THE GEOMETRY OF SEDENION ZERO DIVISORS

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ABSTRACT. The sedenion algebra S is a non-commutative, non-associative, 16-dimensional real algebra with zero divisors. It is obtained from the octonions through the Cayley-Dickson construction. The zero divisors of S can be viewed as the submanifold $\mathcal{Z}(\mathbb{S}) \subset \mathbb{S} \times \mathbb{S}$ of normalized pairs whose product equals zero, or as the submanifold $ZD(\mathbb{S}) \subset \mathbb{S}$ of normalized elements with non-trivial annihilators. We prove that $\mathcal{Z}(\mathbb{S})$ is isometric to the excepcional Lie group G_2 , equipped with a naturally reductive leftinvariant metric. Moreover, $\mathcal{Z}(\mathbb{S})$ is the total space of a Riemannian submersion over the excepcional symmetric space of quaternion subalgebras of the octonion algebra, with fibers that are locally isometric to a product of two round 3-spheres with different radii. Additionally, we prove that $ZD(\mathbb{S})$ is isometric to the Stiefel manifold $V_2(\mathbb{R}^7)$, the space of orthonormal 2-frames in \mathbb{R}^7 , endowed with a specific G_2 -invariant metric. By shrinking this metric along a circle fibration, we construct new examples of an Einstein metric and a family of homogenous metrics on $V_2(\mathbb{R}^7)$ with non-negative sectional curvature.

1. INTRODUCTION

The Cayley-Dickson algebras form a sequence of real algebras \mathbb{A}_n , defined recursively beginning with \mathbb{R} and doubling in dimension with each iteration. The first members of this family are the familiar real division algebras: $\mathbb{R} = \mathbb{A}_0$, $\mathbb{C} = \mathbb{A}_1$, $\mathbb{H} = \mathbb{A}_2$ and $\mathbb{O} = \mathbb{A}_3$. The next algebra in this sequence is the so-called sedenion algebra $\mathbb{S} = \mathbb{A}_4$, which is often overlooked in comparison to its lower-dimensional relatives due to its lack of certain desirable algebraic properties. Nonetheless, this somewhat enigmatic algebra has long intrigued mathematicians and has recently found applications in fields such as theoretical physics [GG19] and machine learning [SA20].

Since S is not a division algebra, it is interesting to understand the structure of its zero divisors. The topology of the sedenion zero divisors is described by the principal bundle

$$\operatorname{SU}(2) \to G_2 \to V_2(\mathbb{R}^7),$$

where G_2 is the exceptional compact Lie group of rank 2 and $V_2(\mathbb{R}^7)$ is the Stiefel manifold of orthonormal 2-frames in \mathbb{R}^7 . Specifically, G_2 is homeomorphic to the submanifold $\mathcal{Z}(\mathbb{S}) \subset \mathbb{S} \times \mathbb{S}$ of normalized sedenion pairs that multiply to zero; $V_2(\mathbb{R}^7)$ is homeomorphic to the submanifold $ZD(\mathbb{S}) \subset \mathbb{S}$ of sedenions with norm $\sqrt{2}$ that have non-trivial annihilators; and, for each $u \in ZD(\mathbb{S})$, the fiber SU(2) corresponds to the sphere of the annihilator subspace of u (see [Mor98, BDI08]). However, little is known regarding the geometry of the sedenion zero divisors.

Both $\mathcal{Z}(\mathbb{S})$ and $ZD(\mathbb{S})$ carry a natural geometry as submanifolds of \mathbb{R}^{32} and \mathbb{R}^{16} , respectively. Furthermore, since the zero divisors of \mathbb{S} are invariant under Aut(\mathbb{S}), whose

Date: December 2, 2024.

²⁰²⁰ Mathematics Subject Classification. 53C30, 17A20.

Key words and phrases. Cayley-Dickson algebras, Sedenion algebra, Zero divisors, Einstein manifolds, Non-negative curvature.

connected component is isomorphic to G_2 , it follows that $\mathcal{Z}(\mathbb{S})$ and $ZD(\mathbb{S})$ are homogeneous submanifolds. In this article, we study the intrinsic geometry of the zero divisors of \mathbb{S} . First, we prove that $\mathcal{Z}(\mathbb{S})$ is isometric to G_2 with a naturally reductive left-invariant metric, forming the total space of a Riemannian submersion over the exceptional symmetric space $G_2/SO(4)$, with fibers locally isometric to a product of two round 3-spheres with different radii.

Next, we analyze the geometry of $ZD(\mathbb{S})$, which is isometric to $V_2(\mathbb{R}^7) = G_2/SU(2)$ with a particular G_2 -invariant metric. At first glance, the geometry of $ZD(\mathbb{S})$ does not seem very interesting; however, by shrinking the metric along a certain circle fibration, we obtain a family g_r (r > 0) of G_2 -invariant metrics on $V_2(\mathbb{R}^7)$, where $g_{\frac{2}{3}}$ represents the original metric. This process reveals several distinguished examples. Specifically, we prove, among other things, that $(V_2(\mathbb{R}^7), g_r)$:

- has positive scalar curvature if and only of $r < \frac{20}{3}$;
- is an Einstein manifold if and only if $r = \frac{5}{9}$;
- has non-negative sectional curvature if and only if $0 < r \leq \frac{4}{9}$.

These results are quite remarkable, as Einstein metrics and metrics with non-negative curvature are very rare. To the best of our knowledge, the examples presented in this article are new. The known homogeneous Einstein metrics on $V_2(\mathbb{R}^7) = G_2/SU(2) = SO(7)/SO(5)$ are limited to the unique SO(7)-invariant Einstein metric discovered by Sagle and the so-called Jensen metrics (see [Sag70, Jen73, BH87, Ker98]). It is worth noticing that the metric $g_{\frac{5}{9}}$ is neither SO(7)-invariant nor a Jensen metric. Regarding metrics with non-negative sectional curvature, we refer to the survey [Zil07]. Typically, examples of homogeneous metrics with non-negative curvature appear as normal homogeneous metrics or are constructed through a Cheeger deformation of a metric already known to have non-negative curvature. Recall that none of the metrics g_r is normal homogeneous (nor even naturally reductive) and that the initial metric $g_{\frac{2}{3}}$ does not possess non-negative sectional curvature.

Let us comment briefly on the proof our main results. In order to study the geometry of $\mathcal{Z}(\mathbb{S})$, it is necessary to "fix an origin" so that the metric can be identified with a left-invariant metric on G_2 . Any choice of such an origin for $\mathcal{Z}(\mathbb{S})$ leads to isometric metrics on G_2 , but a well-chosen origin can greatly simplify computations. We select the origin from among the so-called 84 standard zero divisors of \mathbb{S} . Then, using the results in [DZ79], we show that $\mathcal{Z}(\mathbb{S})$ is a naturally reductive space. A similar approach applies to the study of $ZD(\mathbb{S})$ with the metric g_r . Here, we select another standard zero divisor (different from the previous one) so that the isotropy subgroup of G_2 acts trivially on the usual subalgebra $\mathbb{H} \subset \mathbb{S}$. This choice allows the metric g_r to be expressed in diagonal form with respect to the normal homogeneous metric, making it possible to derive a nice expression for the Ricci tensor of g_r .

The most challenging part is to determine the sign of the sectional curvatures of g_r . Since there is no manageable expression for the curvature (as there is for naturally reductive spaces), using algebraic manipulation proves to be nearly impossible. Indeed, the sectional curvature function F_r of g_r can be interpreted as a homogeneous polynomial of degree 4 in 22 real variables, which, for a generic r, has 285 non-trivial coefficients. To show that F_r is non-negative for $0 < r \leq \frac{4}{9}$, we reduce the problem to proving that both F_0 (the formal extension of F_r at r = 0) and $F_{\frac{4}{9}}$ are non-negative. By using convex optimization techniques, we are able to prove the stronger result that F_0 and $F_{\frac{4}{9}}$ are polynomial sums of squares.

Finally we want to mention that the computations required in the proof of some of our results are often cumbersome and were computer checked using the software SageMath. The code used to verify our results is available at [Reg24].

We believe this work shows that the study of the geometry of Cayley-Dickson algebras, particularly regarding their zero divisors, deserves further attention, as it may have interesting implications in differential geometry of compact homogeneous spaces.

Acknowledgements. This work is supported by CONICET and partially supported by SeCyT-UNR and ANPCyT. The author would like to thank Andreas Arvanitoyeorgos for helpful discussions on homogeneous Einstein metrics on Stiefel manifolds.

2. Preliminaries and notation

The main references for this section are [Mor98, BDI08] on Cayley-Dickson algebras and their zero divisors, [Arv03] on the geometry of homogeneous spaces and [DZ79] on naturally reductive left-invariant metrics on compact Lie groups. Observe that in this section, as well as throughout the rest of the article, we start counting indices from 0.

