NUTs, Bolts, and Spindles

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Abstract

We construct new infinite classes of Euclidean supersymmetric solutions of four dimensional minimal gauged supergravity comprising a $U(1) \times U(1)$-invariant, asymptotically locally hyperbolic, metric on the total space of orbifold line bundles over a spindle (bolt). The conformal boundary is generically a squashed, branched, lens space and the graviphoton gauge field can have either twist or anti-twist through the spindle bolt. Correspondingly, the boundary geometry inherits two types of rigid Killing spinors, that we refer to as twist and anti-twist for the three-dimensional Seifert orbifolds, as well as some specific flat connections for the background gauge field, determined by the data of the spindle bolt. For all our solutions we compute the holographically renormalized on-shell action and compare it to the expression obtained via equivariant localization, uncovering a markedly distinct behaviour in the cases of twist and anti-twist. Our results provide precise predictions for the large N limit of the corresponding localized partition functions of three-dimensional $\mathcal{N} = 2$ superconformal field theories placed on Seifert orbifolds.

Contents

1   Introduction

In the past few years supergravity solutions featuring spindles have significantly enriched the landscape of holographic dualities, indicating that quantum theories of gravity should

be addressed in a context broader than that of smooth manifolds. In almost all cases in the literature so far, the spindle Σ appears as a factor of an AdS$\times\Sigma$ space-time, suggesting that it should be interpreted as the near-horizon geometry of a brane wrapped on a spindle [1, 2, 3]. One of the main novelties of the spindle is that it leads to new possibilities for preserving supersymmetry, with direct implications for quantum field theories with rigid supersymmetry. In particular, a supersymmetric field theory coupled to a background R-symmetry gauge field can be compactified on a spindle in two ways [4], referred to as twist and anti-twist, leading to different types of supersymmetric field theories in two lower dimensions in the IR. In the context of holography the expectation that the low energy limit of the wrapped brane theory is a SCFT is fully vindicated in the case of the supersymmetric accelerating black hole [5] (and its complex non-extremal deformation [6]). In that context one can take the spindle appearing in the near-horizon region all the way to infinity, where (in Euclidean) the geometry at the conformal boundary is $\Sigma \times S^1$. In turn, this led to the study of three-dimensional supersymmetric field theories on such backgrounds [7] and the computation of the corresponding localized partition function, namely the spindle index [8]. The agreement of the large N limit of the spindle index with the entropy of the accelerating black hole was achieved in [9].

In this paper we will construct various classes of explicit supersymmetric solutions of four-dimensional minimal gauged supergravity, where generically the spindle appears in the bulk of an asymptotically locally hyperbolic space. More precisely, our solutions comprise global metrics on the total space of certain orbifold line bundles over the spindle, together with a graviphoton field and Killing spinors. The spindle corresponds to the zero section of these line bundles, meaning that it appears as a *bolt* [10] in the solutions. Generically, the three-dimensional conformal boundaries are squashed lens spaces, possibly with orbifold singularities, which we refer to as squashed, branched, lens spaces [7]. Interestingly, we will show that the topological data of the bulk are encoded in the boundary geometry, through some subtle flat connections for the boundary gauge field. In particular, we will see that the twist/anti-twist for the graviphoton through the spindle bolt is in one-to-one correspondence with two types of rigid supersymmetry in the three-dimensional Seifert orbifolds discussed in [7], that we shall therefore again refer to as *twist and anti-twist*. This information is encoded in some flat connections for the three dimensional gauge field which, surprisingly, turn out to encapsulate the complete data of the spindle in the bulk. The results of this paper were first announced in [11] and presented in [12].

Our analysis includes in special limits several previously known supersymmetric solutions of Euclidean minimal gauged supergravity [13, 14, 15, 16, 6] and sheds new light on them. For example, we will clarify that the 1/4-BPS and 1/2-BPS families of spherical bolt solutions in [15] arise as limits of spindle bolt solutions in the Carter-Plebański family [17, 18], with twist and anti-twist for the graviphoton field, respectively. In turn, the twist and anti-twist for the supersymmetry at the boundary in these two cases correspond to two different choices of spin structures, with the Killing spinors being periodic or anti-periodic along the circle of the Seifert fibration [15, 19]. It is perhaps surprising that the Carter-Plebański family leads to spindles, since its Lorentzian version contains only the Kerr-Newman-NUT-AdS, Reissner-Nordström-NUT-AdS and the AdS-C-metric [20, 21][1],

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[1]Note that in [20] it is incorrectly stated that the Carter-Plebański solution is the most general Plebański-Demianski solution.

whereas the conical singularities of the spindle are associated to a non-zero acceleration, which is present only in the more general Plebański-Demianski family [22][2]. Furthermore, whilst in the accelerating black hole [4] only the anti-twist is realized, the solutions that we find here generically allow for both twist and anti-twist.

With rare exceptions, given a set of boundary conditions and/or topological data, the existence of suitably regular supergravity solutions remains a difficult open problem. For example, building on the results of [27, 28], it has been proved that for toric Sasakian manifolds, a necessary and sufficient condition for the existence of Sasaki-Einstein metrics is that the Sasakian volume is extremized with respect to the allowed Reeb vector fields [29]. More generally, whenever one can formulate an *extremization problem* for a given class of solutions, one hopes that, under suitable conditions, the solution of the extremization problem will be also sufficient for the existence of the supergravity solutions. This paradigm has been reinforced by the recent advances in the techniques for computing on-shell actions using localization methods. In particular, in the context of four dimensional supergravity this problem was initially tackled in [30] (for supersymmetric self-dual solutions of the minimal theory) and in [31] (for generic supersymmetric solutions of the minimal theory) and then fully solved in [32] using equivariant localization[3]. Interestingly, our findings strongly suggest that for the spindle bolt solutions the cases of twist and anti-twist behave differently: in the first case, solutions seem to exist only upon extremizing the localized action with respect to the choices of supersymmetric Killing vector, while in the second case it appears that for any given supersymmetric Killing vector supersymmetric solutions should exist. We will comment more on this aspect of our new solutions in the final section. While this paper was being completed, the paper [34] appeared where general properties of Euclidean supersymmetric solutions of four-dimensional matter-coupled gauged supergravity with $U(1)^2$ isometry are discussed.

The outline of the rest of the paper is as follows. In section 2 we recall the local form of the (Euclidean) Plebański-Demianski family of solutions, review their supersymmetry conditions and summarise some notable old solutions. In section 3 we specialize to the Carter-Plebański class, that we refer to as "non-accelerating". We begin presenting the explicit local Killing spinors for this class of solutions and then we move to the global analysis of the boundary and the bulk in section 3.2. In particular, we show that the with a specific choice of gauge for the graviphoton, the Killing spinor is regular both near to the bolt and at the boundary. This analysis will reveal that all the topological data of the bulk are already contained in the boundary, as summarised in section 3.3. The methods developed in [35, 36] allow us to uncover the underlying toric orbifolds, which may be characterised in terms of two-dimensional labelled polyhedral cones. We show that there exist solutions with both twist and anti-twist in section 3.5 and discuss how these two sub-classes reduce to the old "spherical" bolt solutions of [15], in appropriate limits. In section 4 we discuss the holographically renormalized on-shell action of the non-accelerating class of solutions, comparing the results obtained evaluating the integrals using the explicit solutions with the localization results, pointing out the different role of extremization for the cases of twist and anti-twist. In section 5 we return to the accelerating solutions. We begin presenting the complete holographically renormalized on-shell action for the general class of non-supersymmetric Plebański-Demianski solutions. In order to make analytic

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[2]See also [23, 24, 25, 26].

[3]Extensions to matter-coupled four-dimensional gauged supergravities have been discussed in [33].

progress in the global analysis we make a judicious choice of the parameters, such that the supersymmetry conditions take a form similar to the non-accelerating case. Despite this, we will be able to construct solutions of generic type, where again both twist and anti-twist are realized. In section 6 we discuss our findings and make comments on the implications that these have for holography. Four appendices complete this paper. In appendix A we recall some relevant features of the old 1/4-BPS and 1/2-BPS spherical bolts of [15], which we will use to make comparisons with our solutions. In appendix B we construct a Ricci-flat version of the solutions described in the main body, with nevertheless a spindle (bolt) in the bulk. In appendix C we present some values of the parameters for which the solutions are correctly defined. Finally, appendix D is devoted to global aspects of the uplift of the solutions to eleven dimensions.

2 The Plebański-Demianski solutions

In this paper will consider asymptotically locally (Euclidean) AdS solutions of the Maxwell-Einstein-Λ action. Alternatively, this can be seen as the bosonic sector of the minimal $d = 4$, $\mathcal{N} = 2$ gauged supergravity [37], whose Euclidean action reads

$$ S_E = -\frac{1}{16\pi G_4} \int \mathrm{d}^4 x \sqrt{g} \left(R - F_{\mu\nu}F^{\mu\nu} + \frac{6}{\ell^2} \right). \tag{2.1} $$

Here the cosmological constant Λ is related to the length parameter ℓ via $\Lambda = -3/\ell^2$, R is the Ricci scalar of the four dimensional metric g and $F = \mathrm{d}A$ is the field strength of the abelian graviphoton A. The equations of motion stemming from (2.1) are

$$ R_{\mu\nu} + \frac{3}{\ell^2} g_{\mu\nu} = 2 \left(F_{\mu\rho}F_\nu{}^\rho - \frac{F^2}{4} g_{\mu\nu} \right), \quad \mathrm{d} \star F = 0, \tag{2.2} $$

which may also be rewritten in the equivalent form

$$ \mathrm{d}F = 0, \qquad \mathrm{d} \star F = 0, \qquad [\mathrm{Ric} + 2F \circ F]_0 = 0, \qquad R = -\frac{12}{\ell^2}, \tag{2.3} $$

where Ric is the Ricci tensor, $(F \circ F)_{\mu\nu} \equiv -F_{\mu\rho}F_\nu{}^\rho$ and $[\ ]_0$ denotes the trace-free part with respect to the metric g. Notice that, in particular, all solutions must have constant scalar curvature. The form of the equations (2.3) appears usually in the mathematical literature[4], where typically it is assumed that the underlying manifold (or orbifold) is compact, and then the constant scalar curvature condition can be shown to be implied by the first three equations in (2.3) [38]. In the present paper we shall assume that the space is non-compact and in particular asymptotically locally hyperbolic, but in principle one might also consider the compact case and we shall make some comments about this possibility in the concluding section.

