \Omega$ be the integral curve (geodesic), where $\gamma(0) = x$, $\gamma(t_x) = p$ and $\gamma' = \frac{\nabla f}{|\nabla f|}$. Obviously, $\gamma([0, t_x)) \subset \Omega^\circ$ and γ will stop at the point p. Thus we can define a positive function

$$\phi : \Omega_0^\circ \to \mathbb{R},$$

where $\phi(x) = t_x$. By the above discussion, we also conclude

$$S = \{(x, \phi(x)) \in \widehat{\Omega} \mid x \in \Omega_0^\circ\}.$$

Since the boundary Σ is a smooth manifold, the function ϕ is smooth in Ω_0°. For any point $p = (x, \phi(x)) \in S$, the outward unit normal of Ω is

$$\nu = \frac{(-\nabla_{\widehat{\Omega}_{\phi(x)}} \phi, 1)}{\sqrt{1 + (\cosh\phi)^{-2} |\nabla_{\Omega_0} \phi|^2_{g_{\Omega_0}}}},$$

where $\widehat{\Omega}_{\phi(x)} = \Omega_0 \times \{\phi(x)\}$ (consider ϕ as a function on $\widehat{\Omega}_{\phi(x)}$), $g|_{\widehat{\Omega}_{\phi(x)}} = (\cosh\phi(x))^2 g|_{\Omega_0}$. Since $f = \sinh t$ and $\frac{\partial f}{\partial \nu} = cf$, we then have

$$\begin{aligned} c \sinh\phi = \frac{\partial f}{\partial \nu} = \nabla f \cdot \nu =& (0, \cosh\phi) \cdot \nu \\ =& \frac{\cosh\phi}{\sqrt{1 + (\cosh\phi)^{-2} |\nabla_{\Omega_0} \phi|^2_{g_{\Omega_0}}}}. \end{aligned}$$

Now, we prove $\phi(x) \to 0$ if and only if $x \to \partial\Omega_0$. If there exists a sequence $\{x_m\} \subset \Omega_0^\circ$ such that $d(x_m, \partial\Omega_0) \to 0$ but $\phi(x_m)$ does not converge to 0, then we can find a subsequence (denote also by $\{x_m\}$) such that $x_m \to x \in \partial\Omega_0$ and $\phi(x_m) \to a > 0$. In other words, the sequence $\{(x_m, \phi(x_m))\}$ converges to the point (x, a). Since Σ is compact, we know $(x, a) \in \Sigma$. In particular, $(x, a) \in S \subset \Sigma_+$. However, by Proposition 3.3, we conclude $x \in \Omega_0^\circ$, which is a contradiction. Therefore, $\phi(x) \to 0$ when $x \to \partial\Omega_0$.

Similarly, if there exists a sequence $\{x_m\} \subset \Omega_0^\circ$ such that $\phi(x_m) \to 0$ but $d(x_m, \partial\Omega_0)$ does not converge to 0, then we can find a subsequence (denote also by $\{x_m\}$), where $x_m \to p \in \Omega_0^\circ$ and $\phi(x_m) \to 0$. We then know that $\phi(p) = 0$, which is a contradiction.

Thus, we can extend ϕ to $\partial\Omega_0$ and define the continuous function as following

$$\phi(x) = \begin{cases} t_x, & x \in \Omega_0^\circ, \\ 0, & x \in \partial\Omega_0. \end{cases}$$

Since $\Sigma_+ = S \cup \partial\Omega_0$, we then know that

$$\Sigma_+ = \{(x, \phi(x)) \in \widehat{\Omega} \mid x \in \Omega_0\}.$$

This finishes the proof. □

Since

$$\frac{\cosh\phi}{\sqrt{1 + (\cosh\phi)^{-2}|\nabla_{\Omega_0}\phi|^2_{g|_{\Omega_0}}}} = c\sinh\phi,$$

we know that $0 \leq \phi \leq \coth^{-1}(c) = \theta$. Proposition 3.4 also implies the following

Proposition 3.5. *Σ_+ is homeomorphic to Ω_0 and thus Σ_+ is connected.*

Proof. We define a continuous map

$$\Psi : \Sigma_+ \to \Omega_0,$$

where $\Psi((x, \phi(x)) = x$. By Proposition 3.4, we know that Ψ is bijective and Ψ^{-1} is continuous. Therefore, Σ_+ is homeomorphic to Ω_0. Since Ω_0 is connected, we also know that Σ_+ is connected. □

Let $\Sigma_- = \{p \in \Sigma | f(p) \leq 0\}$. Similarly, we also have

Proposition 3.6. *(1) $\Sigma_- = \{(x, -\psi(x)) \in \widehat{\Omega} | x \in \Omega_0\}$, where $\psi \in C^\infty(\Omega_0^\circ) \cap C(\Omega_0)$ is non-negative and satisfies*

$$\frac{\cosh\psi}{\sqrt{1 + (\cosh\psi)^{-2}|\nabla_{\Omega_0}\psi|^2_{g|_{\Omega_0}}}} = c\sinh\psi, \quad \text{in } \Omega_0^\circ,$$

$$\psi > 0, \quad \text{in } \Omega_0^\circ,$$

and

$$\psi = 0, \quad \text{on } \partial\Omega_0.$$

(2) Σ_- is homeomorphic to Ω_0 and thus Σ_- is connected.