2.1. Cayley-Dickson algebras. The Cayley-Dickson algebras \mathbb{A}_n are a family of real algebras, equipped with an involution $a \mapsto a^*$ (also called conjugation), which are recursively defined starting from $\mathbb{A}_0 = \mathbb{R}$, where $a^* = a$. Each subsequent algebra is defined by setting $\mathbb{A}_n = \mathbb{A}_{n-1} \times \mathbb{A}_{n-1}$ as a vector space, with multiplication given by

$$(a, b)(c, d) = (ac - d^*b, da + bc^*)$$

and involution defined by

$$(a,b)^* = (a^*,-b).$$

Notice that the inclusion $a \mapsto (a, 0)$ is a monomorphism of algebras from \mathbb{A}_{n-1} into \mathbb{A}_n for all $n \geq 0$. It is well known that the first four algebras in the Cayley-Dickson construction are the real division algebras \mathbb{R} , \mathbb{C} , \mathbb{H} and \mathbb{O} , respectively. It is also known that the Cayley-Dickson algebras lose some important properties with each iteration. For example, \mathbb{A}_n is commutative if and only if $n \leq 1$, associative if and only if $n \leq 2$; alterative (i.e., x(xy) = (xx)y and (xy)y = x(yy) for all $x, y \in \mathbb{A}_n$) if and only of $n \leq 3$. On the other hand, every Cayley-Dickson algebra is flexible (i.e., x(yx) = (xy)x for all $x, y \in \mathbb{A}_n$) and power associative (i.e., x^k is well defined for all $x \in \mathbb{A}_n$ and $k \in \mathbb{N}$).

For $x \in A_n$ we define its real and imaginary parts as $\operatorname{Re} x = \frac{1}{2}(x + x^*)$ and $\operatorname{Im} x = \frac{1}{2}(x - x^*)$, respectively. We say that x is *real* (resp. *imaginary*) if $\operatorname{Im} x = 0$ (resp. $\operatorname{Re} x = 0$). Thus, one can recover the usual inner product on $A_n \simeq \mathbb{R}^{2^n}$ by

$$\langle x, y \rangle = \operatorname{Re}(xy^*).$$

In \mathbb{A}_n , one has that $||x||^2 = xx^*$ for all x. However, the identity ||xy|| = ||x|| ||y|| does not hold in general if $n \ge 4$. Recall that \mathbb{A}_n is a division algebra if and only $n \le 3$. If $n \ge 4$, then \mathbb{A}_n has zero divisors. Since, xy = 0 implies yx = 0, the left and right zero divisors of \mathbb{A}_n coincide. Thus, an element $0 \ne u \in \mathbb{A}_n$ is a zero divisor if and only if ann $u \ne 0$, where ann u is the kernel of the \mathbb{R} -linear map $L_u : \mathbb{A}_n \to \mathbb{A}_n$ given by $L_u(x) = ux$. In [Mor98], it is proven that a zero divisor u must be imaginary and dim(ann $u) \equiv 0 \mod 4$. Furthermore, in [BDI08] it is proven that dim ann $u \leq 2^n - 4n + 4$. One can study the zero divisors globally by defining the sets

$$\mathcal{Z}(\mathbb{A}_n) = \{ (u, v) \in \mathbb{A}_n \times \mathbb{A}_n : ||u|| = ||v|| = \sqrt{2} \text{ and } uv = 0 \},\$$
$$ZD(\mathbb{A}_n) = \{ u \in \mathbb{A}_n : (u, v) \in \mathcal{Z}(\mathbb{A}_n) \text{ for some } v \in \mathbb{A}_n \}.$$

Normalizing the zero divisors to $\sqrt{2}$ is not particularly important, but it will be convenient later. When $n \ge 5$, the sets $ZD_k(\mathbb{A}_n) = \{u \in ZD(\mathbb{A}_n) : \dim(\operatorname{ann} u) = k\}$ are also of interest.

For $n \geq 4$, one has that the automorphism group of \mathbb{A}_n is given by

$$\operatorname{Aut}(\mathbb{A}_n) \simeq \operatorname{Aut}(\mathbb{A}_{n-1}) \times S_3 \simeq G_2 \times (S_3)^{n-3},$$

where S_3 is the symmetric group in three elements, and $G_2 = \operatorname{Aut}(\mathbb{O})$ is the 14-dimensional compact simple Lie group of rank 2. Recall that G_2 acts diagonally on \mathbb{A}_n . It follows that $\operatorname{Der}(\mathbb{A}_n) = \mathfrak{g}_2$ for all $n \geq 4$, where \mathfrak{g}_2 is the Lie algebra of G_2 .

2.2. Sedenion zero divisors. From now on we denote the sedenion algebra \mathbb{A}_4 by \mathbb{S} . Let us denote by e_0, \ldots, e_{15} the canonical basis of \mathbb{S} . By making an abuse of notation, we also denote by e_0, \ldots, e_3 and e_0, \ldots, e_7 the canonical basis of \mathbb{H} and \mathbb{O} respectively. The zero divisors of \mathbb{S} have the following form.

Proposition 2.1 (See [BDI08]). An element $(a, b) \in S$ is a zero divisor if and only if a, b are imaginary elements of \mathbb{O} such that $||a|| = ||b|| \neq 0$ and $a \perp b$.

From this result, one can construct the 84 standard zero divisors of S. Namely, the elements of the form $(e_i + e_j, e_k \pm e_l) \in \mathcal{Z}(\mathbb{S})$ such that $1 \leq i \leq 6, 9 \leq j \leq 15, i < k \leq 7$ and $9 \leq l \leq 15$ (see Table 1). Clearly, every automorphism of S maps $\mathcal{Z}(\mathbb{S})$ into itself. Moreover, we have that the connected component of Aut(S) acts simply and transitively on $\mathcal{Z}(\mathbb{S})$:

Theorem 2.2 ([Mor98]). $\mathcal{Z}(S)$ is homeomorphic (and moreover, diffeomorphic) to G_2 .

Given $(u_0, v_0) \in \mathcal{Z}(\mathbb{S})$, we have that $G_2 \cdot u_0 = ZD(\mathbb{S})$. It is not difficult to see that the isotropy subgroup at u_0 is isomorphic to SU(2). Note that $G_2/SU(2)$ is diffeomorphic to the Stiefel manifold $V_2(\mathbb{R}^7)$. In fact, every automorphism of \mathbb{O} is completely determined by its values a, b, c at e_1, e_2, e_4 respectively. Here (a, b, c) can be any triple of pairwise orthonormal imaginary octonions of norm 1 such that $ab \perp c$. Hence the map $(a, b, c) \mapsto$ (a, b) identifies with a transitive action of G_2 in $V_2(\mathbb{R}^7)$, whose isotropy subgroup at (e_1, e_2) are the octonion automorphism that act trivially on \mathbb{H} , and therefore are isomorphic to SU(2). Thus, the topology of the sedenion zero divisors is encoded by the principal bundle

$$\mathrm{SU}(2) \to G_2 \to V_2(\mathbb{R}^7).$$

2.3. The Lie algebra of G_2 . We think of the Lie group $G_2 = \operatorname{Aut}(\mathbb{O})$ as a subgroup of SO(8) in the natural way (since every automorphism of \mathbb{O} fixes e_0 , we have that G_2 is actually a subgroup of SO(7), but we do not use this identification here). So, we have \mathfrak{g}_2 as a subalgebra of $\mathfrak{so}(8)$. Let us consider the bi-invariant metric g_{bi} induced by the inner product on \mathfrak{g}_2 , which we denote with the same symbol, given by

$$g_{\rm bi}(X,Y) = -\operatorname{tr}(XY).$$

Let us denote by $E_{ij} \in \mathfrak{so}(8)$, where $0 \leq i < j \leq 7$, the matrix such that $(E_{ij})_{ij} = -(E_{ij})_{ji} = -1$ and $(E_{ij})_{kl} = 0$ in any other case. We define

 $\begin{aligned} X_0 &= \frac{1}{2}(E_{45} + E_{67}), & X_7 &= \frac{1}{2}(E_{16} + E_{25}), \\ X_1 &= \frac{1}{2}(E_{46} - E_{57}), & X_8 &= -\frac{1}{2}(E_{15} - E_{26}), \\ X_2 &= \frac{1}{2}(E_{47} + E_{56}), & X_9 &= \frac{1}{2}(E_{14} + E_{27}), \\ X_3 &= -\frac{\sqrt{3}}{6}(2E_{23} - E_{45} + E_{67}), & X_{10} &= \frac{\sqrt{3}}{6}(E_{16} - E_{25} + 2E_{34}), \\ X_4 &= \frac{\sqrt{3}}{6}(2E_{13} + E_{46} + E_{57}), & X_{11} &= \frac{\sqrt{3}}{6}(E_{17} + E_{24} + 2E_{35}), \\ X_5 &= -\frac{\sqrt{3}}{6}(2E_{12} - E_{47} + E_{56}), & X_{12} &= -\frac{\sqrt{3}}{6}(E_{14} - E_{27} - 2E_{36}), \\ X_6 &= -\frac{1}{2}(E_{17} - E_{24}), & X_{13} &= -\frac{\sqrt{3}}{6}(E_{15} + E_{26} - 2E_{37}). \end{aligned}$

One can see that X_0, \ldots, X_{13} is an orthonormal basis of \mathfrak{g}_2 with respect to the biinvariant metric. We will denote by X^0, \ldots, X^{13} its dual basis. Define

$$\mathfrak{k}_0 = \bigoplus_{i=0}^2 \mathbb{R}X_i, \qquad \mathfrak{m}_0 = \bigoplus_{i=3}^5 \mathbb{R}X_i, \qquad \mathfrak{m}_1 = \bigoplus_{i=6}^9 \mathbb{R}X_i, \qquad \mathfrak{m}_2 = \bigoplus_{i=10}^{13} \mathbb{R}X_i$$

We have that \mathfrak{k}_0 and \mathfrak{m}_0 are two subalgebras of \mathfrak{g}_2 isomorphic to $\mathfrak{so}(3)$ such that $[\mathfrak{k}_0, \mathfrak{m}_0] = 0$. Moreover, $\mathfrak{k}_0 \oplus \mathfrak{m}_0 \simeq \mathfrak{so}(4)$ is the subalgebra of a maximal subgroup of G_2 isomorphic to SO(4) (cfr. [BLS20]). Such subgroup preserves the orthogonal decomposition $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}^{\perp}$. Furthermore, the subgroup of G_2 with Lie algebra \mathfrak{k}_0 is isomorphic to SU(2) and acts trivially on \mathbb{H} . Recall that $G_2/SO(4)$, with the normal homogeneous metric, is the symmetric space of quaternion subalgebras of \mathbb{O} .