We will be particularly interested in the sub-class of solutions to (2.3) that are *supersymmetric*, namely they admit at least one non-identically zero Dirac spinor ε satisfying the Killing spinor equation

$$ \hat{D}_\mu \varepsilon \equiv \left[D_\mu - \frac{\mathrm{i}}{\ell}A_\mu + \frac{1}{2\ell}\Gamma_\mu + \frac{\mathrm{i}}{4}F_{\nu\rho}\Gamma^{\nu\rho}\Gamma_\mu \right]\varepsilon = 0, \tag{2.4} $$

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[4]The factor of 2 difference with [38] in the third equation is purely conventional and can be reabsorbed by rescaling F.

where $D_\mu \varepsilon = \partial_\mu \varepsilon + 1/4 \omega_{\mu ab} \Gamma^{ab} \varepsilon$ is the standard covariant derivative. Here Γ_a, $a = 1, \ldots, 4$, generate the Clifford algebra Cliff$(4,0)$, so that their curved counterparts satisfy $\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu} \mathrm{Id}_{4\times 4}$. The supersymmetry condition (2.4) can be re-cast equivalently as coupled PDEs for a set of bosonic tensors constructed as bilinears in the Killing spinors, in terms of which the metric and gauge field take a canonical form. This was done in Lorentzian signature in [39] and in Euclidean signature, that is relevant for the present paper, in [31]. However, here we will take advantage of the fact that the supersymmetry conditions for the class of local Plebański–Demianski solutions can be neatly written in terms of some constraints among the parameters of the solutions [21], as we shall review below.

Using the formulas of [40, 41] one can uplift locally any supersymmetric solution of the four-dimensional theory to a supersymmetric solution of M-theory. Global aspects of the uplift to eleven dimensions are discussed in appendix D. As it happens in the context of black holes [42, 43, 6], in our solutions the gauge field and the metric can take complex values. Specifically, we will see that for all the solutions with a twist for the graviphoton the metric will take real values, while for the solutions with anti-twist generically the metric can take complex values. In principle one could investigate systematically for which values of the parameters the metric become real, as was done in [15]. However, from the point of view of holography such requirement does not appear to be motivated and therefore we will not dwell on this presently. From now on we set $\ell = 1$ without loss of generality.

2.1 Non-accelerating vs accelerating solutions

Our starting point is the local form of the solutions found by Plebański-Demianski (PD) [22] almost fifty years ago. This generalized an earlier solution found independently by Carter [17] and Plebański [18] (that we will refer to as Carter-Plebański (CP) solution) by the addition of a parameter that, in Lorentzian signature, may be interpreted as acceleration. Although we will be interested in the class of Euclidean (possibly complex) solutions, we will denote such parameter as $\mathbf{A}$ and continue to refer to this throughout the paper as "acceleration" parameter. We will write the solutions directly in Euclidean signature, using coordinates essentially as in [21], but we introduce the additional free parameter[5] $\mathbf{A}$ from the outset. We will keep track of $\mathbf{A}$, distinguishing between the CP solutions ($\mathbf{A} = 0$) that we will refer to as "non-accelerating", and the PD solutions ($\mathbf{A} \neq 0$) that, by contrast, we will refer to as "accelerating". We start presenting the more general PD solution, but soon we will set $\mathbf{A} = 0$, where we will be able to perform all the relevant computations analytically in full generality, constructing also the Killing spinor ε solving (2.4). Having studied in detail the CP case, we will come back to the PD solutions later on, focussing on a sub-case, where we can make analytic progress.

We start with the general solution solving the equations of motion while we will impose

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[5]The form given in [21] is recovered setting $\mathbf{A} = 1$. The parameter k appearing there should be identified as $k = P^2 - \alpha$. Several authors have studied slightly different parametrization of the PD solutions, see *e.g.* [23, 24, 6, 25, 26]

the supersymmetry conditions later. The metric reads

$$\mathrm{d}s_4^2 = \frac{1}{(1-\mathbf{A}pq)^2}\Bigg\{(q^2-\omega^2p^2)\bigg[\frac{\mathrm{d}q^2}{\mathbf{Q}(q)}+\frac{\mathrm{d}p^2}{-\mathbf{P}(p)}\bigg]$$
$$+\frac{1}{q^2-\omega^2p^2}\bigg[\mathbf{Q}(q)(\mathrm{d}\tau+\omega p^2\mathrm{d}\sigma)^2-\mathbf{P}(p)(\omega\mathrm{d}\tau+q^2\mathrm{d}\sigma)^2\bigg]\Bigg\}\,, \tag{2.5}$$

while the graviphoton is

$$A = \frac{\omega pP - qQ}{q^2-\omega^2p^2}\mathrm{d}\tau + pq\frac{qP-\omega pQ}{q^2-\omega^2p^2}\mathrm{d}\sigma\,, \tag{2.6}$$

where ω is a scaling parameter usually called "twist" [25], which can be set to one. The metric functions are given by

$$\mathbf{P}(p) = \mathcal{P}(p) - 2\mathbf{A}Mp^3 + \mathbf{A}^2\big[-Q^2+\alpha\omega^2-P^2(\omega^2-1)\big]p^4\,,$$
$$\mathbf{Q}(q) = \mathcal{Q}(q) - 2\frac{\mathbf{A}N}{\omega}q^3 + \mathbf{A}^2(-P^2+\alpha)q^4\,, \tag{2.7}$$

where $\mathcal{P}(p)$ and $\mathcal{Q}(q)$ are the "non-accelerating" functions (*cf.* the metric (3.1))

$$\mathcal{P}(p) = \omega^2p^4 + Ep^2 - 2\frac{N}{\omega}p - P^2 + \alpha\,,$$
$$\mathcal{Q}(q) = q^4 + Eq^2 - 2Mq - Q^2 + \alpha\omega^2 - P^2(\omega^2-1)\,. \tag{2.8}$$

Explicitly, we have

$$\mathbf{P}(p) = -P^2 + \alpha - 2\frac{N}{\omega}p + Ep^2 - 2\mathbf{A}Mp^3 + \Big[\mathbf{A}^2\big[\omega^2(-P^2+\alpha)-Q^2+P^2\big]+\omega^2\Big]p^4\,,$$
$$\mathbf{Q}(q) = \big[\omega^2(-P^2+\alpha)-Q^2+P^2\big] - 2Mq + Eq^2 - 2\frac{\mathbf{A}N}{\omega}q^3 + \big[\mathbf{A}^2(-P^2+\alpha)+1\big]q^4\,. \tag{2.9}$$

The CP solution, as considered for example in [20, 16], is obtained by setting $\mathbf{A} = 0$ and $\omega = 1$. With these two conditions, the solution becomes highly symmetric in (p,q), thus allowing for a general analysis. The parameters $(\mathbf{A}, N, M, Q, P, \omega)$ can be loosely identified as the Euclidean counterpart of acceleration, NUT parameter, mass, electric and magnetic charge and rotation, respectively. However, the correct interpretation is more subtle and in the Lorentzian set-up we refer for example to [25, 44] for a more complete classification, where some interesting limits are also discussed. The interpretation of the various parameters in Euclidean signature is again complicated and case-dependent and will be a matter of discussion throughout the paper.

We recall that the solution has a self-dual Weyl tensor if and only if $P = Q$ and $N = M$, and possesses a scaling symmetry

$$(q,p) \to \lambda\,(q,p)\,, \quad (Q,P,E) \to \lambda^2\,(Q,P,E)\,, \quad (N,M) \to \lambda^3\,(N,M)\,,$$
$$\alpha \to \lambda^4\,\alpha\,, \quad \tau \to \lambda^{-1}\,\tau\,, \quad \mathbf{A} \to \lambda^{-2}\,\mathbf{A}\,, \quad \sigma \to \lambda^{-3}\,\sigma\,. \tag{2.10}$$

with $\lambda \in \mathbb{R} \setminus \{0\}$.

In this paper we will be interested in supersymmetric solutions with a holographic interpretation, namely we will require that they are asymptotically locally hyperbolic,

while in the interior, we will allow local conical singularities. As we shall see, this will lead to solutions with the topology of a complex line bundle over a spindle and with a lens space $L(t, 1)$ as conformal boundary[6]. For these reasons, we will need a "radial" coordinate (that we take to be q) and an angular one (that will therefore be p). For this set-up we can take the ranges of definition

$$\mathbf{q}_+ \leq q \leq \frac{1}{\mathbf{A}p}, \quad \mathbf{p}_- \leq p \leq \mathbf{p}_+, \tag{2.11}$$

where $q = 1/(\mathbf{A}p)$ is the location of the conformal boundary, $\mathbf{q}_+$ is the largest real root of $\mathbf{Q}(q)$ and similarly $\mathbf{p}_\pm$ are the largest real roots of $\mathbf{P}(p)$. Moreover, without loss of generality we may take the following conditions in order to ensure the correct signature of the metric

$$q^2 - \omega^2 p^2 > 0, \quad \mathbf{P}(p) < 0, \quad \mathbf{Q}(q) > 0. \tag{2.12}$$

We emphasise that while later we shall allow solutions to take complex values, the procedure we adopt is analogous to that used to study the thermodynamics of supersymmetric black holes, put forward in [42]. Namely, we assume that in the general non-supersymmetric solution the metric and gauge field take real values, so that we can require that for example the metric has correct Euclidean signature. After imposing supersymmetry we will find that in general it may not be possible to keep the metric real. However, we will require that in this more general set-up the Killing spinors have correct global properties and the bosonic fields are regular in the neighbourhood of fixed points[7]. For simplicity, and for future comparison with [16], we (partially) use the scaling symmetry (2.10) to require $\mathbf{p}_- + \mathbf{p}_+ \geq 0$. We thus introduce the ordering

$$\mathbf{p}_- \leq p \leq \mathbf{p}_+ \leq \mathbf{q}_+ \leq q, \quad \mathbf{q}_+^2 - \mathbf{p}_+^2 \geq 0, \quad \mathbf{p}_+^2 - \mathbf{p}_-^2 \geq 0. \tag{2.13}$$

From the signature conditions, we conclude that $\mathbf{Q}'(\mathbf{q}_+) > 0, \mathbf{P}'(\mathbf{p}_+) > 0$ and $\mathbf{P}'(\mathbf{p}_-) < 0$.