By Propositions 3.5 and 3.6, we conclude that Σ is connected and $0 \leq \psi \leq \theta$. Now, we only need to prove $\phi = \psi$.

Proposition 3.7. *There exists a unique solution to the following equation*

$$\frac{\cosh\phi}{\sqrt{1 + (\cosh\phi)^{-2}|\nabla_{\Omega_0}\phi|^2_{g|_{\Omega_0}}}} = c\sinh\phi, \quad \text{in } \Omega_0^\circ,$$

such that

$$0 < \phi \leq \theta, \ \ in\ \Omega_0^\circ,$$

and

$$\phi = 0, \ \ on\ \partial\Omega_0.$$

Proof. Assume that ϕ and ψ are the solutions and $\phi \not\equiv \psi$. Without loss of generality, we assume

$$\max_{\Omega_0} |\phi - \psi| = \phi(p) - \psi(p) > 0,$$

where $p \in \Omega_0^\circ$. Thus

$$0 < \psi(p) < \phi(p) \leq \theta.$$

A direct calculation shows

$$\nabla_{\Omega_0}(\phi - \psi) \cdot \nabla_{\Omega_0}(\phi + \psi) = \frac{\cosh^4 \phi}{c^2 \sinh^2 \phi} - \cosh^2 \phi - \frac{\cosh^4 \psi}{c^2 \sinh^2 \psi} + \cosh^2 \psi.$$

It is elementary to show that the function

$$h(t) = \frac{\cosh^4 t}{c^2 \sinh^2 t} - \cosh^2 t$$

is monotonically decreasing when $t \in (0, \theta)$. Therefore, at the point p, we have

$$\nabla_{\Omega_0}(\phi - \psi) \cdot \nabla_{\Omega_0}(\phi + \psi) = 0$$

and

$$\frac{\cosh^4 \phi}{c^2 \sinh^2 \phi} - \cosh^2 \phi - \frac{\cosh^4 \psi}{c^2 \sinh^2 \psi} + \cosh^2 \psi < 0,$$

which is a contradiction. □

Proof of Theorem 1.1 (2): The second case in Theorem 1.1 follows dirctly from Propositions 3.3–3.7. □

4 The structures of Ω_0 and proofs of Theorems 1.2, 1.3 and 1.4

In this section, we concentrate on the structure of Ω_0. We first determine the structure of Ω_0 in Theorem 1.1 (1) by assumption (K) and the lower bound condition for the diameter of boundary component in Theorem 1.2.

Proposition 4.1. *Let (Ω^{n+1}, g) and f be as in Theorem 1.2. Let S be the boundary component which satisfies the assumption (K). Then we have*

$$Ric^S \geq (n-1)(1 - \frac{1}{c^2}) > 0.$$

Proof. Take an orthonormal frame $\{e_1, e_2, \cdots, e_n\}$ in TS. By Gauss equation, for $i = 2, \cdots, n$, we have

$$R^S(e_1, e_i, e_1, e_i) = R^\Omega(e_1, e_i, e_1, e_i) - h(e_1, e_1)h(e_i, e_i) + h(e_1, e_i)h(e_1, e_i).$$

A direct caculation shows that $h = \frac{1}{c} g|_S$, we then know

$$R^S(e_1, e_i, e_1, e_i) = R^\Omega(e_1, e_i, e_1, e_i) - \frac{1}{c^2}.$$

Since S satisfies the assumption (K), for any subset $\{i_1, i_2, \cdots, i_{k-1}\} \subset \{2, \cdots, n\}$, we have

$$\begin{aligned}\sum_{j=1}^{k-1} -R^S(e_1, e_{i_j}, e_1, e_{i_j}) &\geq (k-1)(1 - \frac{2}{c^2}) + (k-1)\frac{1}{c^2} \\ &= (k-1)(1 - \frac{1}{c^2}).\end{aligned}$$

Obviously, the number of the inequalities discussed above is C_{n-1}^{k-1}. By summing up all these inequalities, we have

$$\sum_{j=2}^{n} -C_{n-2}^{k-2} R^S(e_1, e_j, e_1, e_j) \geq C_{n-1}^{k-1}(k-1)(1 - \frac{1}{c^2}),$$

that is,

$$\mathrm{Ric}^S(e_1, e_1) \geq (n-1)(1 - \frac{1}{c^2}) > 0.$$

□

Proposition 4.2. *Let S be the boundary component discussed above. If we further assume that the diameter d of S satisfies*

$$d \geq \frac{c}{\sqrt{c^2 - 1}}\pi,$$

then Ω_0 is isometric to the standard sphere $\mathbb{S}^n$.

Proof. By Bonnet-Myers theorem and Cheng's theorem, we conclude that S is isometric to a sphere of radius $\cosh\theta$ ($c = \coth\theta$). Since Ω_0 is conformal to S, then a direct calculation shows that Ω_0 is isometric to the standard sphere $\mathbb{S}^n$. □

Proof of Theorem 1.2: Theorem 1.2 follows from Propositions 4.1 and 4.2. □

We now concentrate on the second case in Theorem 1.1. In this case, Ω_0 is a compact manifold with boundary $\partial\Omega_0 = \Omega_0 \cap \Sigma$.

