THE SHARP DIAMETER BOUND OF STABLE MINIMAL SURFACES

QIXUAN HU, GUOYI XU, SHUAI ZHANG

Abstract. For three dimensional complete Riemannian manifolds with scalar curvature no less than one, we obtain the sharp upper bound of complete stable minimal surfaces' diameter.

Mathematics Subject Classification: 53C21, 53C23.

1. Introduction

Let (M^3, g) be a complete 3-dim Riemannian manifold with scalar curvature $R \geq 1$, for any stable minimal surface Σ in M (possibly $\partial\Sigma \neq \emptyset$), and $\gamma \subset \Sigma$ be a closed curve, if

$$U_\rho(\gamma) \cap \partial\Sigma = \emptyset, \qquad \mathrm{Image}[H_1(\gamma) \to H_1(U_\rho(\gamma))] \not\equiv 0. \tag{1.1}$$

where $U_\rho(\gamma) := \{x \in \Sigma : \mathrm{dist}_\Sigma(x, \gamma) \leq \rho\}$, Gromov and Lawson [GL83, Theorem 10.2] showed $\rho \leq \pi$. Furthermore they [GL83, Remark 10.6] conjecture that the best possible conclusion would be $\rho \leq \frac{\pi}{\sqrt{2}}$ which is achieved by $M = \mathbb{S}^2(\sqrt{2}) \times \mathbb{S}^1$, $\Sigma = \mathbb{S}^2(\sqrt{2}) \subset M$, and γ be the great circle in Σ.

On the other hand, Schoen-Yau [SY83, Proof of Proposition 1] showed that $\rho \leq \frac{\sqrt{6}}{3}\pi$.

In this note, we show that $\rho < \frac{\sqrt{6}}{3}\pi$ and it is the sharp upper bound, which is a corollary of the following theorem.

Theorem 1.1. *Let Σ be a complete stable minimal surface in a complete 3-dim Riemannian manifold (M^3, g) with scalar curvature $R(g) \geq 1$. Then*

$$\mathrm{Diam}(\Sigma) < \frac{2\sqrt{6}\pi}{3}. \tag{1.2}$$

The upper bound is sharp in the following sense: there is a sequence of complete 3-dim manifolds $M_k := (S^2 \times S^1, g_k)$ with $R(g_k) \geq 1$ and $k \in \mathbb{Z}^+$, and compact stable minimal surfaces $\Sigma_k \subseteq M_k$ such that $\lim_{k\to\infty} \mathrm{Diam}(\Sigma_k) = \dfrac{2\sqrt{6}}{3}\pi$.

The strictness of (1.2) is obtained by a simple observation, the main contribution of this note is the construction of the example manifolds, which shows the sharpness of this upper bound.

Date: December 4, 2024.

G. Xu was partially supported by NSFC 12141103.

From [Gro20, 4.C], we know

$$\mathrm{FillRad}(M) \leq \frac{1}{2}\mathrm{width}_{n-1}(M) \leq \frac{1}{2}\mathrm{Diam}(M)$$

for complete Riemannian manifold (M^n, g), where $\mathrm{FillRad}(M)$ is the filling radius and $\mathrm{width}_{n-1}(M)$ is the $(n-1)$-th Uryson width, defined as in [Gro20, section 4].

The results in section 3 show that for a complete stable minimal surface Σ in a complete 3-dim Riemannian manifold (M^3, g) with scalar curvature $R \geq 1$, we have $\mathrm{FillRad}(\Sigma) < \frac{\sqrt{6}\pi}{3}$.

Gromov [Gro07] proposed the following conjecture on the filling radius.

Conjecture 1.2. *If (M^n, g) is a complete Riemannian manifold with scalar curvature $R(g) \geq \sigma^2 > 0$, then there exists a universal constant c_n depending only on n such that*

$$\mathrm{FillRad}(M) \leq \frac{c_n}{\sigma}.$$

The conjecture has been partially answered by [WXYZ24] for manifolds with finite asymptotic dimension.

In 3-dim complete manifolds with scalar curvature no less than -1, Munteanu, Sung, Wang [MSW23] obtained the area upper bound of stable minimal surfaces,

In [Gro23, 3.10] Gromov conjectured that a complete manifold with scalar curvature $R(g) \geq 6$ admits a singular foliation by surfaces of area and diameter bounded by a universal constant. The compact case was solved in [LM23], and the non-compact case was solved in [LW23]; also see [WZ23] when M has boundary.

The organization of this paper is as follows. In section 2, we firstly get a general sharp diameter upper bound, for surfaces with the positive first eigenvalue corresponding to Laplace operator with suitable curvature potential term. The example Riemannian surfaces are also constructed in this section. The key is to get the suitable function from some special ODE (see (2.12)), which comes from the symmetrization of the corresponding PDE along the diameter (or 'radius') direction.

In section 3, we use the functions from the example Riemannian surfaces to construct the corresponding 3-dim complete Riemannian manifolds containing the stable minimal surfaces, whose diameter approximates the sharp upper bound.