We now switch gears and study supersymmetry. We do not attempt to solve (2.4) in the general case $\mathbf{A} \neq 0$, even if we will construct explicitly the Killing spinor ε for $\mathbf{A} = 0$ in section 3.1, showing that the solution is 1/4-BPS. The integrability condition $M_{\mu\nu}\varepsilon \equiv [\hat{D}_\mu, \hat{D}_\nu]\varepsilon = 0$ of (2.4) reduces, after some work, to

$$M_{\mu\nu}\varepsilon \equiv \left[\frac{1}{4}R_{\mu\nu}{}^{ab}\Gamma_{ab} + \frac{1}{2}\Gamma_{\mu\nu} - \mathrm{i}F_{\mu\nu} + \frac{\mathrm{i}}{2}\Gamma_{ab}\nabla_{[\mu}F^{ab}\Gamma_{\nu]} + \frac{\mathrm{i}}{4}F_{ab}\Gamma_{[\mu}\Gamma^{ab}\Gamma_{\nu]} \right.$$
$$\left. - \frac{1}{16}F_{ab}F_{cd}[\Gamma^{ab}\Gamma_\mu, \Gamma^{cd}\Gamma_\nu] + \frac{\mathrm{i}}{4}F_{ab}\Gamma^{ab}\Gamma_{\mu\nu} \right]\varepsilon = 0. \tag{2.14}$$

We choose the coordinates to be ordered as (p, τ, σ, q), the vierbein to be

$$e^1 = \frac{1}{(1 - \mathbf{A}pq)}\sqrt{\frac{q^2 - \omega^2 p^2}{-\mathbf{P}(p)}}\mathrm{d}p, \qquad e^2 = \frac{1}{(1 - \mathbf{A}pq)}\sqrt{\frac{-\mathbf{P}(p)}{q^2 - \omega^2 p^2}}(\omega\mathrm{d}\tau + q^2\mathrm{d}\sigma),$$

$$e^3 = \frac{1}{(1 - \mathbf{A}pq)}\sqrt{\frac{\mathbf{Q}(q)}{q^2 - \omega^2 p^2}}(\mathrm{d}\tau + \omega p^2\mathrm{d}\sigma), \quad e^4 = \frac{1}{(1 - \mathbf{A}pq)}\sqrt{\frac{q^2 - \omega^2 p^2}{\mathbf{Q}(q)}}\mathrm{d}q,$$

$$\tag{2.15}$$

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[6]The boundary may be also an orbifold, *i.e.* a branched lens space [7] and in some cases we can also have $t = 0$, corresponding to $\Sigma \times S^1$ topology.

[7]This is the Euclidean counterpart of requiring regularity near the tip of the cigar in the complexified black hole geometries [42].

and adopt the following explicit representation of the Clifford algebra

$$\Gamma_a = \sigma_1 \otimes \sigma_a \,, a = 1, \ldots, 3\,, \quad \Gamma_4 = -\sigma_2 \otimes \mathrm{Id}_2\,, \quad \Gamma_\star = \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4\,, \tag{2.16}$$

with σ_a the standard Pauli matrices. It is then straightforward to show that $\det(M_{1,4}) = 0$ is equivalent to

$$\begin{aligned}
2(q^2 + \omega^2 p^2)(N\Sigma_1 - \omega M\Sigma_2) + 4\omega(P^2 - Q^2)(\mathbf{A}qp + 1)(p\Sigma_1 + q\Sigma_2) \\
- 2pq\omega(2M\Sigma_1 - 2\omega N\Sigma_2 + \mathbf{A}\omega\Sigma_3) - \omega^2(\mathbf{A}^2 p^2 q^2 + 1)\Sigma_3 = 0\,,
\end{aligned} \tag{2.17}$$

where

$$\begin{aligned}
-\Sigma_1 &= 2Q\Omega\omega^2 + \mathbf{A}M\Pi\omega + 2\mathbf{A}^2 N(P^2 - Q^2)\Phi\,, \\
-\Sigma_2 &= 2P\Omega\omega + \mathbf{A}N\Pi + 2\mathbf{A}^2 M(P^2 - Q^2)(-P^2 + \alpha)\omega\,, \\
\Sigma_3 &= \Pi^2 - 4(P^2 - Q^2)^2\big[(-P^2 + \alpha)\omega^2 + P^2 + \mathbf{A}^2(-P^2 + \alpha)\Phi\big]\,,
\end{aligned} \tag{2.18}$$

and we defined

$$\Omega = MP - NQ\,, \quad \Pi = M^2 - N^2 - E(P^2 - Q^2)\,, \quad \Phi = \omega^2(-P^2 + \alpha) - Q^2 + P^2\,. \tag{2.19}$$

Moreover it can be verified that $\Sigma_3 = \sum_{i,j=1}^2 a_{ij}\Sigma_i\Sigma_j$ for some coefficients a_{ij}, thus we only require $\Sigma_{1,2} \equiv 0$. This system is entirely solved by the self-duality condition

$$M = N\,, \quad P = Q\,, \tag{2.20}$$

or, in general, by fixing

$$\begin{aligned}
\alpha &= \frac{-\omega^2(MP - NQ)^2 + \mathbf{A}^2(P^2 - Q^2)\big[N^2(P^2 - Q^2) + (M^2 - N^2)P^2\omega^2\big]}{\mathbf{A}^2\omega^2(P^2 - Q^2)(M^2 - N^2)}\,, \\
E &= \frac{-2(MP - NQ)(NP - MQ)\omega^2 + \mathbf{A}\omega(M^2 - N^2)^2 + 2\mathbf{A}^2 MN(P^2 - Q^2)^2}{\mathbf{A}\omega(P^2 - Q^2)(M^2 - N^2)}\,.
\end{aligned} \tag{2.21}$$

If useful, we can also adopt the following (more complicated) expressions that are perfectly smooth in the limit $\mathbf{A} \to 0$[8]

$$\begin{aligned}
M &= \frac{\omega NPQ + \mathbf{A}N(P^2 - Q^2)\sqrt{P^2 - (P^2 - \alpha)\big[\omega^2 - \mathbf{A}^2\big[-P^2 + Q^2 + \omega^2(P^2 - \alpha)\big]\big]}}{\omega\big[P^2 - \mathbf{A}^2(P^2 - Q^2)(P^2 - \alpha)\big]}\,, \\
E &= \frac{M^2 - N^2}{P^2 - Q^2} - \frac{2\mathbf{A}\big[NP(-P^2 + Q^2) + \omega^2(NP - MQ)(P^2 - \alpha)\big]}{\omega(MP - NQ)}\,.
\end{aligned} \tag{2.22}$$

These are compatible with the results in [21] (with $\mathbf{A} = 1$) and we will refer to (2.21) or (2.22) as the supersymmetry conditions. Even if we will not construct explicitly the spinor for the accelerating case, we believe that these conditions are (necessary and) sufficient for supersymmetry, analogously the CP case. In particular when $\mathbf{A} = 0$, $\Sigma_1 \propto \Sigma_2$ and then Σ_3 becomes independent, and as such it is a genuine constraint. Then the supersymmetry conditions for the non-accelerating case become [20, 13]

$$\mathbf{A} = 0: \quad M = \frac{NQ}{P}\,, \quad E = -\frac{N^2}{P^2} + 2\sqrt{\alpha\omega^2 + P^2(1 - \omega^2)}\,, \tag{2.23}$$

and they further simplify for $\omega = 1$.

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[8]Notice that in general the parameters in Σ_i can be complex, and we have selected a branch for the square root.

2.2 Notable old solutions

In this section we summarize a number of known asymptotically locally hyperbolic solutions sharing some features with the solutions that we will discuss presently. In fact, as we will show later, all of these old solutions can be recovered as special cases or limits from our more general analysis. See figure 1.
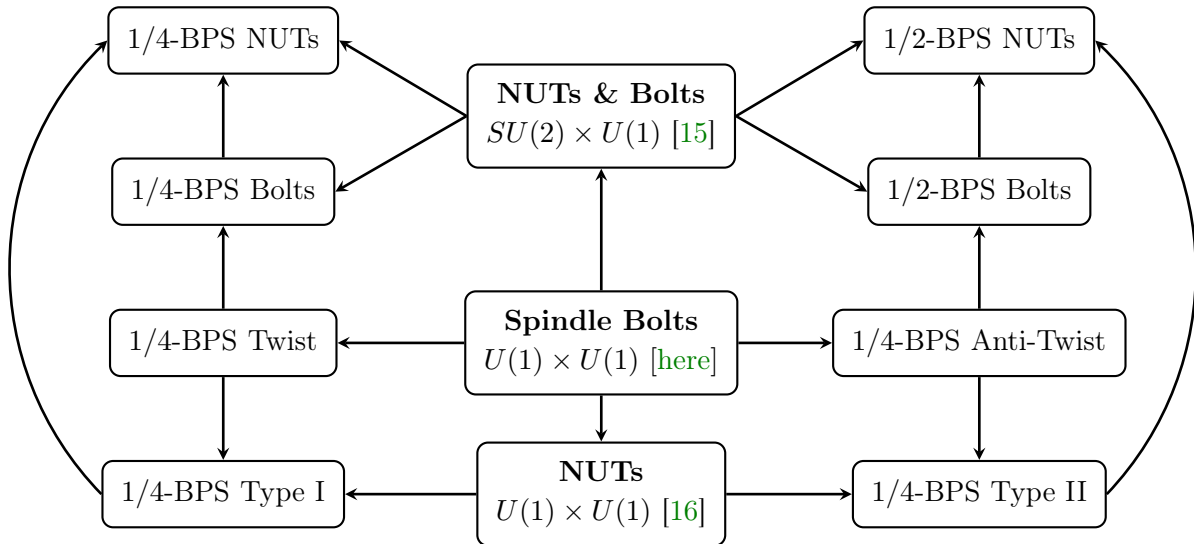

Figure 1: Summary of supersymmetric euclidean AdS_4 solutions to $d = 4$, $\mathcal{N} = 2$ minimal gauged supergravity containing nuts and bolts. In the central column the symmetry and the bulk content are sketched, while the lateral columns show when a certain supersymmetric solution is contained in another one by some limiting procedure which will be explained in detail in section 3.6.