Proof of Theorem 1.3: By the curvature assumptions in Theorem 1.3 and Lemma 2.1 in [10], we know that Σ is connected. Then by Theorem 1.1 (2), we conclude that Ω is a Z_2-symmetric domain in the warped product space

$$\widehat{\Omega} = \Omega_0 \times (-\infty, \infty)_t, \ g = dt^2 + (\cosh t)^2 g|_{\Omega_0},$$

which is bounded by the graph functions $\pm\phi$, where $\phi \in C^\infty(\Omega_0^\circ) \cap C(\Omega_0)$ satisfies

$$\frac{\cosh\phi}{\sqrt{1+(\cosh\phi)^{-2}|\nabla_{\Omega_0}\phi|^2_{g|_{\Omega_0}}}} = c\sinh\phi, \text{ in } \Omega_0^\circ,$$

$$\phi > 0, \text{ in } \Omega_0^\circ$$

and

$$\phi = 0, \text{ on } \partial\Omega_0.$$

In particular, in the coordinate of $\Omega_0 \times (-\infty,\infty)_t$, we have $f = \sinh t$ and $\nabla f = (\cosh t)\frac{\partial}{\partial t}$.

Let $\widehat{\Omega}_t = \Omega_0 \times \{t\}$, where $t \in (-\infty,\infty)$ (obviously, $\widehat{\Omega}_0 = \Omega_0$). We first consider the second fundamental form with respect to $\frac{\partial}{\partial t}$, denoted by $h_{\widehat{\Omega}_t}$. Let $(x_1, x_2, \cdots, x_n)$ be a coordinate in Ω_0, then the metric has the form

$$g = dt^2 + (\cosh t)^2 g_{ij} dx^i dx^j,$$

where $g|_{\Omega_0} = g_{ij}dx^i dx^j$. A direct calculation shows that

$$\begin{aligned} h_{\widehat{\Omega}_t}(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) &= -\langle \nabla^{\widehat{\Omega}}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial t}\rangle \\ &= -\Gamma^t_{ij} \\ &= \frac{\sinh t}{\cosh t}\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\rangle, \end{aligned}$$

which implies that

$$h_{\widehat{\Omega}_t} = \frac{\sinh t}{\cosh t} g|_{\widehat{\Omega}_t}.$$

Therefore, Ω_0 is a totally geodesic hypersurface in $\widehat{\Omega}$ with unit normal $\frac{\partial}{\partial t}$.

Clearly, the graph function ϕ has the following properties. First, $\phi \in [0, \tanh^{-1}(\frac{1}{c})]$ and $\phi|_{\partial\Omega_0} = 0$. Second, the set $\{x \in \Omega_0 \mid \phi(x) = \tanh^{-1}(\frac{1}{c})\}$ (denoted by A) is a compact subset in Ω_0.

Define

$$v = \tanh^{-1}(\frac{1}{c}\sqrt{1-(c^2-1)\sinh^2\phi}).$$

Then v also has the following properties. First, $v \in [0, \tanh^{-1}(\frac{1}{c})]$ and $v|_{\partial\Omega_0} = \tanh^{-1}(\frac{1}{c})$. Second, v is smooth at any x with $v(x) \in (0, \tanh^{-1}(\frac{1}{c}))$. Third, $\{x \in \Omega_0 \mid v(x) = 0\} = A$. In addition, a direct calculation shows that $|\nabla_{\Omega_0} v| \equiv 1$ on $\Omega_0 - (\partial\Omega_0 \cup A)$.

We now consider the level set of v. Set

$$T_t = \{x \in \Omega_0 \mid v(x) = t\},$$

where $t \in [0, \tanh^{-1}(\frac{1}{c})]$. Obviously, $T_0 = A$ and $T_{\tanh^{-1}(\frac{1}{c})} = \partial\Omega_0$. We now prove

$$t \le d(A, T_t),$$

where d is the distance function in Ω_0. Fix $t \in (0, \tanh^{-1}(\frac{1}{c})]$. Let $\widehat{\gamma} : [0, l] \to \Omega_0$ be a minimizing geodesic realizing the distance $d(A, T_t)$ with arc length parameter such that $\widehat{\gamma}(0) \in A$, $\widehat{\gamma}(l) \in T_t$, and $l = d(A, T_t)$. We then have

$$\begin{aligned} t &= \lim_{\epsilon \to 0^+} v(\widehat{\gamma}(s))|_{\epsilon}^{l-\epsilon} \\ &= \lim_{\epsilon \to 0^+} \int_{\epsilon}^{l-\epsilon} \frac{d}{ds}(v(\widehat{\gamma}(s)))ds \\ &= \lim_{\epsilon \to 0^+} \int_{\epsilon}^{l-\epsilon} \langle \nabla_{\Omega_0} v, \gamma' \rangle ds \\ &\le l \\ &= d(A, T_t), \end{aligned}$$

where $\langle \nabla_{\Omega_0} v, \gamma' \rangle \le |\nabla_{\Omega_0} v| = 1$. Therefore, we conclude that $t \le d(A, T_t)$. In particular, $\tanh^{-1}(\frac{1}{c}) \le d(A, T_{\tanh^{-1}(\frac{1}{c})}) = d(A, \partial\Omega_0)$.