2.4. Homogeneous and naturally reductive spaces. Let G be a Lie group and H be a compact subgroup of G. Let us denote by \mathfrak{g} and \mathfrak{h} the Lie algebras of G and H, respectively. Assume that G acts almost effectively on M = G/H and that M is endowed with a G-invariant metric g. Recall that every $X \in \mathfrak{g}$ induces a Killing vector field X^* on M defined as $X_q^* = \frac{d}{dt}\Big|_0 \operatorname{Exp}(tX) \cdot q$. The map $X \mapsto X^*$ from \mathfrak{g} into $\mathfrak{X}(M)$ satisfies

$$[X,Y]^* = -[X^*,Y^*].$$

Let us fix a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ (i.e., \mathfrak{m} is an Ad(H)-invariant subspace of \mathfrak{g} complementary to \mathfrak{h}), which always exists since H is compact. Assume that H is the isotropy subgroup of $p \in M$. Then we can identify $\mathfrak{m} \simeq T_p M$. The geometry of Mis determined by an Ad(H)-invariant inner product on \mathfrak{m} , which we also denote by g, defined such that the map $X \in \mathfrak{m} \mapsto X_p^* \in T_p M$ is a linear isometry. With this setting, we can compute the Levi-Civita connection of M as

$$(\nabla_{X^*}Y^*)_p = -\frac{1}{2}[X,Y]_{\mathfrak{m}} + U(X,Y), \qquad X,Y \in \mathfrak{m},$$
 (2.1)

where U is the algebraic tensor on \mathfrak{m} given by

$$2g(U(X,Y),Z) = g([Z,X]_{\mathfrak{m}},Y) + g(X,[Z,Y]_{\mathfrak{m}}), \qquad X,Y,Z \in \mathfrak{m}$$

Let $R_{X,Y} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ be the curvature tensor of M. The sectional curvature of M is determined by

$$\begin{split} g(R_{X,Y}X,Y) &= -\frac{3}{4} \| [X,Y]_{\mathfrak{m}} \|^{2} - \frac{1}{2} g([X,[X,Y]_{\mathfrak{m}}]_{\mathfrak{m}},Y) - \frac{1}{2} g([Y,[Y,X]_{\mathfrak{m}}]_{\mathfrak{m}},X) \\ &+ \| U(X,Y) \|^{2} - g(U(X,X),U(Y,Y)) + g(Y,[[X,Y]_{\mathfrak{h}},X]_{\mathfrak{m}}) \end{split}$$

for $X, Y \in \mathfrak{m}$. Also, the Ricci tensor of M is determined by

$$\operatorname{Ric}(X, X) = -\frac{1}{2} \sum_{i} \{ \| [X, X_{i}]_{\mathfrak{m}} \|^{2} + g([X, [X, X_{i}]_{\mathfrak{m}}]_{\mathfrak{m}}, X_{i}) + 2g([X, [X, X_{i}]_{\mathfrak{h}}]_{\mathfrak{m}}, X_{i}) \} + \frac{1}{4} \sum_{i,j} g([X_{i}, X_{j}]_{\mathfrak{m}}, X)^{2} - g([Z, X]_{\mathfrak{m}}, X),$$

$$(2.2)$$

for $X \in \mathfrak{m}$, where $\{X_i\}$ is an orthonormal basis of \mathfrak{m} and $Z = \sum_i U(X_i, X_i)$.

Recall that the metric g on M = G/H is *naturally reductive* if and only if $U \equiv 0$. An interesting particular case is when a left-invariant metric on a Lie group is naturally reductive (with respect to a certain transitive Lie group of isometries).

Theorem 2.3 ([DZ79]). Let G be a compact, simple Lie group group endowed with a left-invariant metric g. Let \mathfrak{g} denote the Lie algebra of G and let g_{bi} be a bi-invariant metric on G (which is a negative multiple of the Killing form of \mathfrak{g}). The metric g is naturally reductive if and only if there exists a subalgebra \mathfrak{k} of \mathfrak{g} such that

$$g = g_{\mathfrak{k}_0} \oplus lpha_1 g_{\mathrm{bi}}|_{\mathfrak{k}_1} \oplus \cdots \oplus lpha_r g_{\mathrm{bi}}|_{\mathfrak{k}_r} \oplus lpha g_{\mathrm{bi}}|_{\mathfrak{k}^\perp}$$

where $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_r$, with \mathfrak{k}_0 the center of \mathfrak{k} and $\mathfrak{k}_1, \ldots, \mathfrak{k}_r$ are simple ideals. Here, \mathfrak{k}^{\perp} is the orthogonal complement of \mathfrak{k} with respect to the bi-invariant metric, $g_{\mathfrak{k}_0}$ is an arbitrary inner product on \mathfrak{k}_0 , and $\alpha_1, \ldots, \alpha_r, \alpha$ are positive real numbers.

3. The G_2 -invariant metrics on $\mathcal{Z}(\mathbb{S})$ and $ZD(\mathbb{S})$

Consider $\mathcal{Z}(\mathbb{S})$ as a submanifold of $\mathbb{S} \times \mathbb{S} \simeq \mathbb{R}^{32}$ with the induced metric. Although this reduction is not necessary here, one could lower the codimension of $\mathcal{Z}(\mathbb{S})$. In fact, by Proposition 2.1, $\mathcal{Z}(\mathbb{S})$ is a submanifold of $S^6 \times S^6 \times S^6 \times S^6$. Since $G_2 = \operatorname{Aut}(\mathbb{O}) \subset$ $\operatorname{Aut}(\mathbb{S})$ acts isometrically on $\mathbb{S} \times \mathbb{S}$, we have that $\mathcal{Z}(\mathbb{S})$ is a homogeneous submanifold. Furthermore, by Theorem 2.2, the diffeomorphism $\mathcal{Z}(\mathbb{S}) \simeq G_2$ induces a left-invariant metric g on G_2 .

Theorem 3.1. The metric on $\mathcal{Z}(\mathbb{S})$ is naturally reductive. Furthermore, $\mathcal{Z}(\mathbb{S})$ is the total space of a Riemannian submersion over the exceptional symmetric space $G_2/SO(4)$ with totally geodesic fibers, which are locally isometric to a product of two round 3-spheres with different radii.

Proof. It is sufficient to prove the theorem for the left-invariant metric on G_2 defined in the paragraph preceding the statement. To determine such a metric, one fixes an element $(u_0, v_0) \in \mathcal{Z}(\mathbb{S})$ and computes

$$g(X_i, X_j) = (X_i \cdot (u_0, v_0))^T (X_j \cdot (u_0, v_0)).$$
(3.1)

Note that not every zero divisor pair behaves nicely with respect to the decomposition $\mathfrak{g}_2 = \mathfrak{k}_0 \oplus \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$ given in Subsection 2.3. By running (3.1) over the standard zero

divisors from Table 1, we observe that if $(u_0, v_0) = (e_4 + e_{13}, e_6 + e_{15})$ the metric can be expressed as

$$g = \sum_{i=0}^{2} X^{i} \otimes X^{i} + \frac{1}{3} \sum_{i=3}^{5} X^{i} \otimes X^{i} + \frac{1}{2} \sum_{i=6}^{13} X^{i} \otimes X^{i} = g_{\mathrm{bi}}|_{\mathfrak{k}_{0}} \oplus \frac{1}{3} g_{\mathrm{bi}}|_{\mathfrak{m}_{0}} \oplus \frac{1}{2} g_{\mathrm{bi}}|_{\mathfrak{m}_{1} \oplus \mathfrak{m}_{2}}.$$

From Theorem 2.3, it follows that this metric is naturally reductive. More precisely, this metric is naturally reductive with respect to $G_2 \times SO(4)$, where the second factor acts on the right and the isotropy subgroup is given by diag(SO(4) × SO(4)). Thus, from [DZ79, Theorem 8], the subgroup SO(4) $\subset G_2$, whose Lie algebra is given by $\mathfrak{k}_0 \oplus \mathfrak{m}_0$, is totally geodesic.

Since $g|_{\mathfrak{m}_1\oplus\mathfrak{m}_2}$ is a multiple of the bi-invariant metric, when restricted to $\mathfrak{m}_1\oplus\mathfrak{m}_2$, and $\mathfrak{k}_0\oplus\mathfrak{m}_0\simeq\mathfrak{so}(4)$ is orthogonal to $\mathfrak{m}_1\oplus\mathfrak{m}_2$ with respect to both metrics, we conclude that $(G_2,g)\to G_2/\mathrm{SO}(4)$ is a Riemannian submersion. The fiber of this submersion is isometric to the Lie group SO(4) endowed with the bi-invariant metric $g|_{\mathfrak{k}_0\oplus\mathfrak{m}_0}$, which is obtained by taking two different scalings of the bi-invariant metric on the simple ideals $\mathfrak{so}(3)\simeq\mathfrak{k}_0\simeq\mathfrak{m}_0$ of $\mathfrak{so}(4)$. Hence, the universal cover of SO(4) splits into a product of two round spheres with different radii.