Let us start by considering the $U(1) \times U(1)$-invariant two-parameter Euclidean solutions of [16]. These are 1/4-BPS solutions in the CP family ($\mathbf{A} = 0$) where the conformal boundary is a squashed S^3 and have the topology of the ball, namely they contain a single nut in the bulk. They belong to the class of self-dual solutions [30], but this aspect will not be particularly relevant for our discussion. According to the values of the parameters, there is a splitting into two distinct families, referred to as Type I and Type II, which are neatly distinguished by the form of their on-shell action, which read[9]

$$
\begin{aligned}
\text{Type I}: \quad & S = \frac{\pi}{2G_4}\,, \\
\text{Type II}: \quad & S = \frac{\pi}{8G_4}\left[\beta + \frac{1}{\beta}\right]^2,
\end{aligned}
\tag{2.24}
$$

where β is a combination of the two parameters of the solutions and it can also take complex values. It is interesting to notice that in both families there are two free parameters, which however do not appear in the Type I action, while the Type II action depends on one combination (β). Accordingly, the supersymmetric Killing vectors, up to an irrelevant

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[9]The parameter β is related to the parameter Q that appears in (2.8) as $2Q = (\beta^2 - 1)/(\beta^2 + 1)$.

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normalization constant, read[10]

$$\begin{aligned} \text{Type I}: \quad & \vec{\epsilon} \propto (1, -1)\,, \\ \text{Type II}: \quad & \vec{\epsilon} \propto (\beta^2, 1)\,. \end{aligned} \tag{2.25}$$

Notice that these are consistent with the general results of [30, 45], which showed that for supersymmetric self-dual solutions with the topology of the ball the only values of ϵ_2/ϵ_1 that correspond to non-singular solutions are either $\epsilon_2/\epsilon_1 = -1$ or $\epsilon_2/\epsilon_1 > 0$.

A similar behaviour is present in the one-parameter $SU(2) \times U(1)$-invariant solutions of [15], where the boundary is a squashed lens space $L(\hat{t}, 1) = S^3/\mathbb{Z}_{\hat{t}}$, while the bulk may be either a nut or a bolt, which is always a round S^2. For each type of bulk topology, there are again two distinct families, referred to as 1/4-BPS and 1/2-BPS. The bolt solutions are characterised by their on-shell actions as well as the flux of the graviphoton field through the S^2 bolt, that read

$$\begin{aligned} \tfrac{1}{4}\text{-BPS}: \quad & \frac{1}{2\pi}\int_{S^2} F = -1 + \kappa\frac{\hat{t}}{2}\,, \qquad S_{\text{bolt}} = \frac{\pi}{2G_4}\left[1 - \kappa\frac{\hat{t}}{4}\right]\,, \quad \hat{t} \geq 2\,, \\ \tfrac{1}{2}\text{-BPS}: \quad & \frac{1}{2\pi}\int_{S^2} F = -\kappa\frac{\hat{t}}{2}\,, \qquad\quad S_{\text{bolt}} = \frac{\pi}{2G_4}\left[1 - \kappa\frac{2\sqrt{4s^2 - 1}}{s\hat{t}}\left[s^2 - \frac{\hat{t}^2}{16}\right]\right]\,, \end{aligned} \tag{2.26}$$

where $\hat{t} \in \mathbb{N}$, $\kappa = \pm 1$ is a sign further distinguishing two branches[11] of the solutions and s is the squashing of the lens space boundary, which in [15] was assumed to take real values. Here $\hat{t} \geq 2$ for $\kappa = -1$ and $\hat{t} \geq 3$ for $\kappa = +1$ [15]. Again we see that the on-shell action of the 1/4-BPS bolt is completely fixed by the topology (*i.e.* the value of $\hat{t}$), whilst the free parameter s appears in the 1/2-BPS case. More details about these solutions are given in appendix A. We record here also the supersymmetric Killing vector $\vec{\epsilon}$ for these solutions, which is given (up to an irrelevant normalization constant) by [31]

$$\begin{aligned} \tfrac{1}{4}\text{-BPS}: \quad & \vec{\epsilon} \propto (1, 0)\,, \\ \tfrac{1}{2}\text{-BPS}: \quad & \vec{\epsilon} \propto \left(\frac{\hat{t}}{4s} - 2s - \sqrt{4s^2 - 1}, 2s + \sqrt{4s^2 - 1}\right)\,. \end{aligned} \tag{2.27}$$

For the nut solutions the on-shell actions read

$$\begin{aligned} \tfrac{1}{4}\text{-BPS}: \quad & S_{\text{nut}} = \frac{\pi}{2G_4}\,, \\ \tfrac{1}{2}\text{-BPS}: \quad & S_{\text{nut}} = \frac{\pi}{2G_4}4s^2\,, \end{aligned} \tag{2.28}$$

and again the free parameter s appears only in 1/2-BPS case. Now the supersymmetric Killing vectors read [30]

$$\begin{aligned} \tfrac{1}{4}\text{-BPS}: \quad & \vec{\epsilon} \propto (1, -1)\,, \\ \tfrac{1}{2}\text{-BPS}: \quad & \vec{\epsilon} \propto \left(\frac{1}{4s}, -\frac{1}{4s} + 2s + \sqrt{4s^2 - 1}\right)\,. \end{aligned} \tag{2.29}$$

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[10]Here the vectors $\vec{\epsilon}$ are written in the basis $(-\partial_{\varphi_1}, \partial_{\varphi_2})$ defined in [16].
[11]These were denoted Bolt$_\pm$ in [15].

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For completeness, let us also mention that the 1/4-BPS spherical Bolt solutions have been generalized in [46] to allow bolts with the topology of a Riemann surface with genus g. Correspondingly, the metrics have only $U(1)$ isometry and do not fall in the PD family. The reason why there is no Riemann surface analogue of the 1/2-BPS spherical Bolt solutions will become clear in due course.

Finally we recall the accelerating black hole solution, which has boundary topology $\Sigma \times S^1$ [5]. In this case the underlying metric is PD ($\mathbf{A} \neq 0$) and it is the acceleration itself responsible for the conical singularities of the spindle. Allowing the parameters of the solution to take complex values one obtains a "non-extremal" supersymmetric solution [6] that in the bulk has a bolt with spindle topology[12]. We introduce some standard notation

$$\mu = \frac{m_- + m_+}{m_- - m_+}\,, \quad G_4 Q_m = \frac{m_- - m_+}{4 m_- m_+}\,, \quad \chi_\Sigma = \frac{m_- + m_+}{m_- m_+}\,, \qquad (2.30)$$

in terms of which

$$\frac{1}{2\pi} \int_\Sigma F = 2 G_4 Q_m\,, \quad S_\pm = \pm \frac{1}{2 i G_4} \left[\frac{\varphi^2}{z} + (G_4 Q_m)^2\, z \right]\,, \quad \varphi - \frac{\chi_\Sigma}{4} z = \pm i\pi\,, \qquad (2.31)$$

where z is a complicated function of the two parameters of the solution. This shows that the solution preserves supersymmetry via anti-twist through the spindle, while the on-shell action depends on the single complex parameter z. The supersymmetric Killing vector for this solution is [6]

$$\vec{\epsilon} \propto (2\pi, i z)\,. \qquad (2.32)$$

Notice that the supersymmetric accelerating black hole reduces to the Kerr-Newman-AdS one [21] when the acceleration is turned off. This results in a spherical horizon with $m_- = m_+ = 1$, for which the on-shell action has been computed in [6] and coincides with (2.31) for $Q_m = 0$ and $\chi_{S^2} = 2$.

In all the above cases, the on-shell action of the solutions computed through holographic renormalization coincides with the general expression obtained with localization [31, 32], see (4.19). This has contributions from the fixed points of the supersymmetric Killing vector, which is the single nut for (2.24) and (2.28) and the two poles of the sphere, or the spindle, for (2.26) and (2.31). We will review the results of [31, 32] in Section 4.2.

It is striking that there is an asymmetry in the expressions for the various on-shell actions above. In particular, the actions for the Type I family and for the 1/4-BPS nuts and bolts does not depend on any continuous parameter and can depend only on topological data of the underlying space (namely $\hat{t}$). Instead, for the Type II family, the 1/2-BPS nuts and bolts and the black hole, the action is always written in terms of a single parameter. Throughout the paper we will recover these features from our more general solutions containing a spindle in the bulk and a (possibly branched) lens space in the boundary. In particular, we will show that this asymmetry in the behaviour of the on-shell actions is in 1-1 correspondence with the type of supersymmetry on the spindle: if the twist is realized, the action is fixed in terms of topological data of the underlying space, otherwise, if the anti-twist is realized, the action depends on a single free parameter. In turn, from the point of view of the boundary, this dichotomy between twist and anti-twist can be ascribed to the existence of certain "flat connections" for the

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[12]In the extremal limit the solution in the near horizon becomes $AdS_2 \times \Sigma$.

boundary gauge field, generalising the observations of [15][13]. We conclude this section noticing that while the complex deformation of the accelerating black hole is the only known solution that displays a spindle bolt, we will show that spindle bolts can also arise in the CP family with $\mathbf{A} = 0$.

3 Non-accelerating solutions

In this section we study the non-accelerating solution, *i.e.* the Carter-Plebański case, which is obtained from the general solution (2.5)-(2.8) by setting $\mathbf{A} = 0$ and scaling away the twisting parameter, or $\omega = 1$. We will show that it is possible to obtain, within the CP class of solutions, a bulk topology $\mathbb{C}/\mathbb{Z}_v \hookrightarrow \mathcal{O}(-t) \to \Sigma_{[m_-, m_+]}$. For the ease of the reader we repeat here the metric

$$\mathrm{d}s_4^2 = (q^2 - p^2)\left[\frac{\mathrm{d}q^2}{\mathcal{Q}(q)} + \frac{\mathrm{d}p^2}{-\mathcal{P}(p)}\right] + \frac{1}{(q^2 - p^2)}\left[\mathcal{Q}(q)(\mathrm{d}\tau + p^2\mathrm{d}\sigma)^2 - \mathcal{P}(p)(\mathrm{d}\tau + q^2\mathrm{d}\sigma)^2\right], \quad (3.1)$$

and the graviphoton

$$A = \frac{pP - qQ}{q^2 - p^2}\mathrm{d}\tau + pq\frac{qP - pQ}{q^2 - p^2}\mathrm{d}\sigma. \quad (3.2)$$

The metric functions coming from (2.8) read

$$\begin{aligned}
\mathcal{P}(p) &= p^4 + Ep^2 - 2Np - P^2 + \alpha, \\
\mathcal{Q}(q) &= q^4 + Eq^2 - 2Mq - Q^2 + \alpha,
\end{aligned} \quad (3.3)$$

where the high symmetry in (p, q) is now evident. In fact, it is this symmetry that will allow us to perform a general analysis. The main difference with respect to the accelerating case is that the conformal boundary is now located at infinity, so that

$$q_+ \leq q \leq +\infty, \quad p_- \leq p \leq p_+. \quad (3.4)$$

In appendix B we show that upon sending $\Lambda \to 0$ and setting $A = 0$, we can obtain a Ricci-flat version of this set-up. In this case we will show that is possible again to obtain a bulk topology $\mathcal{O}(-t) \to \Sigma$ but with a Calabi-Yau metric. Firstly, we continue considering local aspects and we construct the Killing spinor ε solving (2.4). Global issues will be addressed later on.