Now, we prove that Ω_0 is a geodesic ball of radius $\tanh^{-1}(\frac{1}{c})$ in the hyperbolic space $\mathbb{H}^n$ by the curvature assumptions.

First, let $e_1 = \frac{\partial}{\partial t}$ and $\{e_2, e_3, \cdots, e_{n+1}\}$ be an orthonormal frame in $T\Omega_0$, then we know that $\{e_1, e_2, \cdots, e_{n+1}\}$ is an orthonormal frame in $T\widehat{\Omega}$. Since Ω_0 is totally geodesic, by the Gauss equation, we have

$$R^{\Omega_0}(e_i, e_j, e_k, e_l) = R^{\widehat{\Omega}}(e_i, e_j, e_k, e_l),$$

where $i, j, k, l \in \{2, 3, \cdots, n+1\}$. In addition, by the Ricci identity, we obtain

$$\begin{aligned} f_k\delta_{ij} - f_j\delta_{ik} &= f_{ijk} - f_{ikj} \\ &= -\sum_{p=1}^{n+1} R^{\widehat{\Omega}}(e_p, e_i, e_j, e_k) f_p \\ &= -R^{\widehat{\Omega}}(e_1, e_i, e_j, e_k) f_1, \end{aligned}$$

which implies

$$R^{\widehat{\Omega}}(e_1, e_j, e_1, e_j) = 1,$$

where $j \in \{2, 3, \cdots, n+1\}$. Therefore, we can deduce

$$\begin{aligned} \mathrm{Ric}^{\Omega_0}(e_i, e_i) &= -\sum_{j=2, j\neq i}^{n+1} R^{\Omega_0}(e_i, e_j, e_i, e_j) \\ &= -\sum_{j=2, j\neq i}^{n+1} R^{\widehat{\Omega}}(e_i, e_j, e_i, e_j) \\ &= -\sum_{j=1, j\neq i}^{n+1} R^{\widehat{\Omega}}(e_i, e_j, e_i, e_j) + 1 \\ &\ge -(n-1). \end{aligned}$$

Next, we prove that the second fundamental form $h_{\partial\Omega_0}$ of $\partial\Omega_0$ in Ω_0 satisfies $h_{\partial\Omega_0} = h|_{\partial\Omega_0}$, where h is the the second fundamental form of Σ in Ω. In fact, $\forall\, P \in \partial\Omega_0$, we have $f_\nu(P) = cf(P) = 0$, that is, $\langle\nabla f, \nu\rangle(P) = 0$, and we know that $\nabla f(P) \in T_P\Sigma$. Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal frame in $T\Sigma$ near the point P such that $e_1(P) = \nabla f(P)$. Since Ω_0 is totally geodesic with constant unit normal $\frac{\partial}{\partial t}$ and $\nabla f(P) = \frac{\partial}{\partial t}(P)$, we know that $\{e_2, e_3, \cdots, e_n\}$ is an orthonormal frame for $\partial\Omega_0$ at the point P and ν is the unit outward normal of $\partial\Omega_0$ in Ω_0. In addition, for $i \in \{2, 3, \cdots, n\}$ we have

$$0 = \nabla^2 f(e_i, \nu) = e_i(\nu f) - \nabla^{\widehat{\Omega}}_{e_i}\nu f = cf_i - \sum_{j=1}^{n} h_{ij} f_j,$$

which implies that $h_{11} = c$ and $h_{1i} = 0$ for $i \in \{2, 3, \cdots, n\}$. So e_1 is a principal direction of Σ at P corresponding to the principal curvature c. Now, let $\{e_2, e_3, \cdots, e_n\}$ be an orthonormal frame in $T\partial\Omega_0$, since Ω_0 is totally geodesic with constant unit normal ∇f, we then have

$$h_{\partial\Omega_0}(e_i, e_j) = -\langle\nabla^{\Omega_0}_{e_i} e_j, \nu\rangle = -\langle\nabla^{\widehat{\Omega}}_{e_i} e_j, \nu\rangle = h(e_i, e_j),$$

where $i, j \in \{2, 3, \cdots, n\}$. Therefore, we conclude that $h_{\partial\Omega_0} = h|_{\partial\Omega_0}$ and we know that the mean curvature of $\partial\Omega_0$ in Ω_0, denoted by $H_{\partial\Omega_0}$, satisfies

$$\begin{aligned} H_{\partial\Omega_0} = \frac{1}{n-1}\mathrm{tr}_{g_{\partial\Omega_0}} h_{\partial\Omega_0} =& \frac{1}{n-1}\mathrm{tr}_{g_{\partial\Omega_0}} h|_{\partial\Omega_0} \\ \geq& \frac{1}{n-1}(nc - c) \\ =& c. \end{aligned}$$

Thus, by the fact that $\tanh^{-1}(\frac{1}{c}) \leq d(A, \partial\Omega_0)$ and Theorem 0.3 in [6], we conclude that Ω_0 is isometric to a geodesic ball of radius $\tanh^{-1}(\frac{1}{c})$ in the hyperbolic space $\mathbb{H}^n$ and A consists of a single point x_0, which is the center of Ω_0. For convenience, we just assume that Ω_0 is the geodesic ball in $\mathbb{H}^n$. Then we know that $\widehat{\Omega}$ is a domain in the hyperbolic space $\mathbb{H}^{n+1}$, and so do Ω.