2. The first eigenvalue and the diameter

Definition 2.1. *For a complete Riemannian manifold (M^n, g) and $w \in C^\infty(M^n)$, we define*

$$\lambda_1(-\Delta + w) := \inf_{f \in H_c^1(M^n)} \frac{\int_M |\nabla f|^2 + w \cdot f^2}{\int_M f^2},$$

where $H_c^1(M)$ is the closure of $C_c^\infty(M^n)$ in $H^1(M^n)$.

Lemma 2.2. *For a complete Riemannian manifold (M^n, g) with $w \in C^\infty(M^n)$, $\lambda \in \mathbb{R}^+$, assume $\lambda_1(-\Delta + w) \geq \lambda$, then there is $\varphi \in C^\infty(M^n)$ satisfying*

$$\varphi > 0, \qquad \text{and} \qquad -\Delta\varphi + (w - \lambda) \cdot \varphi \geq 0.$$

Proof: By definition of λ_1, we get

$$\lambda_1(-\Delta+(w-\lambda)) = \inf_{f\in H_c^{1,2}(M^n)} \frac{\int_M |\nabla f|^2+(w-\lambda)\cdot f^2}{\int_M f^2} = \inf_{f\in H_c^{1,2}(M^n)} \frac{\int_M |\nabla f|^2+w\cdot f^2}{\int_M f^2} - \lambda$$
$$= \lambda_1(-\Delta+w)-\lambda \geq 0. \tag{2.1}$$

The compact case follows from similar argument of [GT83, Theorem 8.38]. The non compact case follows from [FCS80, Theorem 1]. □

Theorem 2.3. *For any $\beta > \frac{1}{4}$ and $\lambda > 0$, assume $\lambda_1(-\Delta_\Sigma+\beta\cdot K_\Sigma)\geq\lambda$ on a complete 2-dim Riemannian manifold (Σ, g), where K_Σ is the sectional curvature of (Σ, g). Then*

$$\mathrm{Diam}(\Sigma) < \frac{2\beta\pi}{\sqrt{\lambda\cdot(4\beta-1)}}.$$

The upper bound is sharp in the following sense: there is a sequence of complete Riemannian manifolds $\Sigma_k = (S^2, g_k)$ with $\lambda_1(-\Delta_{\Sigma_k}+\beta\cdot K_{\Sigma_k})\geq\lambda$ and

$$\lim_{k\to\infty}\mathrm{Diam}(\Sigma_k) = \frac{2\beta\pi}{\sqrt{\lambda\cdot(4\beta-1)}}.$$

Remark 2.4. *Most part of the following proof is similar to* [WXZ, Lemma 2.4] *(also see* [Xu, Appendix A]*) except the last part, we include the complete proof here for readers' convenience. For $\beta = 1$, Theorem 2.3 also answers a question raised in* [WXZ, Remark 2.5].

Proof: **Step (1)**. By Lemma 2.2, we get there exists a positive function $v\in C^\infty(M)$, such that

$$-\Delta v+(\beta\cdot K_\Sigma-\lambda)v\geq 0.$$

Let $u = v^{\frac{1}{\beta}}$, we get

$$-\Delta u+(K_\Sigma-\lambda\beta^{-1})u+(1-\beta)u^{-1}|\nabla u|^2\geq 0. \tag{2.2}$$

Case(i):If Σ is compact, let $p, q\in\Sigma$ be two points with largest distance. Define $I[\gamma]$ by

$$I[\gamma] = \int_\gamma u\mathrm{d}s,$$

where $\mathrm{d}s$ is arclength along γ and γ is any curve from p to q.

Case(ii): If Σ is non compact, chose $p\in\Sigma$ and $R = \frac{2\beta\pi}{\sqrt{\lambda\cdot(4\beta-1)}}$, Define $I[\gamma]$ by

$$I[\gamma] = \int_\gamma u\mathrm{d}s,$$

where ds is arclength along γ and γ is any curve from p to $\partial B_R(p)$.

Since u is positive, for any curve $\gamma : [0, L]\to M$ connecting p and $\partial B_R(p)$, if there exists $L_1\in(0, L)$, such that $\gamma(L_1)\in\partial B_R(p)$. Let γ_1 be the restriction of γ on

$[0, L_1]$, we have

$$I[\gamma] = \int_\gamma u ds > \int_{\gamma_1} u ds = I[\gamma_1].$$

So

$$\inf_{\gamma \subset M} I[\gamma] = \inf_{\mathrm{Int}(\gamma) \subset B_R(p)} I[\gamma],$$

where $\mathrm{Int}(\gamma)$ is the interior of γ.

In both cases, there are at least one minimizer of $I[\cdot]$. Let $\gamma_0 : [0, l] \to M$ be one of the minimizers of $I[\cdot]$ and parametrized with unit speed.