Remark 3.2. Since the metric in $\mathcal{Z}(\mathbb{S})$ is naturally reductive, many geometric properties follow from existing results. For example, the (connected component of the) full isometry group is computed in [DZ79] (see also [OR13]). The so-called index of symmetry of $\mathcal{Z}(\mathbb{S})$, which in this case is trivial, can be computed from the results in [ORT14]. It can also be seen from [DZ79] that the metric on $\mathcal{Z}(\mathbb{S})$ is not Einstein. We verify this fact again in the next proposition by explicitly computing the Ricci tensor, which also allows us to show that the Ricci curvature is positive.

Proposition 3.3. $\mathcal{Z}(\mathbb{S})$ has positive Ricci curvature. Moreover,

$$\operatorname{Ric} = \frac{5}{2} \sum_{i=0}^{2} X^{i} \otimes X^{i} + \frac{29}{54} \sum_{i=3}^{5} X^{i} \otimes X^{i} + \frac{5}{6} \sum_{i=6}^{13} X^{i} \otimes X^{i}.$$
 (3.2)

Proof. It follows from a straightforward computation using the following well-known formula. Let Y_0, \ldots, Y_{13} be an *g*-orthonormal basis of \mathfrak{g}_2 . Then

$$\operatorname{Ric}(Y_{j}, Y_{h}) = \frac{1}{2} \sum_{i,k} \left\{ c_{iki}(c_{kjh} + c_{khj}) + \frac{1}{2} c_{ikh}c_{ikj} - c_{ijk}c_{khi} + c_{iki}c_{jhk} - c_{ijk}c_{ihk} \right\}$$

where $c_{ijk} = g([Y_i, Y_j], Y_k)$. Since g has diagonal form in the basis X_0, \ldots, X_{13} , we can choose $Y_i = g(X_i, X_i)^{-\frac{1}{2}} X_i$. From this, we can show that

$$\operatorname{Ric} = \frac{5}{2} \sum_{i=0}^{2} Y^{i} \otimes Y^{i} + \frac{29}{18} \sum_{i=3}^{5} Y^{i} \otimes Y^{i} + \frac{5}{3} \sum_{i=6}^{13} Y^{i} \otimes Y^{i},$$

which is equivalent to (3.2).

Now we direct our attention to the geometry of $ZD(\mathbb{S})$ with the metric induced from the ambient space $\mathbb{S} \simeq \mathbb{R}^{16}$. Since G_2 acts isometrically and transitively on $ZD(\mathbb{S})$, we have that $ZD(\mathbb{S})$ is isometric to the Stiefel manifold $G_2 \cdot u_0 = G_2/\mathrm{SU}(2) = V_2(\mathbb{R}^7)$, equipped with a certain G_2 -invariant metric, where $\mathrm{SU}(2)$ is the isotropy subgroup of $u_0 \in ZD(\mathbb{S})$. We again denote by g such a metric, which is defined by

$$g(X_i, X_j) = (X_i \cdot u_0)^T (X_j \cdot u_0).$$

Similarly to the case of $\mathcal{Z}(\mathbb{S})$, we can choose u_0 appropriately so that the Lie algebra of SU(2) is \mathfrak{k}_0 . Taking $u_0 = e_1 + e_{10}$, we obtain that

$$\mathfrak{g}_2 = \mathfrak{k}_0 \oplus \mathfrak{m}, \qquad \text{where } \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2,$$

is a reductive decomposition for $G_2/SU(2)$. The corresponding Ad(SU(2))-invariant inner product on \mathfrak{m} is given by

$$g = \frac{1}{3}(X^3 \otimes X^3 + X^4 \otimes X^4) + \frac{2}{3}X^5 \otimes X^5 + \frac{1}{2}\sum_{i=6}^9 X^i \otimes X^i + \frac{1}{6}\sum_{i=10}^{13} X^i \otimes X^i.$$

A detailed study of the isometry group and the curvature of g is given in the next section. Before proceeding, we note a simple fact about the sectional curvatures of g.

Remark 3.4. Let us denote by π_{ij} the 2-dimensional subspace of \mathfrak{m} generated by X_i and X_j , where $3 \leq i < j \leq 13$. Then the sectional curvature of π_{ij} is non-negative if and only if $\pi_{ij} \neq \pi_{34}$. This suggests that one could attempt to modify the metric g along the direction normal to π_{34} inside \mathfrak{m}_1 in order to get some examples of metrics with non-negative sectional curvature. We explore this approach in the next section.

4. A family of G_2 -invariant metrics on $V_2(\mathbb{R}^7)$

For each r > 0, we consider on $V_2(\mathbb{R}^7)$ the family of G_2 -invariant metrics given by

$$g_r = \frac{1}{3} \left(X^3 \otimes X^3 + X^4 \otimes X^4 \right) + r X^5 \otimes X^5 + \frac{1}{2} \sum_{i=6}^9 X^i \otimes X^i + \frac{1}{6} \sum_{i=10}^{13} X^i \otimes X^i.$$
(4.1)

Indeed, g_r gives an Ad(SU(2))-invariant inner product on \mathfrak{m} since \mathfrak{m}_0 is the subspace of fixed points of the isotropy representation of $G_2/SU(2)$ and

$$g_r|_{\mathfrak{m}_1} = \frac{1}{2} g_{\mathrm{bi}}|_{\mathfrak{m}_1}, \qquad \qquad g_r|_{\mathfrak{m}_2} = \frac{1}{6} g_{\mathrm{bi}}|_{\mathfrak{m}_2}.$$

Next, we compute the connected component of the full isometry group of g_r .

Theorem 4.1. $I_0(V_2(\mathbb{R}^7), g_r) \simeq G_2 \times S^1$.

Proof. Since G_2 is a compact simple Lie group, it follows from the results in [Oni92] that $I_0(V_2(\mathbb{R}^7), g_r) \subset I_0(V_2(\mathbb{R}^7), g_{\rm nh})$, where $g_{\rm nh} = g_{\rm bi}|_{\mathfrak{m}}$ is the normal homogeneous metric associated with the homogeneous presentation $V_2(\mathbb{R}^7) = G_2/\mathrm{SU}(2)$. From [Reg10], we have that $I_0(V_2(\mathbb{R}^7), g_{\rm nh}) \simeq G_2 \times K$ (almost direct product) where the Lie algebra of K is given by the G_2 -invariant vector fields, which are identified with the fixed vectors of the isotropy representation. That is, the Lie algebra of K is identified with \mathfrak{m}_0 , but the elements of $K \simeq \mathrm{SU}(2)$ act "on the right". Then, it is not difficult to see that $I_0(V_2(\mathbb{R}^7)) \simeq G_2 \times K'$ (almost direct product) for a compact and connected subgroup K' of K, which in principle depends on r. Since dim K = 3, it is enough to see that $K' \neq K$ and dim $K' \geq 1$.

Now, for $Y \in \mathfrak{m}_0$, let \hat{Y} be the G_2 -invariant vector field induced by Y. Using (2.1) and the fact that $\nabla_{X^*}\hat{Y} = \nabla_{\hat{Y}}X^*$ for all $X \in \mathfrak{m}$, one can see that \hat{Y} is a Killing field for g_r if and only if $[Y, -]_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{m}$ is skew-symmetric with respect to g_r . The $\frac{1}{2}$ - and $\frac{1}{6}$ -scalings of the metric on the irreducible subspaces \mathfrak{m}_1 and \mathfrak{m}_2 prevent \hat{X}_3 and \hat{X}_4 from being Killing fields for g_r . However, one can check that \hat{X}_5 is a Killing field for g_r for any r > 0. Thus $K' = S^1$, which implies $I_0(V_2(\mathbb{R}^7)) = G_2 \times S^1$ is actually a direct product. Observe that we have proved that the S^1 factor is independent of r. Now, we compute the Ricci and scalar curvature of g_r . In particular, we obtain the following result.

Theorem 4.2. (1) The metric g_r is Einstein if and only if $r = \frac{5}{9}$. (2) The metric g_r has positive scalar curvature if and only if $r < \frac{20}{3}$.

Proof. Let Y_3, \ldots, Y_{13} be the g_r -orthonormal basis of \mathfrak{m} obtained from normalizing the basis X_3, \ldots, X_{13} . We can use formula (2.2) to explicitly compute the Ricci tensor Ric_{g_r} of g_r . After lengthy computations, carefully verified using a computer (see [Reg24]), we obtain

$$\operatorname{Ric}_{g_r} = \frac{15r}{2} Y^5 \otimes Y^5 + \left(-\frac{3r}{2} + 5\right) \sum_{i \neq 5} Y^i \otimes Y^i.$$

$$(4.2)$$

Hence, g_r is Einstein if and only if $r = \frac{5}{9}$. Also, from (4.2) we get that the scalar curvature $\operatorname{scal}_{g_r} = 50 - \frac{15}{2}r$ is positive if and only if $r < \frac{20}{3}$.

Remark 4.3. In [Jen73], the construction of remarkable examples of Einstein metrics on the base space of certain principal bundles can be found. Such metrics are now known as Jensen metrics. In particular, there exist G_2 -invariant Einstein metrics on $V_2(\mathbb{R}^7)$, arising from the principal bundle $SU(2) \rightarrow G_2 \rightarrow G_2/SU(2)$, which in our notation takes the form $t^2 g_{\rm bi}|_{\mathfrak{m}_0} \oplus g_{\rm bi}|_{\mathfrak{m}_1 \oplus \mathfrak{m}_2}$ for certain values of t > 0. Notice that the metric $g_{\frac{5}{9}}$ from Theorem 4.2 is not a Jensen metric. Moreover, it is not even bi-invariant when restricted to \mathfrak{m}_0 .