Now, let $t \in (0, \tanh^{-1}(\frac{1}{c}))$, we prove that $T_t = S_t$, where S_t is the geodesic sphere in Ω_0 of radius t centered at x_0. On the one hand, we have known that $t \leq d(x_0, T_t)$. On the other hand, we can use the similar method discussed above to prove that $\tanh^{-1}(\frac{1}{c}) - t \leq d(T_t, \partial\Omega_0)$, and we then have $T_t = S_t$, that is,

$$v(x) = d(x, x_0).$$

Since

$$v = \tanh^{-1}(\frac{1}{c}\sqrt{1 - (c^2 - 1)\sinh^2\phi}),$$

we have

$$\frac{\tanh^2 d(x, x_0)}{\tanh^2\theta} + \frac{\sinh^2\phi(x)}{\sinh^2\theta} = 1,$$

where $\theta = \tanh^{-1}(\frac{1}{c})$. By Proposition 2.3, we know that Σ is a geodesic sphere in the hyperbolic space $\mathbb{H}^{n+1}$ of radius $\tanh^{-1}(\frac{1}{c})$ centered at x_0 and thus Ω is a geodesic ball of radius $\tanh^{-1}(\frac{1}{c})$ in the hyperbolic space $\mathbb{H}^{n+1}$. $\square$

By a similar argument, we can prove Theorem 1.4. In this case, we consider the following Obata type equation

$$\begin{cases} \nabla^2 f + fg = 0, & \text{in}\, \Omega, \\ f_\nu - cf = 0, & \text{on}\, \Sigma, \end{cases}$$

where c is a positive constant. For convenience, we still assume that

$$|\nabla f|^2 + f^2 = 1.$$

Set $\Omega_0 = \{p \in \Omega \mid f(p) = 0\}$, by Theorem 1.3 in [2] and the curvature assumptions in Theorem 1.4, we conclude that Σ is connected (see also [7]) and Ω is a Z_2-symmetric domain in the warped product space

$$\widehat{\Omega} = \Omega_0 \times (-\frac{\pi}{2}, \frac{\pi}{2})_t, \;\; g = dt^2 + (\cos t)^2 g|_{\Omega_0},$$

which is bounded by the graph functions $\pm\phi$, where $\phi \in C^\infty(\Omega_0^\circ) \cap C(\Omega_0)$ satisfies

$$\frac{\cos\phi}{\sqrt{1 + (\cos\phi)^{-2}|\nabla_{\Omega_0}\phi|^2_{g|_{\Omega_0}}}} = c\sin\phi, \;\; \text{in}\; \Omega_0^\circ,$$

$$\phi > 0, \;\; \text{in}\; \Omega_0^\circ,$$

and

$$\phi = 0, \;\; \text{on}\; \partial\Omega_0.$$

In particular, $f = \sin t$.

Let $\widehat{\Omega}_t = \Omega_0 \times \{t\}$, where $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$. We first consider the second fundamental form with respect to $\frac{\partial}{\partial t}$, denoted by $h_{\widehat{\Omega}_t}$. By a similar method in the proof of Theorem 1.3, we have

$$h_{\widehat{\Omega}_t} = -\frac{\sin t}{\cos t} g|_{\widehat{\Omega}_t}.$$

Therefore, $h_{\widehat{\Omega}_t}$ is negative definite when $t > 0$ and Ω_0 is a totally geodesic hypersurface in $\widehat{\Omega}$ with unit normal $\frac{\partial}{\partial t}$.

As for the graph function ϕ. Obviously, $\phi \in [0, \tan^{-1}(\frac{1}{c})]$ and $\phi|_{\partial\Omega_0} = 0$. Here we can provide another way to prove that the set $\{x \in \Omega_0 \mid \phi(x) = \tan^{-1}(\frac{1}{c})\}$ consists of a single point, denoted by x_0, which is different from the method in Theorem 1.3. To see this, we first prove Lemma 4.3. The idea of this lemma comes from Lemma 2.1 in [3].

Lemma 4.3. *Let Ω and f be as in Theorem 1.4. Assume that $p \in \Sigma$ is a critical point of $f|_\Sigma$, then at the point p, the second fundamental form h satisfies*

$$h = cg|_\Sigma.$$

Proof. For convenience, let $z = f|_\Sigma$. Since $|\nabla f|^2 + f^2 = 1$ and $f_\nu = cf$, on Σ, we have

$$1 = |\nabla f|^2 + f^2 = |\nabla_\Sigma z|^2 + (f_\nu)^2 + z^2 = |\nabla_\Sigma z|^2 + (c^2+1)z^2$$

and we know that $z(p) \neq 0$.