Step (2). Let $V(s) = \varphi(s) \cdot \vec{n}$ be the variation vector field along $\gamma_0(s)$, where $\vec{n}$ is the unit normal vector field along γ and

$$\varphi(s) = u^{-\frac{1}{2}}(\gamma_0(s))\psi(s), \qquad \psi(s) = \sin\Big(\frac{\pi}{l} s\Big). \tag{2.3}$$

The first variation formula yields

$$\langle \nabla_{\partial s} \partial s, \vec{n} \rangle = \frac{\partial}{\partial \vec{n}} \ln u. \tag{2.4}$$

The non-negativity of the second variation of I at γ_0 gives

(2.5)
$$0 \le \delta^2 I[\gamma_0] = \int_{\gamma_0} [D^2 u(\vec{n}, \vec{n}) - 2u \cdot (\frac{\partial}{\partial \vec{n}} \ln u)^2] \cdot \varphi^2 + u \cdot \Big((\varphi')^2 - K_\Sigma \cdot \varphi^2\Big) \mathrm{d}s.$$

Note $\Delta_\Sigma u = D^2 u(\vec{n}, \vec{n}) + D^2 u(\partial s, \partial s)$, combining (2.4), we get

$$\Delta_\Sigma u = D^2 u(\vec{n}, \vec{n}) + D^2 u(\partial s, \partial s) = D^2 u(\vec{n}, \vec{n}) + u'' - (\frac{\partial}{\partial \vec{n}} \ln u)^2 \cdot u, \tag{2.6}$$

where $u' := \frac{\mathrm{d}}{\mathrm{d}s} u(\gamma_0(s))$.

Plugging (2.6) into (2.5), using (2.2), we obtain

$$\begin{aligned} 0 &\le \int_{\gamma_0} (\Delta_\Sigma u - u'' - u \cdot (\frac{\partial}{\partial \vec{n}} \ln u)^2) \cdot \varphi^2 + u \cdot \Big((\varphi')^2 - K_\Sigma \cdot \varphi^2\Big) \mathrm{d}s \\ &\le \int_{\gamma_0} \Big((1-\beta)|\nabla u|^2 u^{-1} - u'' - u \cdot (\frac{\partial}{\partial \vec{n}} \ln u)^2\Big) \cdot \varphi^2 + u \cdot \Big((\varphi')^2 - \lambda \beta^{-1} \varphi^2\Big) \mathrm{d}s. \\ &\le \int_{\gamma_0} \Big((1-\beta)(u')^2 u^{-1} - u''\Big) \cdot \varphi^2 + u \cdot \Big((\varphi')^2 - \lambda \beta^{-1} \varphi^2\Big) \mathrm{d}s. \end{aligned} \tag{2.7}$$

Plugging (2.3) into (2.7), and use integration by part, we get

$$
\begin{aligned}
0 &\leq \int_{\gamma_0} \Big[(1-\beta)(u')^2 u^{-2}\psi^2 + u'(-u^{-2}u'\psi^2 + 2u^{-1}\psi\psi') + u(-\frac{1}{2}u^{-\frac{3}{2}}u'\psi + u^{-\frac{1}{2}}\psi')^2 - \lambda\beta^{-1}\psi^2\Big]\mathrm{d}s \\
&= \int_{\gamma_0} \Big[(1+\frac{1}{4}(\beta-\frac{1}{4})^{-1})(\psi')^2 - \lambda\beta^{-1}\psi^2 - \Big((\beta-\frac{1}{4})^{\frac{1}{2}}u'u\psi - \frac{1}{2}(\beta-\frac{1}{4})^{-\frac{1}{2}}\psi'\Big)^2\Big]\mathrm{d}s \\
&\leq \int_{\gamma_0} \Big[(1+\frac{1}{4}(\beta-\frac{1}{4})^{-1})(\psi')^2 - \lambda\beta^{-1}\psi^2\Big]\mathrm{d}s \\
&= \int_0^l \Big[(1+\frac{1}{4}(\beta-\frac{1}{4})^{-1})(\frac{\pi}{l})^2\cos^2(\frac{\pi}{l}s) - \lambda\beta^{-1}\sin^2(\frac{\pi}{l}s)\Big]\mathrm{d}s \\
&= \Big[\frac{4\beta}{4\beta-1}(\frac{\pi}{l})^2 - \lambda\beta^{-1}\Big]\cdot\frac{l}{2}.
\end{aligned}
$$

So we get $l \leq \frac{2\beta\pi}{\sqrt{\lambda\cdot(4\beta-1)}}$. If $l = \frac{2\beta\pi}{\sqrt{\lambda\cdot(4\beta-1)}}$, we get

$$
\int_0^l \Big((\beta-\frac{1}{4})^{\frac{1}{2}}u'u\psi - \frac{1}{2}(\beta-\frac{1}{4})^{-\frac{1}{2}}\psi'\Big)^2\mathrm{d}s = 0. \tag{2.8}
$$

Note $(\beta-\frac{1}{4})^{\frac{1}{2}}u'u\psi - \frac{1}{2}(\beta-\frac{1}{4})^{-\frac{1}{2}}\psi'$ is a continuous function on $[0,l]$, so

$$
(\beta-\frac{1}{4})^{\frac{1}{2}}u'u\psi - \frac{1}{2}(\beta-\frac{1}{4})^{-\frac{1}{2}}\psi' \equiv 0, \quad \forall s\in[0,l]. \tag{2.9}
$$

However, for $s=0$, we have $\psi(0)=0$ and $\psi'(0)=\frac{\pi}{l}$, which is the contradiction.

Now we have $l < \frac{2\beta\pi}{\sqrt{\lambda\cdot(4\beta-1)}}$. If Σ is not compact, from the definition of l, we know that

$$
l \geq d(p, \partial B_R(p)) = \frac{2\beta\pi}{\sqrt{\lambda\cdot(4\beta-1)}},
$$

which is the contradiction.