Theorem 4.4. The metric g_r has non-negative sectional curvature if and only if $r \leq \frac{4}{9}$.

In order to prove our theorem, we will need the following result, which is a particular case of Theorem 1 in [PW98] (see also [CLR95]).

Lemma 4.5. Let $F \in \mathbb{R}[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree 4. Then F is a (polynomial) sum of squares if and only if there exists a symmetric positive semi-definite matrix H such that

$$F = \boldsymbol{x}^T H \boldsymbol{x} \tag{4.3}$$

where $\boldsymbol{x} = (x_0^2, x_0 x_1, \dots, x_{n-1} x_n, x_n^2)^T$ is the vector of monomials of degree 2.

Let us mention that the vector \boldsymbol{x} has $\frac{(n+2)(n+1)}{2}$ coordinates and the subspace of, not necessarily positive semi-definite, matrices H satisfying (4.3) has dimension $\frac{(n+2)(n+1)^2n}{12}$. Thus, finding an exact (positive semi-definite) solution H for equation (4.3) can be quite difficult, even for relatively small values of n.

Proof of Theorem 4.4. It is not hard to see that if $\pi_{34} = \mathbb{R}X_3 \oplus \mathbb{R}X_4$, then the sectional curvature of the plane π_{34} is

$$\kappa_{g_r}(\pi_{34}) = -\frac{9}{4}r + 1.$$

Thus, g_r does not have non-negative sectional curvature for $r > \frac{4}{9}$. Let Y_3, \ldots, Y_{13} be the orthonormal basis of \mathfrak{m} defined in the proof of Theorem 4.2 and write

$$X = \sum_{i=3}^{13} x_{i-3} Y_i, \qquad Y = \sum_{i=3}^{13} x_{i+8} Y_i.$$

For each r, consider the polynomial

$$F_r = g_r(R_{X,Y}^{g_r}X,Y) \in \mathbb{R}[x_0,\ldots,x_{21}],$$

where R^{g_r} denotes the curvature tensor of g_r . Observe that we can formally extend the polynomial F_r to every $r \in \mathbb{R}$ (even when g_r does not make sense for $r \leq 0$). Moreover, from the explicit formula for F_r , which can be found in the Appendix A.2, we see that fixing x_0, \ldots, x_{21} , the map $r \mapsto F_r(x_0, \ldots, x_{21})$ defines a linear function on r. Thus, it is enough to prove that the polynomials F_0 and $F_{\frac{4}{9}}$ are non-negative. We will use Lemma 4.5 to prove the stronger statement that F_0 and $F_{\frac{4}{9}}$ are polynomial sums of squares. Since F_r is obtained from computing sectional curvatures, every monomial $x_i x_j x_k x_l$ with nontrivial coefficient in F_r satisfies $0 \leq i \leq j \leq 10 < k \leq l \leq 21$. Hence, we do not lose generality replacing \boldsymbol{x}^T in Lemma 4.5 with

$$\boldsymbol{x}^{T} = (x_0 x_{11}, \dots, x_0 x_{21}, \dots, x_{10} x_{11}, \dots, x_{10} x_{21}, x_0^2, \dots, x_{21}^2)$$

This change substantially reduces the size of the system (4.3) from 253×253 to 143×143 . Now we are looking for symmetric positive semi-definite matrices H_{α} such that

$$F_{\alpha} = \boldsymbol{x}^T H_{\alpha} \boldsymbol{x}, \qquad \qquad \alpha \in \{0, \frac{4}{9}\}$$

This is a convex optimization problem, which, thanks to the reduction of the dimension mentioned above, can be successfully solved by the Python solver CVXOPT. We implemented the computer code in SageMath through two instances of SemidefiniteProgram(). However, this only yields numerical solutions, and since the condition of being positive semi-definite is a closed one, an exact solution is not guaranteed. Nonetheless, since the polynomial sums of squares are dense in the set of non-negative polynomials, exact solutions are expected to exist. Moreover, since F_r has relatively few non-trivial coefficients, one can expect to find sparse solutions H_0 and $H_{\frac{4}{9}}$. This is indeed the case, since rounding the numerical solutions lead us to the exact solution described as follows.

Define the index subsets

$$\begin{split} &I_{0,-2} = \{(39, 69), (49, 59), (87, 117), (97, 107)\}, \\ &I_{0,-1} = \{(1, 11), (1, 39), (1, 59), (1, 87), (1, 107), (11, 49), (11, 69), (11, 97), (11, 117), \\ & (39, 49), (39, 117), (49, 107), (59, 69), (59, 97), (69, 87), (87, 97), (107, 117)\}, \\ &I_{0,-\frac{1}{2}} = \{(3, 17), (3, 33), (3, 88), (3, 100), (4, 7), (4, 44), (4, 56), (4, 111), (5, 10), (5, 15), \\ & (5, 18), (5, 55), (6, 19), (6, 34), (6, 66), (6, 99), (7, 16), (7, 21), (7, 77), (8, 17), \\ & (8, 33), (8, 88), (8, 100), (9, 19), (9, 34), (9, 66), (9, 99), (10, 45), (10, 78), (10, 110), \\ & (14, 19), (14, 34), (14, 66), (14, 99), (15, 45), (15, 78), (15, 110), (16, 44), (16, 56), \\ & (16, 111), (17, 20), (17, 67), (18, 45), (18, 78), (18, 110), (19, 89), (20, 33), (20, 88), \\ & (20, 100), (21, 44), (21, 56), (21, 111), (33, 67), (34, 89), (37, 47), (37, 71), (37, 95), \\ & (37, 119), (38, 50), (38, 58), (38, 98), (38, 106), (40, 80), (40, 92), (40, 104), \\ & (40, 116), (43, 53), (43, 73), (43, 93), (43, 113), (44, 77), (45, 55), (47, 61), (47, 85), \\ & (47, 109), (50, 70), (50, 86), (50, 118), (52, 80), (52, 92), (52, 104), (52, 116), \\ & (53, 63), (53, 83), (53, 103), (55, 78), (55, 110), (56, 77), (58, 70), (58, 86), (58, 118), \\ & (61, 71), (61, 95), (61, 119), (63, 73), (63, 93), (63, 113), (64, 80), (64, 92), (64, 104), \\ & (64, 116), (66, 89), (67, 88), (67, 100), (70, 98), (70, 106), (71, 85), (71, 109), \\ & (73, 83), (73, 103), (76, 80), (76, 92), (76, 104), (76, 116), (77, 111), (83, 93), \\ & (83, 113), (85, 95), (85, 119), (86, 98), (86, 106), (89, 99), (93, 103), (95, 109), \\ \end{split}$$

$$\begin{array}{l} (98,118), (103,113), (106,118), (109,119) \}, \\ I_{0,\frac{1}{2}} = \{(3,3), (3,8), (3,20), (3,67), (4,4), (4,16), (4,21), (4,77), (5,5), (5,45), (5,78), \\ (5,110), (6,6), (6,9), (6,14), (6,89), (7,7), (7,44), (7,56), (7,111), (8,8), (8,20), \\ (8,67), (9,9), (9,14), (9,89), (10,10), (10,15), (10,18), (10,55), (14,14), (14,89), \\ (15,15), (15,18), (15,55), (16,16), (16,21), (16,77), (17,17), (17,33), (17,88), \\ (17,100), (18,18), (18,55), (19,19), (19,34), (19,66), (19,99), (20,20), (20,67), \\ (21,21), (21,77), (33,33), (33,88), (33,100), (34,34), (34,66), (34,99), (37,37), \\ (37,61), (37,85), (37,109), (38,38), (38,70), (38,86), (38,118), (40,40), (40,52), \\ (40,64), (40,76), (43,43), (43,63), (43,83), (43,103), (44,44), (44,56), (44,111), \\ (45,45), (45,78), (45,110), (47,47), (47,71), (47,95), (47,119), (50,50), (50,58), \\ (50,98), (50,106), (52,52), (52,64), (52,76), (53,53), (53,73), (53,93), (53,113), \\ (55,55), (56,56), (56,111), (58,58), (58,98), (58,106), (61,61), (61,85), (61,109), \\ (63,63), (63,83), (63,103), (64,64), (64,76), (66,66), (66,99), (67,67), (70,70), \\ (70,86), (70,118), (71,71), (71,95), (71,119), (73,73), (73,93), (73,113), (76,76), \\ (77,77), (78,78), (78,110), (80,80), (80,92), (80,104), (80,116), (83,83), \\ (83,103), (85,85), (85,109), (86,86), (86,118), (88,88), (88,100), (89,89), \\ (92,92), (92,104), (92,116), (93,93), (93,113), (95,95), (95,119), (98,98), \\ (98,106), (99,99), (100,100), (103,103), (104,104), (104,116), (106,106), \\ (109,109), (110,110), (111,111), (113,113), (116,116), (118,118), (119,119) \}, \\ I_{0,1} = \{(1,1), (1,49), (1,69), (1,97), (1,117), (11,11), (11,39), (11,59), (11,87), (11,107), \\ (39,59), (39,87), (49,69), (49,97), (59,107), (69,117), (87,107), (97,117) \}, \\ I_{0,2} = \{(39,39), (49,49), (59,59), (69,69), (87,87), (97,97), (107,107), (117,117) \}, \\ \end{array}$$

and let ${\cal H}_0$ be the symmetric matrix defined as

$$(H_0)_{ij} = \begin{cases} a, & (i,j) \in I_{0,a}, \\ 0, & (i,j) \notin I_{0,\pm\frac{1}{2}} \cup I_{0,\pm1} \cup I_{0,\pm2}. \end{cases}$$