3.1 Local Killing spinors

Recall that the supersymmetry conditions (2.23) for the Euclidean CP solution with $\omega = 1$ read[14]

$$MP = NQ, \quad E = -\frac{N^2}{P^2} + 2\sqrt{\alpha}. \quad (3.5)$$

In the following we will assume $N \neq 0$. In this case (3.5) are the correct BPS conditions. When instead $N = 0$, from the first equation in (3.5) wee see that it is possible to choose

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[13]For the 1/4-BPS spherical Bolt solutions the role of the flat connection has been emphasised in the very recent paper [47].

[14]Recall that, for simplicity, we have chosen a specific branch for the square root $\sqrt{\alpha}$.

$M = 0$ or $P = 0$ independently, and the analysis splits into two sub-cases. The $N = 0$ situation will be discussed separately, in section 3.5.3. In the following we will construct the spinor in the general case, with all the parameters different from zero.

The solution presented in [16] corresponds to the sub-case in which $P = Q$ and $N = M$ by regularity of the metric, so that $\Omega = 0$ identically (see (2.19)). As we will discuss, we are in the general situation for which $\Omega = 0$ is a constraint for having a supersymmetric solution. Using these expressions to eliminate M and E in favour of the other parameters, the metric functions (3.3) factorize as

$$P^2 \mathcal{P}(p) = \mathcal{P}_-(p)\mathcal{P}_+(p), \quad P^2 \mathcal{Q}(q) = \mathcal{Q}_-(q)\mathcal{Q}_+(q), \tag{3.6}$$

where

$$\mathcal{P}_\pm(p) = Pp^2 \mp Np \mp P^2 + P\sqrt{\alpha}, \quad \mathcal{Q}_\pm(q) = Pq^2 \mp Nq \mp PQ + P\sqrt{\alpha}. \tag{3.7}$$

Writing the four-dimensional Dirac spinor in terms of its chiral components $\varepsilon_\pm$ as

$$\varepsilon = \begin{pmatrix} \varepsilon_+ \\ \varepsilon_- \end{pmatrix} \tag{3.8}$$

and employing (2.16), the integrability condition (2.14) can be used to relate ε_- to ε_+. In particular we find the relation

$$(X^+ + Y^+)\varepsilon_+ + (X^- + Y^-)\varepsilon_- = 0, \tag{3.9}$$

where $X^\pm$ and $Y^\pm$ are 2×2 matrices with non-vanishing elements

$$X_{1,2}^\pm = \pm\,\mathrm{i}\,\frac{2\big[\mathcal{P}(p) - \mathcal{Q}(q)\big] + (p \pm q)\Big\{2(p \mp q)\big[(q \pm p)^2 - P \mp Q\big] - \mathcal{P}'(p) \pm \mathcal{Q}'(q)\Big\}}{4(p \pm q)^2 \sqrt{-\mathcal{P}(p)}\sqrt{\mathcal{Q}(q)}},$$

$$X_{2,1}^\pm = \pm\,\mathrm{i}\,\frac{2\big[\mathcal{P}(p) - \mathcal{Q}(q)\big] + (p \pm q)\Big\{2(p \mp q)\big[(q \pm p)^2 + P \mp Q\big] - \mathcal{P}'(p) \pm \mathcal{Q}'(q)\Big\}}{4(p \pm q)^2 \sqrt{-\mathcal{P}(p)}\sqrt{\mathcal{Q}(q)}},$$
$$\tag{3.10}$$

and

$$Y^\pm = \frac{(p \mp q)(P \pm Q)}{2(q \pm p)^2 \sqrt{q^2 - p^2}} \begin{pmatrix} \mp\mathrm{i}/\sqrt{\mathcal{Q}(q)} & \pm 1/\sqrt{-\mathcal{P}(p)} \\ \mp 1/\sqrt{-\mathcal{P}(p)} & \pm\mathrm{i}/\sqrt{\mathcal{Q}(q)} \end{pmatrix}. \tag{3.11}$$

Using these relations it is easy to construct the Killing spinor, that is

$$\varepsilon = c \begin{pmatrix} -\mathrm{i}\sqrt{\mathcal{P}_-(p)}\sqrt{\mathcal{Q}_-(q)}/\sqrt{q+p} \\ -\mathrm{i}\sqrt{\mathcal{P}_+(p)}\sqrt{\mathcal{Q}_+(q)}/\sqrt{q+p} \\ \langle P\rangle\sqrt{\mathcal{P}_-(p)}\sqrt{\mathcal{Q}_+(q)}/\sqrt{q-p} \\ \langle P\rangle\sqrt{\mathcal{P}_+(p)}\sqrt{\mathcal{Q}_-(q)}/\sqrt{q-p} \end{pmatrix} e^{\mathrm{i}\frac{N}{2P}(\tau + \sqrt{\alpha}\sigma)}, \tag{3.12}$$

where $\langle P\rangle \equiv P/\sqrt{P^2} \in \mathbb{C} \setminus \{0\}$ and $c \in \mathbb{C}$ a normalization constant. Notice that it takes remarkably the same (implicit) form as in [16], and is then the most general form of the Killing spinor for the Euclidean CP solution even when $M \neq N$ and $P \neq Q$.

Before tackling the regularity of the metric and the spinor, let us express the parameters of the solution (3.3) in terms of the real roots $p_\pm$ and q_+. For p_+ to be a root of

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$\mathcal{P}(p)$, it must be a zero of one between $\mathcal{P}_+(p)$ or $\mathcal{P}_-(p)$ in (3.7). We then introduce a sign $\eta = \pm 1$ such that $\mathcal{P}_\eta(p_+) = 0$. Similarly for p_- we impose $\mathcal{P}_\delta(p_-) = 0$, with $\delta = \pm 1$. The combinations $\mathcal{P}_\eta(p_+)\eta - \mathcal{P}_\delta(p_-)\delta = 0$ and $p_-\mathcal{P}_\eta(p_+)\eta - p_+\mathcal{P}_\delta(p_-)\delta = 0$ can be inverted to give[15]

$$N = \eta P \frac{(p_+ + p_-)(p_+ + \sigma p_-) + \eta P(\sigma - 1)}{p_+ + p_-} \,,$$

$$\sqrt{\alpha} = \sigma \frac{\eta P(p_+ - p_-)(p_+ + \sigma p_-) + p_+ p_-(p_+^2 - p_-^2)}{p_+^2 - p_-^2} \,. \tag{3.13}$$

where we introduced $\sigma \equiv \eta\delta = \pm 1$. Similarly, the condition $\mathcal{Q}_\lambda(q_+) = 0$ with $\lambda = \pm 1$, supplemented with (3.13), is equivalent to

$$Q = \lambda \frac{(p_+ + p_-)\big[q_+^2 + \sigma p_+ p_- - q_+(p_+ + \sigma p_-)\kappa\big] + \eta P\big[\sigma p_+ + p_- + \kappa\sigma q_+(\sigma - 1)\big]}{p_+ + p_-} \,, \tag{3.14}$$

where similarly we introduced $\kappa \equiv \eta\lambda = \pm 1$. It is evident from the previous expressions that the system is distinguished by the value of σ. Indeed, in the next sections we will see that $\sigma = \pm 1$ correspond to the twist or anti-twist realization of the supersymmetry for the spindle [4]. The sign κ instead leads to different branches that, as we will see later, generalize those present in [15, 16].

For each value of the signs η, δ and λ, it holds

$$\mathcal{P}'(p_+) = \frac{2\,(p_+ - p_-)\big[N + p_+(p_+ + p_-)^2\big]}{p_+ + p_-} \,, \tag{3.15}$$

where recall that $p_+ + p_- > 0$ from the scaling symmetry. A necessary and sufficient condition for this to be positive (as required by the signature of the metric) is $N \in \mathbb{R}_+$[16]. When $\sigma = 1$, $0 < N = (p_+ + p_-)\eta P$ from (3.13), and we conclude that

$$\sigma = 1: \quad P \in \mathbb{R}, \quad \eta = \operatorname{sign} P \,. \tag{3.16}$$

In the case $\sigma = -1$, from

$$\sigma = -1: \quad N = \frac{\eta P}{p_+ + p_-}\big[(p_+^2 - p_-^2) - 2\eta P\big] \,, \tag{3.17}$$

we see that in order for N to be real and positive it is necessary that

$$\sigma = -1: \quad \begin{aligned} P \in \mathbb{C} &\implies \operatorname{Re}(P) = \frac{\eta(p_+^2 - p_-^2)}{4} \,, \quad \eta = \operatorname{sign}\operatorname{Re}(P) \,, \\ P \in \mathbb{R} &\implies 0 < \eta P < \frac{(p_+^2 - p_-^2)}{2} \,, \quad \eta = \operatorname{sign} P \,. \end{aligned} \tag{3.18}$$

When $P \in \mathbb{R}$, which can happen for both $\sigma = \pm 1$, all the parameters of the solution are real from (3.13) and (3.14). Then also the metric and the gauge field are real. However,

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[15]In writing the following formulas we assume $p_+ + p_- > 0$. The $p_+ = -p_-$ case is treated at the end of section 3.2.1.

[16]$N = 0$ is also possible, but we will continue in the general case $N > 0$ for the moment. See section 3.5.3 for the reality analysis when $N = 0$.

only for $\sigma = -1$, P can be complex with its real part fixed by (3.18). However, in this case one can verify that the parameter E and the combination $-P^2 + \alpha$ appearing in the structure function $\mathcal{P}(p)$ in (3.3) are nevertheless real. Instead M and $-Q^2 + \alpha$ are complex, unless the real root q_+ satisfies the further condition $q_+ = p_+$ with $\kappa = 1$. Finally, $Q \in \mathbb{C}$ iff $P \in \mathbb{C}$.