Let $\{e_1, e_2, \cdots, e_{n+1}\}$ be an orthonormal frame near the boundary satisfying $e_{n+1}|_\Sigma = \nu$. $\forall\, i, j \in \{1, 2, \cdots, n\}$, we have

$$\begin{aligned} z_{ij} &= e_i(e_j z) - \nabla^\Sigma_{e_i} e_j z \\ &= e_i(e_j z) - \nabla_{e_i} e_j z - h_{ij} f_\nu \\ &= \nabla^2 f(e_i, e_j) - h_{ij} f_\nu \\ &= -z\delta_{ij} - ch_{ij} z \\ &= -(ch_{ij} + \delta_{ij}) z \end{aligned}$$

and

$$\begin{aligned} cz_i &= e_i(e_{n+1} z) - \nabla_{e_i} e_{n+1} f + \nabla_{e_i} e_{n+1} f \\ &= \nabla^2 f(e_i, e_{n+1}) + \sum_{j=1}^n h_{ij} z_j \\ &= \sum_{j=1}^n h_{ij} z_j. \end{aligned}$$

By taking covariant differentiation with respect to e_k ($k \in \{1, 2, \cdots, n\}$) at p, we have

$$\begin{aligned} -c(ch_{ik} + \delta_{ik}) z &= cz_{ik} \\ &= \sum_{j=1}^n e_k h_{ij} z_j + \sum_{j=1}^n h_{ij} z_{jk} \\ &= \sum_{j=1}^n -h_{ij}(ch_{jk} + \delta_{jk}) z, \end{aligned}$$

which implies

$$c^2 h(p) + cI = ch^2(p) + h(p),$$

where $h(p)$ is the $n \times n$ matrix (h_{ij}) and I is the identity matrix. We then have

$$(ch(p) + I)(h(p) - cI) = 0.$$

Since the mean curvature $H \geq c$, we obtain that

$$h(p) = cI.$$

This finishes the proof of the lemma. ∎

Now, we can prove that $\{x \in \Omega_0 \mid \phi(x) = \tan^{-1}(\frac{1}{c})\} = \{x_0\}$.

Proposition 4.4. *The set* $\{x \in \Omega_0 \mid \phi(x) = \tan^{-1}(\frac{1}{c})\}$ *consists of a single point, denoted by* x_0.

Proof. Since $f = \sin t$, we only need to prove that $\{x \in \Sigma \mid f(x) = \frac{1}{\sqrt{c^2+1}}\}$ consists of a single point. Since $|\nabla_\Sigma f|^2 + (c^2+1)f^2 = 1$, we know that $f|_\Sigma$ is a transnormal function on Σ. Then by Theorem A in [17], we know that $\{x \in \Sigma | f(x) = \frac{1}{\sqrt{c^2+1}}\}$ is a submanifold of Σ, denoted by M. In particular, we know that any points in M are critical points of $f|_\Sigma$ and the outward unit normal ν satisfies $\nu = \frac{\partial}{\partial t}$ on M. Assume that $\dim M \geq 1$, then we can consider a smooth curve $\gamma : [0, l] \to M$ with arc length parameter. Obviously, γ is a smooth curve in Σ and we also know that $\gamma([0, l]) \subset \widehat{\Omega}_{\tan^{-1}(\frac{1}{c})}$. We then have

$$
\begin{aligned}
h(\gamma', \gamma') &= -\langle \nabla^{\widehat{\Omega}}_{\gamma'} \gamma', \nu \rangle \\
&= -\langle \nabla^{\widehat{\Omega}}_{\gamma'} \gamma', \frac{\partial}{\partial t} \rangle \\
&= h_{\widehat{\Omega}_{\tan^{-1}(\frac{1}{c})}}(\gamma', \gamma') \\
&= -\frac{1}{c} \\
&< 0.
\end{aligned}
$$

However, by Lemma 4.3, we have

$$h(\gamma', \gamma') = c > 0,$$

which is a contradiction. Therefore, $\dim M = 0$. Then by Lemma 2.6 in [2], we know that M is connected and consists of a single point. Hence, $\{x \in M_0 \mid \phi(x) = \tan^{-1}(\frac{1}{c})\} = \{x_0\}$. □

The remaining part of the proof of Theorem 1.4 is similar to that of Theorem 1.3.

Proof of Theorem 1.4: Set

$$v = \tan^{-1}\left(\frac{1}{c}\sqrt{1 - (c^2+1)\sin^2\phi}\right),$$

v then has the following properties

(1) $v \in [0, \tan^{-1}(\frac{1}{c})]$ and $v|_{\partial\Omega_0} = \tan^{-1}(\frac{1}{c})$.

(2) v is smooth at any x with $v(x) \in (0, \tan^{-1}(\frac{1}{c}))$.

(3) $\{x \in \Omega_0 | v(x) = 0\} = \{x_0\}$.

(4) On $\Omega_0 - (\partial\Omega_0 \cup \{x_0\})$, $|\nabla_{\Omega_0} v| \equiv 1$.