Hence Σ must be compact, and from the definition of l in compact case we get

$$
\mathrm{Diam}(\Sigma) < \frac{2\beta\pi}{\sqrt{\lambda\cdot(4\beta-1)}}.
$$

Step (3). In the rest argument, we construct Σ_k. We assume $k\in\mathbb{Z}^+$ with

$$
k^{-1} \in \Big(0, \frac{\pi}{4}\sqrt{\frac{\beta}{\lambda}}\Big). \tag{2.10}
$$

Define

$$
\phi(x) = \begin{cases} e^{-\frac{1}{x}}, & x>0, \\ 0, & x\leq 0. \end{cases}
$$

$$
\eta_k(x) = \frac{\phi(2k^{-1}-x)}{\phi(x-k^{-1})+\phi(2k^{-1}-x)}, \qquad \forall x\in\mathbb{R}. \tag{2.11}
$$

Note η_k is a smooth function on $\mathbb{R}$.

We define

$$\begin{cases} \psi_k'(t) = -\frac{1}{\beta}\psi_k^2 + \frac{1-\eta_k^2}{4\beta^2}\psi_k^2 - \lambda, & \forall t \in [k^{-1}, b_k), \\ \psi_k(t) = \sqrt{\beta\lambda} \cdot \cot(\sqrt{\frac{\lambda}{\beta}}t), & \forall t \in (0, k^{-1}], \end{cases} \tag{2.12}$$

where $[k^{-1}, b_k)$ is the largest interval for the solution to (2.12).

By (2.12), note $\beta > \frac{1}{4}$, we get

$$\psi_k'(t) < 0, \qquad \forall t \in [0, b_k). \tag{2.13}$$

Using (2.12) , note $|\eta_k| \leq 1$, we get

$$\begin{cases} \psi_k'(t) \geq -\frac{1}{\beta}\psi_k^2 - \lambda\,, & \forall t \in [k^{-1}, 2k^{-1}], \\ \psi_k(k^{-1}) = \sqrt{\beta\lambda} \cdot \cot(\sqrt{\frac{\lambda}{\beta}}k^{-1}). \end{cases} \tag{2.14}$$

From (2.14), taking integral from k^{-1} to $2k^{-1}$ with respect to t, we get

$$\psi_k(2k^{-1}) \geq \sqrt{\lambda\beta}\cot\Big\{\sqrt{\frac{\lambda}{\beta}}k^{-1} + \cot^{-1}(\frac{\psi_k(k^{-1})}{\sqrt{\lambda\beta}})\Big\} = \sqrt{\lambda\beta}\cot\Big(2\sqrt{\frac{\lambda}{\beta}}k^{-1}\Big) > 0, \tag{2.15}$$

where the last inequality follows from (2.10).

Step (4). Now we consider $\psi_k(t)$ for $t \in [2k^{-1}, b_k)$, and we have

$$\psi_k'(t) = -(\frac{1}{\beta} - \frac{1}{4\beta^2})\psi_k^2 - \lambda\,, \qquad \forall t \in [2k^{-1}, b_k). \tag{2.16}$$

Note $\beta > \frac{1}{4}$, hence $\frac{1}{\beta} - \frac{1}{4\beta^2} > 0$. Direct integration yields

$$\psi_k(t) = 2\beta\sqrt{\frac{\lambda}{4\beta-1}} \cdot \cot\Big\{\frac{\sqrt{(4\beta-1)\lambda}}{2\beta}(t - 2k^{-1}) + \cot^{-1}(\frac{1}{2\beta}\sqrt{\frac{4\beta-1}{\lambda}}\psi_k(2k^{-1}))\Big\}, \quad \forall t \in [2k^{-1}, b_k). \tag{2.17}$$

From (2.17), we get $b_k < \infty$ and $\lim\limits_{t\to b_k-} \psi_k(t) = -\infty$.

Define $l_k = \min\{t \in (0, b_k) : \psi_k(t) = 0\}$, from (2.15) and (2.13), we get

$$l_k \geq 2k^{-1}, \qquad \psi_k(l_k) = 0.$$

From (2.17) we get

$$\frac{\sqrt{(4\beta-1)\lambda}}{2\beta}(l_k - 2k^{-1}) + \cot^{-1}(\frac{1}{2\beta}\sqrt{\frac{4\beta-1}{\lambda}}\psi_k(2k^{-1})) = \frac{\pi}{2},$$

which implies

$$\begin{aligned} l_k &= \frac{2\beta}{\sqrt{(4\beta-1)\lambda}}\Big\{\frac{\pi}{2} - \cot^{-1}(\frac{1}{2\beta}\sqrt{\frac{4\beta-1}{\lambda}}\psi_k(2k^{-1}))\Big\} + 2k^{-1} \\ &= \frac{\beta\pi}{\sqrt{(4\beta-1)\lambda}} - \frac{2\beta}{\sqrt{(4\beta-1)\lambda}}\cot^{-1}(\frac{1}{2\beta}\sqrt{\frac{4\beta-1}{\lambda}}\psi_k(2k^{-1})) + 2k^{-1}. \end{aligned} \tag{2.18}$$