Summarizing, for $\sigma = 1$ the solution is real, whilst for $\sigma = -1$ the function $\mathcal{P}(p)$ is always real but the metric remains complex except if $q_+ = p_+$. This sub-case corresponds to the solution presented in [16], where the metric was always real but the gauge field can be complex if P (and therefore Q) is complex. Reality properties of the solution when $N = 0$ will be addressed throughout section 3.5.3.

3.2 Global analysis

Having established the local form of the solutions of interest, we now examine their global properties. Following closely [48, 16], we divide the regularity analysis into two steps: firstly we require the boundary to be a lens space $L(t, 1)$ (eventually branched [7]) and then we consider the conditions under which the bulk topology is that of $\mathcal{M}_4 = \mathcal{O}(-t) \to \Sigma$, a (complex) line bundle over the spindle. Finally, we will show the regularity of the Killing spinor (3.12).

3.2.1 Metric at the boundary

We now study the regularity of the boundary metric $\mathrm{d}s_b^2$, obtained as the leading order in $q \to +\infty$. In [16], the boundary was topologically an S^3, while more generally here we require it to be a (branched) lens space. As it is well known [19, 5, 4, 7], the metric on the very same S^3 (or orbifold thereof) can be written in different ways, more precisely as a standard Hopf fibration over S^2, or as a weighted $U(1)$ fibration over the spindle. For these reasons, it is convenient to adopt coordinates in which the boundary is already written as a fibration over $\Sigma_{[m_-, m_+]}$, differently from [16]. To this end, it proves useful to write the solution (3.1) in coordinates adapted to the degenerate Killing vectors at the end-points of the coordinate range, namely $p_\pm$ and q_+. A generic Killing vector $K = a_1 \partial_\tau + a_2 \partial_\sigma$, in the full four-dimensional metric (3.1), has norm

$$K_\mu K^\mu = \frac{\left[-(a_1 + a_2 q^2)^2 \mathcal{P}(p) + (a_1 + a_2 p^2)^2 \mathcal{Q}(q) \right]}{q^2 - p^2}, \tag{3.19}$$

and since it is the sum of positive quantities, they should be nil separately. At this point it is worth emphasizing that, in order to describe correctly global issues, we need to introduce patches on $\mathcal{M}_4$. In particular we consider two patches $U_\pm$ which include $p_\pm$ respectively, so that $\mathcal{M}_4 = U_+ \cup U_-$. A generic Killing vector is more precisely written as $K = a_1^\pm \partial_{\tau^\pm} + a_2^\pm \partial_{\sigma^\pm}$, where $(\tau^\pm, \sigma^\pm)$ are coordinates on $U_\pm$. We then take $K_{q_+} = (q_+^2 \partial_{\tau^\pm} - \partial_{\sigma^\pm})$ and $K_\pm = K_{p_\pm} = (-p_\pm^2 \partial_{\tau^\pm} + \partial_{\sigma^\pm})$, which degenerate at q_+ and at $p_\pm$, respectively[17]. These Killing vectors can be conveniently written as $K_{q_+} = \partial_{\theta_1^\pm}$ and $K_\pm = \partial_{\theta_2^\pm}$, with again $(\theta_1^\pm, \theta_2^\pm)$ on $U_\pm$, if we change coordinates as

$$U_\pm : \quad \tau^\pm = q_+^2 \theta_1^\pm - p_\pm^2 \theta_2^\pm, \quad \sigma^\pm = -\theta_1^\pm + \theta_2^\pm. \tag{3.20}$$

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[17]Notice that if we take $a_1 = 0$, then or $a_2 = 0$ or $q_+ = p_\pm$, which are both degenerate cases. Similarly it happens if we start with $a_2 = 0$, so we continue assuming $a_i \neq 0$.

Since the Jacobian of (3.20) is $q_+^2 - p_\pm^2$, the periodicities are related simply as

$$\Delta\tau^+\Delta\sigma^+ = \Delta\tau^-\Delta\sigma^- = (q_+^2 - p_+^2)\Delta\theta_1^+\Delta\theta_2^+ = (q_+^2 - p_-^2)\Delta\theta_1^-\Delta\theta_2^- \,. \tag{3.21}$$

In these new coordinates $(q, p, \theta_1^\pm, \theta_2^\pm)$, the four-dimensional metric reads

$$U_\pm : \quad \mathrm{d}s_4^2 = \frac{q^2 - p^2}{-\mathcal{P}(p)}\mathrm{d}p^2 - \frac{(q^2 - p_\pm^2)^2 \mathcal{P}(p)}{q^2 - p^2}\mathrm{d}(\theta_2^\pm)^2 + \frac{q^2 - p^2}{\mathcal{Q}(q)}\mathrm{d}q^2$$
$$+ (q^2 - q_+^2)\frac{\mathrm{d}(\theta_1^\pm)^2(q^2 - q_+^2) - 2(q^2 - p_\pm^2)\mathrm{d}\theta_1^\pm\mathrm{d}\theta_2^\pm}{p^2 - q^2}\mathcal{P}(p) \tag{3.22}$$
$$- \frac{\left[(p^2 - p_\pm^2)\mathrm{d}\theta_2^\pm + (q_+^2 - p^2)\mathrm{d}\theta_1^\pm\right]^2}{p^2 - q^2}\mathcal{Q}(q) \,.$$

We are now in the position to analyze the boundary of (3.22), which is easily found to be

$$U_\pm : \quad \mathrm{d}s_b^2 = \frac{\mathrm{d}p^2}{-\mathcal{P}(p)} - \frac{(q_+^2 - p_\pm^2)^2 \mathcal{P}(p)}{(q_+^2 - p^2)^2 - \mathcal{P}(p)}\mathrm{d}(\theta_2^\pm)^2$$
$$+ [(q_+^2 - p^2)^2 - \mathcal{P}(p)]\left\{\mathrm{d}\theta_1^\pm + \frac{(q_+^2 - p^2)(p^2 - p_\pm^2) + \mathcal{P}(p)}{[(q_+^2 - p^2)^2 - \mathcal{P}(p)]}\mathrm{d}\theta_2^\pm\right\}^2 \,. \tag{3.23}$$

Notice that the boundary metric (3.23) contains only the function $\mathcal{P}(p)$ and is therefore real as a consequence of the discussion at the end of the previous section. We now zoom in near the zeros of $\mathcal{P}(p)$ introducing a new coordinate $r_\pm$

$$U_\pm : \quad p = p_\pm - \frac{\mathcal{P}'(p_\pm)}{4}r_\pm^2 \,, \tag{3.24}$$

for which the boundary metric, at the leading order in $r_\pm$, reads

$$U_\pm : \quad \mathrm{d}s_b^2 \underset{p\to p_\pm}{\simeq} \mathrm{d}r_\pm^2 + \frac{\mathcal{P}'(p_\pm)^2}{4}r_\pm^2\mathrm{d}(\theta_2^\pm)^2 + (q_+^2 - p_\pm^2)^2\mathrm{d}(\theta_1^\pm)^2 \,. \tag{3.25}$$

Provided

$$U_\pm : \quad \frac{|\mathcal{P}'(p_\pm)|}{2}\Delta\theta_2^\pm = \frac{2\pi}{m_\pm} \,, \tag{3.26}$$

the base of this fibration in (3.23) can be made topologically a spindle $\mathbb{\Sigma}_{[m_-,m_+]}$, for some co-prime positive integers $m_\pm$, with $m_- > m_+$. Recall also that $\mathcal{P}'(p_-) < 0$, for which we take the absolute value in (3.26). This condition is equivalent to

$$m_- > m_+ \implies (q_+^2 - p_+^2)\mathcal{P}'(p_-) + (q_+^2 - p_-^2)\mathcal{P}'(p_+) > 0 \,. \tag{3.27}$$

We will refer to this spindle as $\mathbb{\Sigma}_{[m_-,m_+]}^\infty$, since its shape is different for different values of q, but the topology will remain the same, i.e. the volume is a function of q, but the conical deficits remain the same along the flow. We stress that in the boundary the spindle is a topologically trivial cycle, but nevertheless we will see that it plays an important role. Indeed, with reference to the two dimensional base

$$U_\pm : \quad \mathrm{d}s_{\mathbb{\Sigma}_\infty}^2 = \frac{\mathrm{d}p^2}{-\mathcal{P}(p)} - \frac{(q_+^2 - p_\pm^2)^2 \mathcal{P}(p)}{(q_+^2 - p^2)^2 - \mathcal{P}(p)}\mathrm{d}(\theta_2^\pm)^2 \,, \tag{3.28}$$

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it can be seen that

$$\sqrt{g_{\Sigma_\infty}}R_{\Sigma_\infty} = (q_+^2 - p_\pm^2)\partial_p\left\{(q_+^2 - p^2)\left[4\mathcal{P}(p)p + (q_+^2 - p^2)\mathcal{P}'(p)\right]\left[(q_+^2 - p^2)^2 - \mathcal{P}(p)\right]^{-3/2}\right\},$$
(3.29)

and as a consequence

$$U_\pm: \quad \frac{1}{4\pi}\int_{\Sigma_\infty} dp\, d\theta_2^\pm \sqrt{g_{\Sigma_\infty}}R_{\Sigma_\infty} = \frac{q_+^2 - p_\pm^2}{4\pi}\left.\frac{\mathcal{P}'(p)}{q_+^2 - p^2}\right|_{p_-}^{p_+}\Delta\theta_2^\pm.$$
(3.30)

This computation must agree in the two patches $U_\pm$, and using (3.21) we see that necessarily $\Delta\theta_1^+ = \Delta\theta_1^-$. In turn, this implies

$$\frac{1}{4\pi}\int_{\Sigma_\infty} dp\, d\theta_2^\pm \sqrt{g_{\Sigma_\infty}}R_{\Sigma_\infty} = \frac{m_- + m_+}{m_- m_+} \equiv \chi_\Sigma,$$
(3.31)

where χ_Σ is the (orbifold) Euler characteristic of the spindle. Then, with (3.26) and

$$\Delta\theta_1 \equiv \Delta\theta_1^+ = \Delta\theta_1^-,$$
(3.32)

the metric (3.28) represents indeed $\Sigma_{[m_-,m_+]}^\infty$. With these conditions imposed, it is also useful to introduce

$$\Delta \equiv (q_+^2 - p_+^2)\Delta\theta_2^+ = (q_+^2 - p_-^2)\Delta\theta_2^-,$$
(3.33)

in terms of which (3.26) is rewritten as

$$\pm\frac{\mathcal{P}'(p_\pm)}{2(q_+^2 - p_\pm^2)}\Delta = \frac{2\pi}{m_\pm}.$$
(3.34)