By a similar argument in the proof of Theorem 1.3, we conclude that Ω_0 is a geodesic ball of radius $\tan^{-1}(\frac{1}{c})$ in the standard sphere $\mathbb{S}^n$ centered at x_0 and the graph function ϕ satisfies

$$\frac{\tan^2 d(x, x_0)}{\tan^2 r} + \frac{\sin^2 \phi(x)}{\sin^2 r} = 1,$$

where $r = \tan^{-1}(\frac{1}{c})$. By Proposition 2.4, we know that Σ is a geodesic sphere in the standard sphere $\mathbb{S}^{n+1}$ of radius $\tan^{-1}(\frac{1}{c})$ centered at x_0 and thus Ω is a geodesic ball of radius $\tan^{-1}(\frac{1}{c})$ in the standard sphere $\mathbb{S}^{n+1}$. □

As the end of this section, we give some applications of the Theorems 1.3 and 1.4 as follows.

Let (Ω^{n+1}, g) be an $(n+1)$-dimensional smooth compact connected Riemannian manifold with smooth boundary Σ. Assume that the Ricci curvature of Ω satisfies $\mathrm{Ric}^{\Omega} \geq -n$ and the principal curvatures of Σ are bounded from below by a positive constant $c > 1$. Then we know that Σ is connected. Let u be an eigenfunction corresponding to the first nonzero eigenvalue λ_1 of the Laplacian on Σ.

Then we know that the Dirichlet problem

$$\begin{cases} \Delta f = (n+1)f, & \text{in}\,\Omega, \\ f = u, & \text{on}\,\Sigma \end{cases}$$

has a unique solution $f \in C^{\infty}(\Omega)$. We then have the following inequality and rigidity result.

Corollary 4.5. *Let Ω and f be the manifold and smooth function discussed above. If we further assume that the mean curvature of Σ is bounded from below by $\frac{\lambda_1+n}{nc}$. Then we have*

$$c||u||^2_{L^2(\Sigma)} \geq (u, f_{\nu})_{L^2(\Sigma)}.$$

Moreover, equality holds if and only if Ω is isometric to a geodesic ball of radius $\tanh^{-1}(\frac{1}{c})$ in the hyperbolic space $\mathbb{H}^{n+1}$.

Proof. Since u is a non-constant eigenfunction, we have

$$\int_{\Sigma} |\nabla_{\Sigma} u|^2 dv = -\int_{\Sigma} (\Delta_{\Sigma} u) u dv = \lambda_1 \int_{\Sigma} u^2 dv.$$

By the Reilly-type formula in [14], we have

$$\begin{aligned} &\int_{\Omega} \left([\Delta f - (n+1)f]^2 - |\nabla^2 f - fg|^2 \right) dV \\ =& \int_{\Sigma} [2(\Delta_{\Sigma} u) f_{\nu} + nH(f_{\nu})^2 + h(\nabla_{\Sigma} u, \nabla_{\Sigma} u) - 2nuf_{\nu}] dv \\ &+ \int_{\Omega} [Ric^{\Omega}(\nabla f, \nabla f) + 2n|\nabla f|^2 + n(n+1)f^2] dV \end{aligned}$$

Since $h \geq cI$ and $H \geq \frac{\lambda_1+n}{nc} > 0$, we conclude that

$$\begin{aligned}
0 &\geq \int_\Omega \left([\Delta f - (n+1)f]^2 - |\nabla^2 f - fg|^2 \right) dV \\
&\geq \int_\Sigma [-2\lambda_1 u f_\nu + \frac{\lambda_1 + n}{c}(f_\nu)^2 + c\lambda_1 u^2 - 2nuf_\nu] dv \\
&+ \int_\Omega [n|\nabla f|^2 + n(n+1)f^2] dV \\
&= \int_\Sigma [\frac{\lambda_1 + n}{c}(f_\nu)^2 - (n+2\lambda_1)uf_\nu + c\lambda_1 u^2] dv,
\end{aligned}$$

where the last equality is the divergence theorem. Therefore, we have

$$0 \geq \int_\Sigma [(\lambda_1 + n)(\frac{f_\nu}{\sqrt{c}} - \sqrt{c}u)^2 - ncu^2 + nuf_\nu] dv.$$

Then we conclude that

$$c||u||^2_{L^2(\Sigma)} \geq (u, f_\nu)_{L^2(\Sigma)}.$$

If Ω is isometric to a geodesic ball of radius $\tanh^{-1}(\frac{1}{c})$ in the hyperbolic space $\mathbb{H}^{n+1}$, it is easy for us to check that $c||u||^2_{L^2(\Sigma)} = (u, f_\nu)_{L^2(\Sigma)}$. Now we assume that $c||u||^2_{L^2(\Sigma)} = (u, f_\nu)_{L^2(\Sigma)}$. Obviously, the above inequalities must take equality sign, we then have

$$\begin{cases} \nabla^2 f - fg = 0, & \text{in } \Omega, \\ f_\nu - cf = 0, & \text{on } \Sigma. \end{cases}$$

By Theorem 1.3, we know that Ω is isometric to a geodesic ball of radius $\tanh^{-1}(\frac{1}{c})$ in the hyperbolic space $\mathbb{H}^{n+1}$. □

Similarly, we also have the following

Let (Ω^{n+1}, g) be an $(n+1)$-dimensional smooth compact connected Riemannian manifold with smooth boundary Σ. Assume that the Ricci curvature of Ω satisfies $\mathrm{Ric}^\Omega \geq n$ and the principal curvatures of Σ are bounded from below by a positive constant c. Then Σ is connected. Let u be an eigenfunction corresponding to the first nonzero eigenvalue λ_1 of the Laplacian on Σ.