From (2.15) and (2.18), we have

$$l_k \geq \frac{\beta\pi}{\sqrt{(4\beta-1)\lambda}} - \frac{2\beta}{\sqrt{(4\beta-1)\lambda}}\cot^{-1}\Big(\sqrt{1-\frac{1}{4\beta}}\cot\Big(2\sqrt{\frac{\lambda}{\beta}}k^{-1}\Big)\Big) + 2k^{-1}. \tag{2.19}$$

One key fact is that

$$\varliminf_{k\to\infty} l_k \geq \frac{\beta\pi}{\sqrt{(4\beta-1)\lambda}}. \tag{2.20}$$

Step (5). Now we define $f_k : [0, 2l_k] \to \mathbb{R}$ by

$$\begin{cases} (\ln f_k)'(t) = \frac{2\beta+\eta_k(t)-1}{2\beta^2}\psi_k(t)\ , & \forall t \in (0, l_k]; \\ f_k(k^{-1}) = \sqrt{\frac{\beta}{\lambda}}\sin(\sqrt{\frac{\lambda}{\beta}}k^{-1})\ ; & \\ f_k(t) = f_k(2l_k - t)\ , & \forall t \in [l_k, 2l_k); \\ f_k(0) = f_k(2l_k) = 0. & \end{cases} \tag{2.21}$$

From the property of f_k, we get the complete Riemannian surface $\Sigma_k = (S^2, dr^2 + f_k(r)^2 d\theta)$ for $r \in [0, 2l_k]$, where $\theta \in \mathbb{S}^1$.

Now we define $\zeta_k : [0, 2l_k] \to \mathbb{R}$ by

$$\begin{cases} (\ln \zeta_k)'(t) = \frac{1-\eta_k(t)}{2\beta}\psi_k(t)\ , & \forall t \in (0, l_k]; \\ \zeta_k(k^{-1}) = 1\ ; & \\ \zeta_k(t) = \zeta_k(2l_k - t)\ , & \forall t \in [l_k, 2l_k); \\ \zeta_k(0) = \zeta_k(2l_k) = 1. & \end{cases} \tag{2.22}$$

Note $\psi_k(t) \geq 0$ for $t \in (0, l_k]$ by the choice of l_k, then from (2.22), we get $\zeta_k(t) > 0$ for any $t \in [0, 2l_k]$.

Define $u_k(r, \theta) = \zeta_k(r)$, from (2.22) we know that $u_k \in C^\infty(\Sigma_k)$. From (2.12), (2.21) and (2.22), we get that for $r \in (0, l_k]$,

$$\begin{aligned} -\Delta_{\Sigma_k} u_k + \beta \cdot K_{\Sigma_k} u_k - \lambda u_k &= -(\zeta_k'' + \frac{f_k'}{f_k}\zeta_k') + \beta(-\frac{f_k''}{f_k})\zeta_k - \lambda\zeta_k \\ &= -\zeta_k\Big\{\psi_k' + \frac{1}{\beta}\psi_k^2 - \frac{1-\eta_k^2}{4\beta^2}\psi_k^2 + \lambda\Big\} = 0. \end{aligned} \tag{2.23}$$

By the symmetry definition of u_k, f_k, we get (2.23) also holds for $r \in [l_k, 2l_k)$. So

$$-\Delta_{\Sigma_k} u_k + \beta \cdot K_{\Sigma_k} u_k = \lambda u_k.$$

From (2.20) and Theorem 2.3, we have

$$\frac{2\beta\pi}{\sqrt{(4\beta-1)\lambda}} \geq \varlimsup_{k\to\infty} \mathrm{Diam}(\Sigma_k) \geq \varliminf_{k\to\infty} \mathrm{Diam}(\Sigma_k) \geq 2\varliminf_{k\to\infty} l_k \geq \frac{2\beta\pi}{\sqrt{(4\beta-1)\lambda}}.$$

□

Remark 2.5. *We shall point out that in our example, for the case of $\beta = \frac{1}{2}$, from (2.12) we have*

$$\psi_k(t) \leq \psi_k(k^{-1}) = \sqrt{\frac{\lambda}{2}} \cdot \cot(\sqrt{2\lambda}k^{-1}) \leq \frac{k}{2}, \quad t \in (k^{-1}, 2k^{-1}].$$

From (2.21) we have

$$\begin{cases} (\ln f_k)'(t) \leq 2\psi_k(t) \leq k\,, & \forall t \in (k^{-1}, 2k^{-1}], \\ (\ln f_k)'(t) = 0\,, & \forall t \in (2k^{-1}, l_k], \\ f_k(k^{-1}) = \sqrt{\frac{1}{2\lambda}}\sin(\sqrt{2\lambda}k^{-1}). & \end{cases} \tag{2.24}$$

So

$$f_k(t) \equiv f_k(2k^{-1}) \leq e \cdot f_k(k^{-1}) \leq e \cdot k^{-1}, \quad t \in [2k^{-1}, 2l_k - 2k^{-1}].$$

and $\Sigma_k = (S^2, g_k)$ *converges to a segment* $[0, 2\pi]$ *as* $k \to \infty$. *So* f_k *converges to* 0 *as* $k \to \infty$*, which implies* Σ_k *converges to a segment (see Figure 1).*

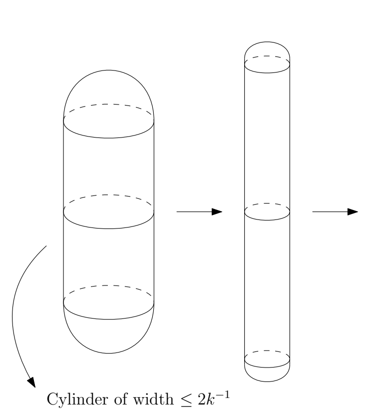

FIGURE 1. Figure of Σ_k as $k \to \infty$.