Moreover we require that the fibration is well-defined in the orbifold sense [4], that is

$$\frac{1}{2\pi}\int_{\Sigma_\infty}\frac{2\pi}{\Delta\theta_1}d\left\{\frac{(q_+^2 - p^2)(p^2 - p_\pm^2) + \mathcal{P}(p)}{[(q_+^2 - p^2)^2 - \mathcal{P}(p)]}d\theta_2^\pm\right\} \equiv \frac{t}{m_- m_+},$$
(3.35)

for $t \in \mathbb{N}$, which is equivalent to

$$U_\pm: \quad t = m_- m_+ \frac{p_+^2 - p_-^2}{q_+^2 - p_\mp^2}\frac{\Delta\theta_2^\pm}{\Delta\theta_1} \implies t = m_- m_+ \frac{p_+^2 - p_-^2}{(q_+^2 - p_-^2)(q_+^2 - p_+^2)}\frac{\Delta}{\Delta\theta_1}.$$
(3.36)

Notice that combining (3.21), (3.32), (3.33), (3.34) and (3.36) we can write

$$\Delta\tau^\pm\Delta\sigma^\pm = \Delta\theta_1\Delta = \frac{m_- m_+}{t}\Delta^2\frac{p_+^2 - p_-^2}{(q_+^2 - p_-^2)(p_+^2 - p_-^2)} = \frac{(4\pi)^2}{t}\frac{p_+^2 - p_-^2}{-\mathcal{P}'(p_-)\mathcal{P}'(p_+)},$$
(3.37)

where the last two expressions are valid only for $t \neq 0$.

To summarise the analysis so far, we have shown that by chosing suitably the parameters of the local solutions, the boundary is a circle orbifold bundle over the spindle $\Sigma_{[m_-,m_+]}$, namely the bundle $S^1 \hookrightarrow \mathcal{M}_3 \to \Sigma_{[m_-,m_+]}$, which is a *Seifert orbifold*. It is in fact known from general results on Seifert orbifolds [49, 19] that topologically $\mathcal{M}_3$ must be a lens space, possibly with residual orbifold singularities. More precisely $\mathcal{M}_3$ is a smooth lens space $L(t,1) = S^3/\mathbb{Z}_t$ if $\gcd(t,m_\pm) = 1$. When t and $m_\pm$ have a common divisor there

are various sub-cases that we do not discuss in detail here [7], but the simplest example in this context is perhaps when $t = \bar{t}\, m_+\, m_-$. This symmetric situation corresponds to an "orbifold lens space" $\mathbb{L}_{[m_-,m_+]}(t,1)$, also referred to as branched lens space, which consists in a three-sphere with orbifold poles of local topology $\mathbb{C}/\mathbb{Z}_{m_\pm}$ and with a further global $\mathbb{Z}_t$ quotient. For simplicity, we assume $\gcd(t, m_\pm) = 1$ for the time being.

When $t = 0$, the boundary is generically a direct product $\Sigma \times S^1$. Since $t \propto (p_+ + p_-)$, we can study separately the case where $p_+ = -p_- \equiv r$. Indeed, the discussion around (3.13) assumed $p_+ + p_- > 0$. We can repeat now that analysis, requiring a priori $t = 0$. Imposing again $\mathcal{P}_\eta(p_+)\eta - \mathcal{P}_\delta(p_-)\delta = 0$, $p_-\mathcal{P}_\eta(p_+)\eta - p_+\mathcal{P}_\delta(p_-)\delta = 0$ and $\mathcal{Q}_\lambda(q_+) = 0$ we find

$$
\begin{aligned}
\sigma = +1: \quad & N = 0\,, \quad \sqrt{\alpha} = -r^2 + \eta P\,, \quad Q = \lambda(q_+^2 - r^2 + \eta P)\,, \\
\sigma = -1: \quad & N = 0\,, \quad P = 0\,.
\end{aligned}
\tag{3.38}
$$

with $\sigma = \eta\delta = \pm 1$ as before. We see that $t = 0 \implies N = 0$, but the vice-versa is not true. The case $t = 0$ is thus excluded from the generic analysis, and will be treated in all details in section 3.5.3.

Let us conclude this section by working out explicitly the coordinate change between U_+ and U_- from (3.20). To this end it is useful to write the Killing vectors expressed in the two patches

$$
\begin{aligned}
K_{p_+} &= \left[-\frac{p_+^2 - p_-^2}{q_+^2 - p_-^2}\partial_{\theta_1^-} + \frac{q_+^2 - p_+^2}{q_+^2 - p_-^2}\partial_{\theta_2^-} \right] = \partial_{\theta_2^+}\,, \\
K_{q_+} &= \partial_{\theta_1^-} = \partial_{\theta_1^+}\,, \\
K_{p_-} &= \partial_{\theta_2^-} = \left[\frac{p_+^2 - p_-^2}{q_+^2 - p_+^2}\partial_{\theta_1^+} + \frac{q_+^2 - p_-^2}{q_+^2 - p_+^2}\partial_{\theta_2^+} \right]\,,
\end{aligned}
\tag{3.39}
$$

which follow from

$$
\theta_1^+ = \theta_1^- + \frac{p_+^2 - p_-^2}{q_+^2 - p_+^2}\theta_2^-\,, \quad \theta_2^+ = \frac{q_+^2 - p_-^2}{q_+^2 - p_+^2}\theta_2^-\,.
\tag{3.40}
$$

These can be rewritten introducing 2π-periodic coordinates

$$
\varphi_1^\pm = \frac{2\pi}{\Delta\theta_1}\theta_1^\pm\,, \quad \varphi_2^\pm = \frac{2\pi}{\Delta\theta_2^\pm}\theta_2^\pm\,,
\tag{3.41}
$$

and using (3.32), (3.33) and (3.36) to obtain

$$
\varphi_1^+ = \varphi_1^- + \frac{t}{m_- m_+}\varphi_2^-\,, \quad \varphi_2^+ = \varphi_2^-\,.
\tag{3.42}
$$

Thus we can define

$$
\varphi_2 \equiv \varphi_2^+ = \varphi_2^-\,,
\tag{3.43}
$$

which is identified in the gluing and use it independently of the patch $U_\pm$. Notice that when $p_- = -p_+$, i.e. when $t = 0$, we can always choose $\theta_1^+ = \theta_1^- \equiv \theta_1$ and $\theta_2^+ = \theta_2^- \equiv \theta_2$ from (3.40). This is the reason for which, for example in [16, 6], it is not needed the description in patches.

3.2.2 Metric in the bulk

We now study the regularity of the bulk, with a finite value of the radial coordinate q. In principle one can try to consider the case $p_+ = q_+$, as in [16]. However, this is incompatible with a bulk topology[18] $\mathcal{O}(-t) \to \Sigma_{[m_-, m_+]}$. Indeed in that case the Killing vector K_{q_+} has a zero-dimensional fixed point (called a *nut*), but we want it to develop a two-dimensional fixed submanifold $\Sigma \subset \mathcal{M}_4$ (called a *bolt*), using a standard terminology [10]. In the regularity analysis of [16], for $p_+ = q_+$, the authors obtained that $N = M$, and in turn $P = Q$. Indeed the point $p = q = p_+ = q_+$ contains a curvature singularity unless $P = Q$. For example, the square root of the Ricci tensor is

$$R_{\mu\nu}R^{\mu\nu} = 4\frac{\left[9(q^2 - p^2)^4 + (P^2 - Q^2)^2\right]}{(q^2 - p^2)^4}\,. \tag{3.44}$$

From the previous discussion, we continue with $Q \neq P$ and $N \neq M$. The situation is summarized as

$$\begin{aligned}
\mathcal{Q}(q) \geq 0\,, \quad \mathcal{P}(p) \leq 0\,, \quad q^2 > p^2\,, \\
p_- \leq p \leq p_+ < q_+ \leq q\,, \quad q_+^2 > p_\pm^2\,,
\end{aligned} \tag{3.45}$$

where the range of p and q is in general disjoint. In order to obtain a complete metric, the space must necessarily "close off" at the largest root of $\mathcal{Q}(q)$, the location of the bolt.

We can now look at (3.22). Noticing that for $q = q_+$ the second line is zero, we argue that $(p, \theta_2^\pm)$, when q is fixed at q_+, are again the coordinates on (a different shaped) spindle $\Sigma_{[m_-, m_+]}^{q_+}$. Since q_+ is the position of the bolt, we will refer to this $\Sigma_{[m_-, m_+]}^{q_+}$ as a "spindle bolt". Near the points $(p_\pm, q_+)$ we use

$$U_\pm: \quad q = q_+ + \frac{\mathcal{Q}'(q_+)}{4(q_+^2 - p_\pm^2)}r^2\,, \quad p = p_\pm - \frac{\mathcal{P}'(p_\pm)}{4(q_+^2 - p_\pm^2)}r_\pm^2\,, \tag{3.46}$$

on (3.22), obtaining at the leading order

$$U_\pm: \quad \mathrm{d}s_4^2 \underset{q_+, p_\pm}{\simeq} \mathrm{d}r^2 + \frac{\mathcal{Q}'(q_+)^2}{4}r^2\mathrm{d}(\theta_1^\pm)^2 + \mathrm{d}r_\pm^2 + \frac{r_\pm^2}{m_\pm^2}\mathrm{d}\varphi_2^2\,, \tag{3.47}$$

where we already used (3.41), (3.43) and (3.26). Locally this metric describes $\mathbb{C}/\mathbb{Z}_v \times \mathbb{C}/\mathbb{Z}_{m_\pm}$, once we impose

$$\frac{\mathcal{Q}'(q_+)}{2}\Delta\theta_1 = \frac{2\pi}{v}\,, \tag{3.48}$$

which is the analogous of (3.26) and recall that $\Delta\theta_1 = \Delta\theta_1^+ = \Delta\theta_1^-$. Summarizing, equations (3.34), (3.36) and (3.48) are "quantization conditions", which we will study in detail in section 3.5.