Now it is routine to verify that H_0 is positive semi-definite and satisfies $F_0 = \boldsymbol{x}^T H_0 \boldsymbol{x}$. Similarly, if

$$\begin{split} I_{\frac{4}{9},-1} &= \{(39,69),(39,107),(49,59),(49,117),(59,87),(69,97),(87,117),(97,107)\},\\ I_{\frac{4}{9},-\frac{1}{2}} &= \{(3,17),(3,33),(3,88),(3,100),(4,7),(4,44),(4,56),(4,111),(5,10),(5,15),\\ &(5,18),(5,55),(6,19),(6,34),(6,66),(6,99),(7,16),(7,21),(7,77),(8,17),\\ &(8,33),(8,88),(8,100),(9,19),(9,34),(9,66),(9,99),(10,45),(10,78),(10,110),\\ &(14,19),(14,34),(14,66),(14,99),(15,45),(15,78),(15,110),(16,44),(16,56),\\ &(16,111),(17,20),(17,67),(18,45),(18,78),(18,110),(19,89),(20,33),(20,88),\\ &(20,100),(21,44),(21,56),(21,111),(33,67),(34,89),(37,47),(37,71),(37,95),\\ &(37,119),(38,50),(38,58),(38,98),(38,106),(40,80),(40,92),(40,104),\\ &(40,116),(43,53),(43,73),(43,93),(43,113),(44,77),(45,55),(47,61),(47,85), \end{split}$$

 $(47, 109), (50, 70), (50, 86), (50, 118), (52, 80), (52, 92), (52, 104), (52, 116), (53, 63), (53, 83), (53, 103), (55, 78), (55, 110), (56, 77), (58, 70), (58, 86), (58, 118), (61, 71), (61, 95), (61, 119), (63, 73), (63, 93), (63, 113), (64, 80), (64, 92), (64, 104), (64, 116), (66, 89), (67, 88), (67, 100), (70, 98), (70, 106), (71, 85), (71, 109), (73, 83), (73, 103), (76, 80), (76, 92), (76, 104), (76, 116), (77, 111), (83, 93), (83, 113), (85, 95), (85, 119), (86, 98), (86, 106), (89, 99), (93, 103), (95, 109), (98, 118), (103, 113), (106, 118), (109, 119)\},$

- $$\begin{split} I_{\frac{4}{9},-\frac{1}{3}} &= \{(2,22),(13,23),(25,35),(26,46),(27,57),(28,68),(29,79),(30,90),(31,101),\\ &\quad (32,112)\}, \end{split}$$
 - $$\begin{split} I_{\frac{4}{9},\frac{1}{3}} &= \{(2,2),(13,13),(22,22),(23,23),(25,25),(26,26),(27,27),(28,28),(29,29),\\ &\quad (30,30),(31,31),(32,32),(35,35),(46,46),(57,57),(68,68),(79,79),(90,90),\\ &\quad (101,101),(112,112)\}, \end{split}$$
- $I_{\frac{4}{2},\frac{1}{2}} = \{(3,3), (3,8), (3,20), (3,67), (4,4), (4,16), (4,21), (4,77), (5,5), (5,45), (5,78), ($ (5, 110), (6, 6), (6, 9), (6, 14), (6, 89), (7, 7), (7, 44), (7, 56), (7, 111), (8, 8), (8, 20),(8, 67), (9, 9), (9, 14), (9, 89), (10, 10), (10, 15), (10, 18), (10, 55), (14, 14), (14, 89),(15, 15), (15, 18), (15, 55), (16, 16), (16, 21), (16, 77), (17, 17), (17, 33), (17, 88),(17, 100), (18, 18), (18, 55), (19, 19), (19, 34), (19, 66), (19, 99), (20, 20), (20, 67),(21, 21), (21, 77), (33, 33), (33, 88), (33, 100), (34, 34), (34, 66), (34, 99), (37, 37),(37, 61), (37, 85), (37, 109), (38, 38), (38, 70), (38, 86), (38, 118), (40, 40), (40, 52),(40, 64), (40, 76), (43, 43), (43, 63), (43, 83), (43, 103), (44, 44), (44, 56), (44, 111),(45, 45), (45, 78), (45, 110), (47, 47), (47, 71), (47, 95), (47, 119), (50, 50), (50, 58),(50, 98), (50, 106), (52, 52), (52, 64), (52, 76), (53, 53), (53, 73), (53, 93), (53, 113),(55, 55), (56, 56), (56, 111), (58, 58), (58, 98), (58, 106), (61, 61), (61, 85), (61, 109),(63, 63), (63, 83), (63, 103), (64, 64), (64, 76), (66, 66), (66, 99), (67, 67), (70, 70),(70, 86), (70, 118), (71, 71), (71, 95), (71, 119), (73, 73), (73, 93), (73, 113), (76, 76),(77, 77), (78, 78), (78, 110), (80, 80), (80, 92), (80, 104), (80, 116), (83, 83),(83, 103), (85, 85), (85, 109), (86, 86), (86, 118), (88, 88), (88, 100), (89, 89),(92, 92), (92, 104), (92, 116), (93, 93), (93, 113), (95, 95), (95, 119), (98, 98),(98, 106), (99, 99), (100, 100), (103, 103), (104, 104), (104, 116), (106, 106),(109, 109), (110, 110), (111, 111), (113, 113), (116, 116), (118, 118), (119, 119) $I_{\frac{4}{2},1} = \{(39,39), (39,97), (49,49), (49,87), (59,59), (59,117), (69,69), (69,107), (87,87), (69,107), (87,87), (10,107), (10,1$ $(97, 97), (107, 107), (117, 117)\},\$

then

$$(H_{\frac{4}{9}})_{ij} = \begin{cases} a, & (i,j) \in I_{\frac{4}{9},a}, \\ 0, & (i,j) \notin I_{\frac{4}{9},\pm\frac{1}{3}} \cup I_{\frac{4}{9},\pm\frac{1}{2}} \cup I_{\frac{4}{9},\pm1}, \end{cases}$$

defines a symmetric positive semi-definite matrix satisfying $F_{\frac{4}{9}} = \boldsymbol{x}^T H_{\frac{4}{9}} \boldsymbol{x}$. This concludes the proof of the theorem.

Remark 4.6. We are not certain if F_r is a polynomial sum of squares for $0 < r < \frac{4}{9}$. Although it is not needed in the proof of Theorem 4.4, it would be interesting to know if this is indeed the case.

Remark 4.7. The definition (4.1) of the metric g_r resembles the construction of Berger spheres from the Hopf fibration by shrinking the metrics along the fibers. Moreover, if we restrict g_r to $\mathfrak{m}_0 \simeq \mathfrak{su}(2)$, then $S_r^3 = (\mathrm{SU}(2), g_r|_{\mathfrak{m}_0})$ is a Berger sphere. Recall that S_r^3 has (strictly) positive sectional curvature if and only if $0 < r < \frac{4}{9}$.

Appendix A.

A.1. Standard zero divisors. In this appendix, we include Table 1 with the standard zero divisors of the sedenion algebra.