As an corollary, we get a sharp version of [CL, Lemma 16].

Corollary 2.6. *For compact surface* (Σ^2, g)*, if there is a smooth function* $u > 0$ *such that*

$$\frac{\Delta u}{u} \leq (K_\Sigma - 2^{-1}) + \frac{1}{2}\frac{|\nabla_\Sigma u|^2}{u^2}, \tag{2.25}$$

then the diameter of Σ *satisfies the sharp upper bound* $\mathrm{Diam}(\Sigma) < 2\pi$.

Proof: Let $v = \sqrt{u} > 0$, from (2.25), we get

$$-\Delta v + (\frac{1}{2}K_\Sigma - \frac{1}{4}) \cdot v \geq 0. \tag{2.26}$$

Using (2.26) in the proof of Theorem 2.3, we get the conclusion. □

3. The diameter of stable minimal surfaces in 3-dim manifolds

Lemma 3.1. *Let* Σ *be a complete stable minimal surface in a complete* 3*-dim Riemannian manifold* (M^3, g) *with scalar curvature* $R \geq 1$*, then* $\lambda_1(-\Delta_\Sigma + K_\Sigma) \geq \frac{1}{2}$*, where* K_Σ *is the sectional curvature of* Σ*.*

Proof: Let e_1, e_2, e_3 be an orthonormal frame defined locally on Σ with e_1, e_2 tangential and e_3 be unit normal. Since Σ is a stable minimal surface, for all $f \in C_c^\infty(\Sigma)$ we have

$$\int_\Sigma |\nabla f|^2 - (Rc(e_3, e_3) + \sum h_{ij}^2) f^2 \geq 0, \tag{3.1}$$

where $h_{ij} =< \nabla_{e_i} e_j, e_3 >$ is the second fundamental form of Σ. (see [Li12, Chapter 1])

Since Σ is minimal, we have $h_{11} + h_{22} = 0$. By Gauss curvature equation, we have $K_\Sigma = R_{1212} + h_{11}h_{22} - h_{12}^2$, and R_{ijkl} is the Ricci curvature tensor of M.

So (3.1) can be written as

$$\int_\Sigma |\nabla f|^2 - (\frac{1}{2} R - K_\Sigma + \frac{1}{2} \sum h_{ij}^2) f^2 \geq 0.$$

Since $R \geq 1$, we get

$$\int_\Sigma |\nabla f|^2 + (K_\Sigma - \frac{1}{2}) f^2 \geq 0,$$

which implies $\lambda_1(-\Delta + K_\Sigma) \geq \frac{1}{2}$. □

Theorem 3.2. *Let Σ be a complete stable minimal surface in a complete 3-dim Riemannian manifold (M^3, g) with scalar curvature $R \geq 1$, then*

$$\mathrm{Diam}(\Sigma) < \frac{2\sqrt{6}\pi}{3}. \tag{3.2}$$

The upper bound is sharp in the following sense: there is a sequence of complete 3-dim manifolds $M_k := (S^2 \times S^1, g_k)$ with $R \geq 1$ and $k \in \mathbb{Z}^+$, and compact stable minimal surfaces $\Sigma_k \subseteq M_k$ such that $\lim_{k\to\infty} \mathrm{Diam}(\Sigma_k) = \frac{2\sqrt{6}}{3}\pi$.

Proof: **Step (1)**. The inequality (3.2) follows from Lemma 3.1 and Theorem 2.3 for the case $\beta = 1, \lambda = \frac{1}{2}$.

Let $g_k = \mathrm{d}r^2 + f_k^2(r)\mathrm{d}\theta^2 + \zeta_k^2(r)\mathrm{d}\phi^2$, where (r, θ) is the polar coordinate on S^2, ϕ is the coordinate on S^1, and f_k, ζ_k are defined as in (2.21) and (2.22).

Note $r \in [0, 2l_k]$, then by (2.20) and (3.2), we have

$$\lim_{k\to\infty} \mathrm{Diam}(\Sigma_k) = \frac{2\sqrt{6}}{3}\pi.$$

Define $\Sigma_k = S^2 \times \{\phi_0\} \subseteq M_k$ in the rest argument.

If the r-coordinate of $p \in M_k$ is 0 or $2l_k$, by the definition of M_k, the k^{-1}-neighborhood of p in M_k is an open set in $\mathbb{S}^2(\sqrt{2}) \times \mathbb{S}^1(1)$; where $\mathbb{S}^2(\sqrt{2})$ is a round sphere with radius $\sqrt{2}$, $\mathbb{S}^1(1)$ is a round unit circle and $\mathbb{S}^2(\sqrt{2}) \times \mathbb{S}^1(1)$ has the product metric.