By Theorem 4 in [16], we know that the Dirichlet problem

$$\begin{cases} \Delta f + (n+1)f = 0, & \text{in } \Omega, \\ f = u, & \text{on } \Sigma \end{cases}$$

has a unique solution $f \in C^\infty(\Omega)$. We then have the following inequality and rigidity result.

Corollary 4.6. *Let Ω and f be the manifold and smooth function discussed above. If we further assume that the mean curvature of Σ is bounded from below by $\frac{\lambda_1 - n}{nc} > 0$, we then have*

$$(u, f_\nu)_{L^2(\Sigma)} \geq c||u||^2_{L^2(\Sigma)}.$$

Moreover, equality holds if and only if Ω is isometric to a geodesic ball of radius $\tan^{-1}\left(\frac{1}{c}\right)$ in the standard sphere $\mathbb{S}^{n+1}$.

The proof of Corollary 4.6 is similar to that of Corollary 4.5, so we omit it.

References

[1] S. Almaraz and E. Barbosa. Rigidity on an eigenvalue problem with mixed boundary condition. *arXiv preprint arXiv:1710.06701*, 2017.

[2] X. Z. Chen, M. J. Lai, and F. Wang. The Obata equation with Robin boundary condition. *Revista matemática iberoamericana*, 37(2):643–670, 2020.

[3] M. P. do Carmo and C. Y. Xia. Rigidity theorems for manifolds with boundary and nonnegative Ricci curvature. *Results in Mathematics*, 40:122–129, 2001.

[4] J. Escobar. Uniqueness theorems on conformal deformation of metrics, Sobolev inequalities, and an eigenvalue estimate. *Comm. Pure Appl. Math.*, 43(7):857–883, 1990.

[5] G. J. Galloway and H. C. Jang. Some scalar curvature warped product splitting theorems. *Proc. Amer. Math. Soc.*, 148(6):2617–2629, 2020.

[6] J. Ge. Comparison theorems for manifolds with mean convex boundary. *Commun. Contemp. Math.*, 17(5):1550010, 12, 2015.

[7] Ryosuke Ichida. Riemannian manifolds with compact boundary. *Yokohama Math. J*, 29(2):169–177, 1981.

[8] M. Kanai. On a differential equation characterizing a Riemannian structure of a manifold. *Tokyo J. Math.*, 6(1):143–151, 1983.

[9] M. J. Lai and H. H. Zhou. A note on Obata equations on manifolds with boundary. *J. Math. Study*, 55(3):242–253, 2022.

[10] H. Z. Li and Y. Wei. Rigidity theorems for diameter estimates of compact manifold with boundary. *Int. Math. Res. Not. IMRN*, (11):3651–3668, 2015.

[11] A. Lichnerowicz. Géométrie des groupes de transformations. *Travaux et Recherches Mathématiques III*, 1958.

[12] Yiwei Liu and Yi-Hu Yang. Lower bound estimates of the first eigenvalue for boundary of compact manifolds. *Acta Mathematica Sinica, English series*, 2024.

[13] M. Obata. Certain conditions for a Riemannian manifold to be isometric with a sphere. *J. Math. Soc. Japan*, 14:333–340, 1962.

[14] G. H. Qiu and C. Xia. A generalization of Reilly's formula and its applications to a new Heintze–Karcher type inequality. *International Mathematics Research Notices*, 2015(17):7608–7619, 2015.

[15] S. Raulot and A. Savo. On the first eigenvalue of the Dirichlet-to-Neumann operator on forms. *Journal of Functional Analysis*, 262(3):889–914, 2012.

[16] R. Reilly. Applications of the Hessian operator in a Riemannian manifold. *Indiana University Mathematics Journal*, 26(3):459–472, 1977.

[17] Q. M. Wang. Isoparametric functions on Riemannian manifolds. I. *Math. Ann.*, 277(4):639–646, 1987.

[18] G. Q. Wu and R. G. Ye. A note on Obata's rigidity theorem. *Commun. Math. Stat.*, 2(3-4):231–252, 2014.

[19] C. Xia and C. W. Xiong. Escobar's conjecture on a sharp lower bound for the first nonzero Steklov eigenvalue. *Peking Mathematical Journal*, pages 1–20, 2023.

[20] C. Y. Xia. The first nonzero eigenvalue for manifolds with Ricci curvature having positive lower bound. *Chinese mathematics into the 21st century*, 1991.

Yiwei Liu:
School of Mathematical Sciences, Shanghai Jiao Tong University
email: lyw201611012@sjtu.edu.cn

Yi-Hu Yang:
School of Mathematical Sciences, Shanghai Jiao Tong University
email: yangyihu@sjtu.edu.cn