$(e_1 + e_{10}, e_4 - e_{15})$	$(e_1 + e_{10}, e_5 + e_{14})$	$(e_1 + e_{10}, e_6 - e_{13})$	$(e_1 + e_{10}, e_7 + e_{12})$
$(e_1 + e_{11}, e_4 + e_{14})$	$(e_1 + e_{11}, e_5 + e_{15})$	$(e_1 + e_{11}, e_6 - e_{12})$	$(e_1 + e_{11}, e_7 - e_{13})$
$(e_1 + e_{12}, e_2 + e_{15})$	$(e_1 + e_{12}, e_3 - e_{14})$	$(e_1 + e_{12}, e_6 + e_{11})$	$(e_1 + e_{12}, e_7 - e_{10})$
$(e_1 + e_{13}, e_2 - e_{14})$	$(e_1 + e_{13}, e_3 - e_{15})$	$(e_1 + e_{13}, e_6 + e_{10})$	$(e_1 + e_{13}, e_7 + e_{11})$
$(e_1 + e_{14}, e_2 + e_{13})$	$(e_1 + e_{14}, e_3 + e_{12})$	$(e_1 + e_{14}, e_4 - e_{11})$	$(e_1 + e_{14}, e_5 - e_{10})$
$(e_1 + e_{15}, e_2 - e_{12})$	$(e_1 + e_{15}, e_3 + e_{13})$	$(e_1 + e_{15}, e_4 + e_{10})$	$(e_1 + e_{15}, e_5 - e_{11})$
$(e_2 + e_9, e_4 + e_{15})$	$(e_2 + e_9, e_5 - e_{14})$	$(e_2 + e_9, e_6 + e_{13})$	$(e_2 + e_9, e_7 - e_{12})$
$(e_2 + e_{11}, e_4 - e_{13})$	$(e_2 + e_{11}, e_5 + e_{12})$	$(e_2 + e_{11}, e_6 + e_{15})$	$(e_2 + e_{11}, e_7 - e_{14})$
$(e_2 + e_{12}, e_3 + e_{13})$	$(e_2 + e_{12}, e_5 - e_{11})$	$(e_2 + e_{12}, e_7 + e_9)$	$(e_2 + e_{13}, e_3 - e_{12})$
$(e_2 + e_{13}, e_4 + e_{11})$	$(e_2 + e_{13}, e_6 - e_9)$	$(e_2 + e_{14}, e_3 - e_{15})$	$(e_2 + e_{14}, e_5 + e_9)$
$(e_2 + e_{14}, e_7 + e_{11})$	$(e_2 + e_{15}, e_3 + e_{14})$	$(e_2 + e_{15}, e_4 - e_9)$	$(e_2 + e_{15}, e_6 - e_{11})$
$(e_3 + e_9, e_4 - e_{14})$	$(e_3 + e_9, e_5 - e_{15})$	$(e_3 + e_9, e_6 + e_{12})$	$(e_3 + e_9, e_7 + e_{13})$
$(e_3 + e_{10}, e_4 + e_{13})$	$(e_3 + e_{10}, e_5 - e_{12})$	$(e_3 + e_{10}, e_6 - e_{15})$	$(e_3 + e_{10}, e_7 + e_{14})$
$(e_3 + e_{12}, e_5 + e_{10})$	$(e_3 + e_{12}, e_6 - e_9)$	$(e_3 + e_{13}, e_4 - e_{10})$	$(e_3 + e_{13}, e_7 - e_9)$
$(e_3 + e_{14}, e_4 + e_9)$	$(e_3 + e_{14}, e_7 - e_{10})$	$(e_3 + e_{15}, e_5 + e_9)$	$(e_3 + e_{15}, e_6 + e_{10})$
$(e_4 + e_9, e_6 - e_{11})$	$(e_4 + e_9, e_7 + e_{10})$	$(e_4 + e_{10}, e_5 + e_{11})$	$(e_4 + e_{10}, e_7 - e_9)$
$(e_4 + e_{11}, e_5 - e_{10})$	$(e_4 + e_{11}, e_6 + e_9)$	$(e_4 + e_{13}, e_6 + e_{15})$	$(e_4 + e_{13}, e_7 - e_{14})$
$(e_4 + e_{14}, e_5 - e_{15})$	$(e_4 + e_{14}, e_7 + e_{13})$	$(e_4 + e_{15}, e_5 + e_{14})$	$(e_4 + e_{15}, e_6 - e_{13})$
$(e_5 + e_9, e_6 - e_{10})$	$(e_5 + e_9, e_7 - e_{11})$	$(e_5 + e_{10}, e_6 + e_9)$	$(e_5 + e_{11}, e_7 + e_9)$
$(e_5 + e_{12}, e_6 - e_{15})$	$(e_5 + e_{12}, e_7 + e_{14})$	$(e_5 + e_{14}, e_7 - e_{12})$	$(e_5 + e_{15}, e_6 + e_{12})$
$(e_6 + e_{10}, e_7 - e_{11})$	$(e_6 + e_{11}, e_7 + e_{10})$	$(e_6 + e_{12}, e_7 - e_{13})$	$(e_6 + e_{13}, e_7 + e_{12})$

TABLE 1. The 84 standard zero divisors of \mathbb{S}

A.2. Expression for the sectional curvature. We write down the polynomial F_r used in the proof of Theorem 4.4.

$$\begin{split} F_r &= (-\frac{9}{4}r+1)x_0^2 x_{12}^2 + \frac{3}{4}rx_0^2 x_{13}^2 + \frac{1}{2}x_0^2 x_{14}^2 + x_0^2 x_{14} x_{19} + \frac{1}{2}x_0^2 x_{15}^2 - x_0^2 x_{15} x_{18} + \frac{1}{2}x_0^2 x_{16}^2 \\ &\quad -x_0^2 x_{16} x_{21} + \frac{1}{2}x_0^2 x_{17}^2 + x_0^2 x_{17} x_{20} + \frac{1}{2}x_0^2 x_{18}^2 + \frac{1}{2}x_0^2 x_{19}^2 + \frac{1}{2}x_0^2 x_{20}^2 + \frac{1}{2}x_0^2 x_{21}^2 \\ &\quad + (\frac{9}{2}r-2)x_0 x_1 x_{11} x_{12} + 2x_0 x_1 x_{14} x_{20} + 2x_0 x_1 x_{15} x_{21} - 2x_0 x_1 x_{16} x_{18} \\ &\quad - 2x_0 x_1 x_{17} x_{19} - \frac{3}{2}r x_0 x_2 x_{11} x_{13} - x_0 x_3 x_{11} x_{14} - x_0 x_3 x_{11} x_{19} \end{split}$$

 $+ \left(\frac{9}{2}r - 3\right)x_0x_3x_{12}x_{17} - x_0x_3x_{12}x_{20} - x_0x_4x_{11}x_{15} + x_0x_4x_{11}x_{18}$ $+ \left(-\frac{9}{2}r + 3\right)x_0x_4x_{12}x_{16} - x_0x_4x_{12}x_{21} - x_0x_5x_{11}x_{16} + x_0x_5x_{11}x_{21}$ $+ \left(\frac{9}{2}r - 3\right)x_0x_5x_{12}x_{15} + x_0x_5x_{12}x_{18} - x_0x_6x_{11}x_{17} - x_0x_6x_{11}x_{20}$ $+ \left(-\frac{9}{2}r+3\right)x_0x_6x_{12}x_{14} + x_0x_6x_{12}x_{19} + x_0x_7x_{11}x_{15} - x_0x_7x_{11}x_{18} + x_0x_7x_{12}x_{16$ $+(\frac{9}{2}r-3)x_0x_7x_{12}x_{21}-x_0x_8x_{11}x_{14}-x_0x_8x_{11}x_{19}+x_0x_8x_{12}x_{17}$ $+\left(-\frac{9}{2}r+3\right)x_{0}x_{8}x_{12}x_{20}-x_{0}x_{9}x_{11}x_{17}-x_{0}x_{9}x_{11}x_{20}-x_{0}x_{9}x_{12}x_{14}$ $+(\frac{9}{2}r-3)x_0x_9x_{12}x_{19}+x_0x_{10}x_{11}x_{16}-x_0x_{10}x_{11}x_{21}-x_0x_{10}x_{12}x_{15}$ $+(-\frac{9}{2}r+3)x_0x_{10}x_{12}x_{18}+(-\frac{9}{4}r+1)x_1^2x_{11}^2+\frac{3}{4}rx_1^2x_{13}^2+\frac{1}{2}x_1^2x_{14}^2-x_1^2x_{14}x_{19}$ $+\frac{1}{2}x_1^2x_{15}^2+x_1^2x_{15}x_{18}+\frac{1}{2}x_1^2x_{16}^2+x_1^2x_{16}x_{21}+\frac{1}{2}x_1^2x_{17}^2-x_1^2x_{17}x_{20}+\frac{1}{2}x_1^2x_{18}^2$ $+\frac{1}{2}x_{1}^{2}x_{19}^{2}+\frac{1}{2}x_{1}^{2}x_{20}^{2}+\frac{1}{2}x_{1}^{2}x_{21}^{2}-\frac{3}{2}rx_{1}x_{2}x_{12}x_{13}+(-\frac{9}{2}r+3)x_{1}x_{3}x_{11}x_{17}$ $-x_1x_3x_{11}x_{20} - x_1x_3x_{12}x_{14} + x_1x_3x_{12}x_{19} + (\frac{9}{2}r - 3)x_1x_4x_{11}x_{16} - x_1x_4x_{11}x_{21}$ $-x_1x_4x_{12}x_{15} - x_1x_4x_{12}x_{18} + (-\frac{9}{2}r+3)x_1x_5x_{11}x_{15} + x_1x_5x_{11}x_{18}$ $-x_1x_5x_{12}x_{16} - x_1x_5x_{12}x_{21} + (\frac{9}{2}r - 3)x_1x_6x_{11}x_{14} + x_1x_6x_{11}x_{19}$ $-x_1x_6x_{12}x_{17} + x_1x_6x_{12}x_{20} + x_1x_7x_{11}x_{16} + (-\frac{9}{2}r+3)x_1x_7x_{11}x_{21}$ $-x_1x_7x_{12}x_{15} - x_1x_7x_{12}x_{18} + x_1x_8x_{11}x_{17} + (\frac{9}{2}r - 3)x_1x_8x_{11}x_{20}$ $+x_1x_8x_{12}x_{14} - x_1x_8x_{12}x_{19} - x_1x_9x_{11}x_{14} + (-\frac{9}{2}r+3)x_1x_9x_{11}x_{19} + x_1x_9x_{12}x_{17}$ $-x_1x_9x_{12}x_{20} - x_1x_{10}x_{11}x_{15} + (\frac{9}{2}r - 3)x_1x_{10}x_{11}x_{18} - x_1x_{10}x_{12}x_{16} - x_1x_{10}x_{12}x_{21}$ $+ \frac{3}{4}rx_2^2x_{11}^2 + \frac{3}{4}rx_2^2x_{12}^2 + \frac{3}{4}rx_2^2x_{14}^2 + \frac{3}{4}rx_2^2x_{15}^2 + \frac{3}{4}rx_2^2x_{16}^2 + \frac{3}{4}rx_2^2x_{17}^2 + \frac{3}{4}rx_2^2x_{18}^2$ $+\frac{3}{4}rx_{2}^{2}x_{10}^{2}+\frac{3}{4}rx_{2}^{2}x_{20}^{2}+\frac{3}{4}rx_{2}^{2}x_{21}^{2}-\frac{3}{2}rx_{2}x_{3}x_{13}x_{14}-\frac{3}{2}rx_{2}x_{4}x_{13}x_{15}-\frac{3}{2}rx_{2}x_{5}x_{13}x_{16}$ $-\frac{3}{2}rx_2x_6x_{13}x_{17} - \frac{3}{2}rx_2x_7x_{13}x_{18} - \frac{3}{2}rx_2x_8x_{13}x_{19} - \frac{3}{2}rx_2x_9x_{13}x_{20} - \frac{3}{2}rx_2x_{10}x_{13}x_{21}$ $+ \frac{1}{2}x_3^2x_{11}^2 + \frac{1}{2}x_3^2x_{12}^2 + \frac{3}{4}rx_3^2x_{13}^2 + \frac{1}{2}x_3^2x_{15}^2 + \frac{1}{2}x_3^2x_{16}^2 + (-\frac{9}{4}r+2)x_3^2x_{17}^2 + \frac{1}{2}x_3^2x_{18}^2$ $+\frac{1}{2}x_3^2x_{21}^2 - x_3x_4x_{14}x_{15} + (\frac{9}{2}r - 3)x_3x_4x_{16}x_{17} + x_3x_4x_{18}x_{19} - x_3x_4x_{20}x_{21}$ $-x_3x_5x_{14}x_{16} + (-\frac{9}{2}r+3)x_3x_5x_{15}x_{17} + x_3x_5x_{18}x_{20} + x_3x_5x_{19}x_{21}$ $+(\frac{9}{2}r-4)x_3x_6x_{14}x_{17}-x_3x_7x_{14}x_{18}+x_3x_7x_{15}x_{19}+x_3x_7x_{16}x_{20}$ $+\left(-\frac{9}{2}r+3\right)x_{3}x_{7}x_{17}x_{21}+x_{3}x_{8}x_{11}^{2}-x_{3}x_{8}x_{12}^{2}-2x_{3}x_{8}x_{15}x_{18}-2x_{3}x_{8}x_{16}x_{21}$ $+\frac{9}{2}rx_3x_8x_{17}x_{20}+2x_3x_9x_{11}x_{12}+2x_3x_9x_{15}x_{21}-2x_3x_9x_{16}x_{18}-\frac{9}{2}rx_3x_9x_{17}x_{19}$ $-x_3x_{10}x_{14}x_{21} - x_3x_{10}x_{15}x_{20} + x_3x_{10}x_{16}x_{19} + (\frac{9}{2}r - 3)x_3x_{10}x_{17}x_{18} + \frac{1}{2}x_4^2x_{11}^2$ $+ \frac{1}{2}x_4^2x_{12}^2 + \frac{3}{4}rx_4^2x_{13}^2 + \frac{1}{2}x_4^2x_{14}^2 + (-\frac{9}{4}r + 2)x_4^2x_{16}^2 + \frac{1}{2}x_4^2x_{17}^2 + \frac{1}{2}x_4^2x_{19}^2 + \frac{1}{2}x_4^2x_{20}^2$ $+(\frac{9}{2}r-4)x_4x_5x_{15}x_{16}+(-\frac{9}{2}r+3)x_4x_6x_{14}x_{16}-x_4x_6x_{15}x_{17}+x_4x_6x_{18}x_{20}$ $+ x_4 x_6 x_{19} x_{21} - x_4 x_7 x_{11}^2 + x_4 x_7 x_{12}^2 - 2 x_4 x_7 x_{14} x_{19} + \frac{9}{2} r x_4 x_7 x_{16} x_{21}$ $-2x_4x_7x_{17}x_{20} + x_4x_8x_{14}x_{18} - x_4x_8x_{15}x_{19} + (-\frac{9}{2}r+3)x_4x_8x_{16}x_{20} + x_4x_8x_{17}x_{21}$ $-x_4x_9x_{14}x_{21} - x_4x_9x_{15}x_{20} + (\frac{9}{2}r - 3)x_4x_9x_{16}x_{19} + x_4x_9x_{17}x_{18} + 2x_4x_{10}x_{11}x_{12}$ $+2x_4x_{10}x_{14}x_{20} - \frac{9}{2}rx_4x_{10}x_{16}x_{18} - 2x_4x_{10}x_{17}x_{19} + \frac{1}{2}x_5^2x_{11}^2 + \frac{1}{2}x_5^2x_{12}^2 + \frac{3}{4}rx_5^2x_{13}^2$ $+\frac{1}{2}x_5^2x_{14}^2 + (-\frac{9}{4}r+2)x_5^2x_{15}^2 + \frac{1}{2}x_5^2x_{17}^2 + \frac{1}{2}x_5^2x_{19}^2 + \frac{1}{2}x_5^2x_{20}^2 + (\frac{9}{2}r-3)x_5x_6x_{14}x_{15}$ $-x_5x_6x_{16}x_{17} - x_5x_6x_{18}x_{19} + x_5x_6x_{20}x_{21} - 2x_5x_7x_{11}x_{12} - 2x_5x_7x_{14}x_{20}$