The k^{-1}-neighborhood of p in Σ is an open set of $\mathbb{S}^2(\sqrt{2}) \times \{\phi_0\} \subset \mathbb{S}^2(\sqrt{2}) \times \mathbb{S}^1(1)$ for some $\phi_0 \in \mathbb{S}^1$.

Since $\mathbb{S}^2(\sqrt{2})$ is a minimal surface of $\mathbb{S}^2(\sqrt{2}) \times \mathbb{S}^1(1)$ with the product metric, we get the mean curvature of Σ_k is 0 in a neighborhood of p.

Step (2). Now we assume $p \in \Sigma_k$, where $\phi_0 \in S^1$, and the r-coordinate of p is not equal to 0 or $2l_k$.

Let $\{e_1, e_2, e_3\} = \{\frac{\partial}{\partial r}, \frac{1}{f_k}\frac{\partial}{\partial \theta}, \frac{1}{\zeta_k}\frac{\partial}{\partial \phi}\}$ be a local orthonormal frame on a neighborhood of p in M_k, we have e_1, e_2 is tangential and e_3 is unit normal. The second fundamental form of Σ_k is defined by symmetric quadratic tensor $h_{ij} =< \nabla_{e_i} e_j, e_3 >$, $i, j \in \{1, 2\}$, where ∇ is the Riemannian connection of M_k.

Let $\Gamma_{ij}^k = \frac{1}{2} g^{lk} \left(\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} \right)$ be the Christoffel symbol, where $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}$ are $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}$ respectively, and (g_{ij}) is the matrix corresponding to the metric of M_k.

By direct computation, we get $\Gamma_{33}^1 = -\zeta_k \frac{\partial \zeta_k}{\partial r}$, $\Gamma_{13}^3 = \Gamma_{31}^3 = \frac{1}{\zeta_k}\frac{\partial \zeta_k}{\partial r}$, $\Gamma_{22}^1 = -f_k \frac{\partial f_k}{\partial r}$, and $\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{f_k}\frac{\partial f_k}{\partial r}$. Other Christoffel symbols are zero.

We have

$$
\begin{aligned}
\nabla_{e_1} e_2 &= \nabla_{\frac{\partial}{\partial r}} (\frac{1}{f_k}\frac{\partial}{\partial \theta}) = \frac{\partial}{\partial r}(\frac{1}{f_k})\frac{\partial}{\partial \theta} + \frac{1}{f_k^2}\frac{\partial f_k}{\partial r}\frac{\partial}{\partial \theta} = 0, \\
\nabla_{e_1} e_1 &= \nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r} = 0, \\
\nabla_{e_2} e_2 &= \nabla_{(\frac{1}{f_k}\frac{\partial}{\partial \theta})} (\frac{1}{f_k}\frac{\partial}{\partial \theta}) = \frac{1}{f_k}\frac{\partial}{\partial \theta}(\frac{1}{f_k})\frac{\partial}{\partial \theta} - \frac{1}{f_k}\frac{\partial f_k}{\partial r}\frac{\partial}{\partial r}.
\end{aligned}
$$

We get $\nabla_{e_i} e_j$ is in the tangent plane of Σ_k, and

$$
h_{ij} = 0, \qquad \forall i, j \in \{1, 2\}.
$$

So on a neighborhood of p in Σ_k, we have the mean curvature of Σ_k is 0.

So we get Σ_k is a minimal surface.

Step (3). Since $h_{ij} = 0$, we get the scalar curvature of M_k is $2K_{\Sigma_k} + 2Rc(e_3, e_3)$, where K_{Σ_k} is the sectional curvature of Σ_k.

By the definition of Ricci curvature tensor, we have

$$
R_{ij} = \frac{\partial}{\partial x_k}\Gamma_{ij}^k - \frac{\partial}{\partial x_j}\Gamma_{ik}^k + \Gamma_{ij}^s \Gamma_{sk}^k - \Gamma_{ik}^s \Gamma_{sj}^k.
$$

We get

$$
Rc(e_3, e_3) = \frac{1}{\zeta_k^2} Rc(\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi}) = -\frac{1}{\zeta_k} \cdot (\frac{\partial^2 \zeta_k}{\partial r^2} + \frac{1}{f_k}\frac{\partial f_k}{\partial r}\frac{\partial \zeta_k}{\partial r}).
$$

On Σ_k, we have $\Delta_{\Sigma_k} = \frac{\partial^2}{\partial r^2} + \frac{f_k'}{f_k}\frac{\partial}{\partial r} + \frac{1}{f_k^2}\frac{\partial^2}{\partial \theta^2}$, and $K_{\Sigma_k} = -\frac{f_k''}{f_k}$, where Δ_{Σ_k} is the Laplacian operator on Σ_k.

Define $u_k(r, \theta) = \zeta_k(r)$, then $u_k \in C^\infty(\Sigma_k)$. So $Rc(e_3, e_3) = -\frac{\Delta_{\Sigma_k} u_k}{u_k}$.