THE GEOMETRY OF SEDENION ZERO DIVISORS

$$\begin{split} &-\frac{9}{2}rx_5x_7x_{15}x_{21}+2x_5x_7x_{17}x_{19}+x_5x_8x_{14}x_{21}+\left(\frac{9}{2}r-3\right)x_5x_8x_{15}x_{20}\\ &-x_5x_8x_{16}x_{19}-x_5x_8x_{17}x_{18}+x_5x_9x_{14}x_{18}+\left(-\frac{9}{2}r+3\right)x_5x_9x_{15}x_{19}-x_5x_9x_{16}x_{20}\\ &+x_5x_9x_{17}x_{21}-x_5x_{10}x_{11}^2+x_5x_{10}x_{12}^2-2x_5x_{10}x_{14}x_{19}+\frac{9}{2}rx_5x_{10}x_{15}x_{18}\\ &-2x_5x_{10}x_{17}x_{20}+\frac{1}{2}x_6^2x_{11}^2+\frac{1}{2}x_6^2x_{12}^2+\frac{3}{4}rx_6^2x_{13}^2+\left(-\frac{9}{4}r+2\right)x_6^2x_{14}^2+\frac{1}{2}x_6^2x_{15}^2\\ &+\frac{1}{2}x_6^2x_{16}^2+\frac{1}{2}x_6^2x_{18}^2+\frac{1}{2}x_6^2x_{21}^2+\left(\frac{9}{2}r-3\right)x_6x_7x_{14}x_{21}+x_6x_7x_{15}x_{20}-x_6x_7x_{16}x_{19}\\ &-x_6x_7x_{17}x_{18}-2x_6x_8x_{11}x_{12}-\frac{9}{2}rx_6x_8x_{14}x_{20}-2x_6x_8x_{15}x_{21}+2x_6x_8x_{16}x_{18}\\ &+x_6x_9x_{11}^2-x_6x_9x_{12}^2+\frac{9}{2}rx_6x_9x_{14}x_{19}-2x_6x_9x_{15}x_{18}-2x_6x_9x_{16}x_{21}\\ &+\left(-\frac{9}{2}r+3\right)x_6x_{10}x_{14}x_{18}+x_6x_{10}x_{15}x_{19}+x_6x_{10}x_{16}x_{20}-x_6x_{10}x_{17}x_{21}+\frac{1}{2}x_7^2x_{21}^2\\ &+\frac{1}{2}x_7^2x_{12}^2+\frac{3}{4}rx_7^2x_{13}^2+\frac{1}{2}x_7^2x_{14}^2+\frac{1}{2}x_7^2x_{17}^2+\frac{1}{2}x_7^2x_{29}^2+\left(-\frac{9}{4}r+2\right)x_7^2x_{21}^2\\ &+x_7x_8x_{14}x_{15}-x_7x_8x_{16}x_{17}-x_7x_8x_{18}x_{19}+\left(\frac{9}{2}r-3\right)x_7x_8x_{20}x_{21}+x_7x_9x_{14}x_{16}\\ &+x_7x_9x_{15}x_{17}-x_7x_9x_{18}x_{20}+\left(-\frac{9}{2}r+3\right)x_7x_9x_{19}x_{21}+\left(\frac{9}{2}r-4\right)x_7x_{10}x_{18}x_{21}\\ &+\frac{1}{2}x_8^2x_{21}^2+\left(\frac{9}{2}r-4\right)x_8x_9x_{19}x_{20}+x_8x_{10}x_{14}x_{16}+x_8x_{10}x_{15}x_{17}\\ &+\left(-\frac{9}{2}r+3\right)x_8x_{10}x_{18}x_{20}-x_8x_{10}x_{19}x_{21}+\frac{1}{2}x_9^2x_{12}^2+\frac{3}{4}rx_9^2x_{13}^2+\frac{1}{2}x_9^2x_{15}^2\\ &+\frac{1}{2}x_9^2x_{16}^2+\frac{1}{2}x_9^2x_{18}^2+\left(-\frac{9}{4}r+2\right)x_9^2x_{19}^2+\frac{1}{2}x_9^2x_{21}^2-x_9x_{10}x_{16}x_{17}\\ &+\left(\frac{9}{2}r-3\right)x_9x_{10}x_{18}x_{19}-x_9x_{10}x_{20}x_{21}+\frac{1}{2}x_{10}^2x_{11}^2+\frac{1}{2}x_{10}^2x_{12}^2+\frac{3}{4}rx_{10}^2x_{13}^2+\frac{1}{2}x_{10}^2x_{14}^2\\ &+\frac{1}{2}x_{10}^2x_{17}^2+\left(-\frac{9}{4}r+2\right)x_{10}^2x_{18}^2+\frac{1}{2}x_{10}^2x_{12}^2+\frac{1}{2}x_{10}^2x_{20}^2. \end{split}$$

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