By (2.23), we get

$$
-\Delta_{\Sigma_k} u_k + (K_{\Sigma_k} - \frac{1}{2}) u_k = 0. \tag{3.3}
$$

We get the scalar curvature of M_k is

$$
R(g_k) = 2K_{\Sigma_k} - 2\frac{\Delta_{\Sigma_k} u_k}{u_k} = 1.
$$

Step (4). Let $w_k = \ln u_k$, from (3.3) we get

$$\Delta_{\Sigma_k} w_k = \frac{\Delta_{\Sigma_k} g}{g} - |\frac{\nabla g}{g}|^2 = K_{\Sigma_k} - \frac{1}{2} - |\nabla w_k|^2. \tag{3.4}$$

For any $\varphi \in C^\infty(\Sigma_k)$, we multiply both sides of (3.4) by φ^2 and take integration on Σ_k to get

$$\int_{\Sigma_k} -2\varphi\nabla\varphi \cdot \nabla w_k = \int_{\Sigma_k} (K_{\Sigma_k} - \frac{1}{2})\varphi^2 - \varphi^2 \cdot |\nabla w_k|^2. \tag{3.5}$$

Note

$$-2\varphi\nabla\varphi \cdot \nabla w_k \geq -|\nabla\varphi|^2 - \varphi^2 \cdot |\nabla w_k|^2. \tag{3.6}$$

By (3.6) and (3.5), we obtain

$$\int_{\Sigma_k} |\nabla\varphi|^2 + (K_{\Sigma_k} - \frac{1}{2})\varphi^2 \geq 0, \qquad \forall \varphi \in C^\infty(\Sigma_k).$$

Since $K_{\Sigma_k} - \frac{1}{2} = \frac{\Delta_{\Sigma_k} u_k}{u_k} = -Rc(e_3, e_3)$ and $h_{ij} = 0$, we get

$$\int_{\Sigma_k} |\nabla\varphi|^2 - (Rc(e_3, e_3) + \sum h_{ij}^2)\varphi^2 \geq 0, \qquad \forall \varphi \in C^\infty(\Sigma_k).$$

So Σ_k is a stable minimal surface of M_k. □

Acknowledgments

We thank Bo Zhu for his comments and suggestion on the earlier version of this paper.

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interests The authors declare that they have no conflict of interest.

References

[CL] Otis Chodosh and Chao Li, *Generalized soap bubbles and the topology of manifolds with positive scalar curvature*, arXiv:2008.11888v3 [math.DG].

[FCS80] Doris Fischer-Colbrie and Richard Schoen, *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math. **33** (1980), no. 2, 199–211, DOI 10.1002/cpa.3160330206.

[GT83] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order* **224** (1983), xiii+513, DOI 10.1007/978-3-642-61798-0.

[GL83] Mikhael Gromov and Jr. Lawson H. Blaine, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Inst. Hautes Études Sci. Publ. Math. **58** (1983), 83–196 (1984).

[Gro07] Misha Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, English, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2007. Based on the 1981 French original, With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.

[Gro20] Misha Gromov, *No metrics with Positive Scalar Curvatures on Aspherical 5-Manifolds*, 2020.

[Gro23] Misha Gromov, *Four lectures on scalar curvature*, Perspectives in scalar curvature. Vol. 1, 2023, pp. 1–514.

[Li12] Peter Li, *Geometric analysis*, Cambridge Studies in Advanced Mathematics, 134. Cambridge University Press, Cambridge, x+406 pp, 2012.

[LM23] Yevgeny Liokumovich and Davi Maximo, *Waist inequality for 3-manifolds with positive scalar curvature*, Perspectives in scalar curvature. Vol. 2, 2023, pp. 799–831.

[LW23] Yevgeny Liokumovich and Zhichao Wang, *On the waist and width inequality in complete 3-manifolds with positive scalar curvature*, 2023.

[MSW23] Ovidiu Munteanu, Chiung-Jue Anna Sung, and Jiaping Wang, *Area and spectrum estimates for stable minimal surfaces*, J. Geom. Anal. **33** (2023), no. 2, Paper No. 40, 34, DOI 10.1007/s12220-022-01076-x.

[SY83] Richard Schoen and S. T. Yau, *The existence of a black hole due to condensation of matter*, Comm. Math. Phys. **90** (1983), no. 4, 575–579.

[WXYZ24] Jinmin Wang, Zhizhang Xie, Guoliang Yu, and Bo Zhu, *Filling Radius, Quantitative K-theory and Positive Scalar Curvature*, 2024.

[WZ23] Zhichao Wang and Bo Zhu, *Uryson width of three dimensional mean convex domain with non-negative Ricci curvature*, J. Funct. Anal. **285** (2023), no. 8, Paper No. 110062, 25, DOI 10.1016/j.jfa.2023.110062.

[WXZ] Guodong Wei, Guoyi Xu, and Shuai Zhang, *Volume growth and positive scalar curvature*, To appear in Transactions of the American Mathematical Society.

[Xu] Kai Xu, *On closed surfaces with nonnegative curvature in the spectral sense*, arXiv:2211.11715v2 [math.DG].

Qixuan Hu, Department of Mathematical Sciences, Tsinghua University, Beijing, P. R. China
Email address: huqx20@mails.tsinghua.edu.cn

Guoyi Xu, Department of Mathematical Sciences, Tsinghua University, Beijing, P. R. China
Email address: guoyixu@tsinghua.edu.cn

Shuai Zhang, Department of Mathematical Sciences, Tsinghua University, Beijing, P. R. China
Email address: zhangshu22@mails.tsinghua.edu.cn