Defining the (real-valued) spindle bolt metric to be the first line of (3.22) in $q = q_+$

$$U_\pm \quad \mathrm{d}s_{\Sigma_{q_+}}^2 = \frac{q_+^2 - p^2}{-\mathcal{P}(p)}\mathrm{d}p^2 - \frac{(q_+^2 - p_\pm^2)^2\mathcal{P}(p)}{q_+^2 - p^2}\mathrm{d}(\theta_2^\pm)^2\,, \tag{3.49}$$

---

[18]With a standard abuse of notation we will omit often the fiber, since it is $\mathbb{C}$ or orbifold thereof.

we can observe that

$$U_\pm: \quad \sqrt{g_{\Sigma_{q_+}}} R_{\Sigma_{q_+}} = (q_+^2 - p_\pm^2)\partial_p \left[\frac{2p\mathcal{P}(p) + (q_+^2 - p^2)\mathcal{P}'(p)}{(q_+^2 - p^2)^2}\right], \qquad (3.50)$$

and the (orbifold) Euler characteristic is, again, found to be

$$\chi_\Sigma = \frac{1}{4\pi} \int \sqrt{g_{\Sigma_{q_+}}} R_{\Sigma_{q_+}} \, dp \, d\theta_2^\pm = \frac{m_- + m_+}{m_- m_+} \,. \qquad (3.51)$$

The metric of the spindle (3.49) is different from (3.28), but it describes topologically the same spindle, with conical singularities $\mathbb{C}/\mathbb{Z}_{m_\pm}$. The four-dimensional metric (3.22), with (3.34), (3.36) and (3.48) describes the bundle $\mathbb{C}/\mathbb{Z}_v \hookrightarrow \mathcal{O}(-t) \to \Sigma_{[m_-,m_+]}^{q_+}$, which has a normal singularity $\mathbb{C}/\mathbb{Z}_v$.

We conclude this section by considering the collapse of the metric at $q = q_0 > q_+$ fixed. To zoom in near $p_\pm$ we moreover use $p = p_\pm - \frac{\mathcal{P}'(p_\pm)}{4(q_0^2 - p_\pm^2)} R_\pm^2$ for which (3.22) becomes

$$U_\pm: \quad ds^2\big|_{q=q_0} \underset{p \to p_\pm}{\simeq} dR_\pm^2 + \frac{R_\pm^2}{m_\pm^2}(d\varphi_2 + c_\pm(q_0)d\theta_1^\pm)^2 + \frac{(q_+^2 - p_\pm^2)^2}{q_0^2 - p_\pm^2}\mathcal{Q}(q_0)d(\theta_1^\pm)^2 \,, \quad (3.52)$$

where $c_\pm(q_0)$ is an irrelevant constant that goes to zero as q_0 approaches q_+. Then, the regularity follows from the previous conditions imposed to the topology of the boundary.

3.2.3 Gauge field and Killing spinors in the bulk

Let us come back to the study of the regularity of the spinor (3.12). The analysis is divided into two parts: firstly, we pick a gauge in which the graviphoton (3.2) is regular on the spindle bolt, then we choose an adapted frame in which the spinor is smooth and well-defined. In the patch $U_\pm$ which contains $p_\pm$ we perform the gauge transformation

$$U_\pm: \quad A'_\pm = A + q_+ Q d\theta_1^\pm - p_\pm P d\theta_2^\pm \,, \qquad (3.53)$$

so that $A'_\pm(q_+, p_\pm) = 0$. Correspondingly, the spinor (3.12) acquires a phase. Summing up the contributions, the overall phase $\Phi_\pm$ in the 2π-periodic coordinates $\varphi_i^\pm$ can be written as

$$U_\pm: \quad \Phi_\pm = \frac{2q_+ PQ + N(q_+^2 - \sqrt{\alpha})}{2P} \frac{\Delta\theta_1}{2\pi}\varphi_1^\pm - \frac{2p_\pm PQ + N(p_\pm^2 - \sqrt{\alpha})}{2P} \frac{\Delta\theta_2^\pm}{2\pi}\varphi_2$$
$$= \frac{\lambda \mathcal{Q}'(q_+)}{4} \frac{\Delta\theta_1}{2\pi}\varphi_1^\pm - \frac{\rho^\pm \mathcal{P}'(p_\pm)}{4} \frac{\Delta\theta_2^\pm}{2\pi}\varphi_2 = \frac{\lambda}{2v}\varphi_1^\pm \mp \frac{\rho_\pm}{2m_\pm}\varphi_2 \,, \qquad (3.54)$$

where we have used (3.41), (3.43), (3.6), (3.7) and (3.26), (3.48) in the last step. Moreover we have introduced the sign $\rho_\pm$, defined as $\rho_+ = \eta$ and $\rho_- = \delta$, so that the product $\rho_+\rho_- = \eta\delta = \sigma$. Next, we notice that the frame (2.15) is singular at the poles of the bolt. It is therefore convenient to write the frame in $\varphi_i^\pm$ coordinates and focus on $(q_+, p_\pm)$ by means of (3.46). We obviously obtain (see (3.47))

$$U_\pm: \quad e^a \simeq \left\{\mp dr_\pm, \frac{r_\pm}{m_\pm}d\varphi_2, \frac{r}{v}d\varphi_1^\pm, dr\right\}, \qquad (3.55)$$

where the sign in e^1 is due to the fact that approaching p_+ (p_-) the coordinate p is increasing (decreasing), while $r_\pm$ is decreasing (increasing). A well-defined frame on the spindle, in the patch $U_\pm$, is given by

$$U_\pm: \quad (\hat{e}^1) + \mathrm{i}(\hat{e}^2) = \mathrm{d}\left[\mp r_\pm \exp\left(\mp\mathrm{i}\frac{\varphi_2}{m_\pm}\right)\right] = \exp\left(\mp\mathrm{i}\frac{\varphi_2}{m_\pm}\right)(e^1 + \mathrm{i}e^2)\,, \tag{3.56}$$

which is obtained by means of a $U(1) \simeq SO(2)$ rotation. Similarly it happens for the transverse part described by (e^3, e^4), so that the whole $SO(2) \times SO(2)$ transformation is

$$U_\pm: \quad \begin{pmatrix} \hat{e}^1 \\ \hat{e}^2 \\ \hat{e}^3 \\ \hat{e}^4 \end{pmatrix} = \begin{pmatrix} \cos\left[\mp\varphi_2/m_\pm\right] & -\sin\left[\mp\varphi_2/m_\pm\right] & 0 & 0 \\ \sin\left[\mp\varphi_2/m_\pm\right] & \cos\left[\mp\varphi_2/m_\pm\right] & 0 & 0 \\ 0 & 0 & \cos\left[\varphi_1^\pm/v\right] & \sin\left[\varphi_1^\pm/v\right] \\ 0 & 0 & -\sin\left[\varphi_1^\pm/v\right] & \cos\left[\varphi_1^\pm/v\right] \end{pmatrix} \begin{pmatrix} e^1 \\ e^2 \\ e^3 \\ e^4 \end{pmatrix}, \tag{3.57}$$

Since under a transformation of the vielbein $\hat{e}^a = \exp(\lambda^a{}_b)\,e^b$ a four-dimensional Dirac spinor ψ transforms in the spin representation as $\psi' = \exp(1/2\lambda^{ab}\Gamma_{ab})\psi$, with the Γ_a given in (2.16), we have that the four dimensional spinor (3.12) becomes

$$U_\pm: \quad \varepsilon' = \exp\left[\frac{\mathrm{i}}{2}\mathrm{diag}\left\{\pm\frac{\varphi_2}{m_\pm} - \frac{\varphi_1^\pm}{v}, \mp\frac{\varphi_2}{m_\pm} + \frac{\varphi_1^\pm}{v}, \pm\frac{\varphi_2}{m_\pm} + \frac{\varphi_1^\pm}{v}, \mp\frac{\varphi_2}{m_\pm} - \frac{\varphi_1^\pm}{v}\right\}\right]\varepsilon\,. \tag{3.58}$$

Explicitly, summing the contributions from (3.54) and (3.58), the spinor (3.12) in the regular gauge and frame reads

$$U_\pm: \quad \varepsilon'(q, p, \varphi_1^\pm, \varphi_2) = c \begin{pmatrix} -\mathrm{i}\dfrac{\sqrt{\mathcal{P}_-(p)}\sqrt{\mathcal{Q}_-(q)}}{\sqrt{q+p}}\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda-1}{v}\varphi_1^\pm \mp \frac{\rho^\pm-1}{m_\pm}\varphi_2]} \\ -\mathrm{i}\dfrac{\sqrt{\mathcal{P}_+(p)}\sqrt{\mathcal{Q}_+(q)}}{\sqrt{q+p}}\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda+1}{v}\varphi_1^\pm \mp \frac{\rho^\pm+1}{m_\pm}\varphi_2]} \\ \langle P\rangle\dfrac{\sqrt{\mathcal{P}_-(p)}\sqrt{\mathcal{Q}_+(q)}}{\sqrt{q-p}}\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda+1}{v}\varphi_1^\pm \mp \frac{\rho^\pm-1}{m_\pm}\varphi_2]} \\ \langle P\rangle\dfrac{\sqrt{\mathcal{P}_+(p)}\sqrt{\mathcal{Q}_-(q)}}{\sqrt{q-p}}\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda-1}{v}\varphi_1^\pm \mp \frac{\rho^\pm+1}{m_\pm}\varphi_2]} \end{pmatrix}. \tag{3.59}$$

We can now see that the spinor in (3.59) is smooth everywhere by noticing that, at the fixed points, it can be written as

$$\sigma = 1: \quad \varepsilon'(q_+, p_\pm) = \frac{c}{4} \begin{pmatrix} -\mathrm{i}\dfrac{\sqrt{\mathcal{P}_-(p_\pm)}\sqrt{\mathcal{Q}_-(q_+)}}{\sqrt{q_+ + p_\pm}}(1+\eta)(1+\lambda)\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda-1}{v}\varphi_1^\pm \mp \frac{\eta-1}{m_\pm}\varphi_2]} \\ -\mathrm{i}\dfrac{\sqrt{\mathcal{P}_+(p_\pm)}\sqrt{\mathcal{Q}_+(q_+)}}{\sqrt{q_+ + p_\pm}}(1-\eta)(1-\lambda)\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda+1}{v}\varphi_1^\pm \mp \frac{\eta+1}{m_\pm}\varphi_2]} \\ \langle P\rangle\dfrac{\sqrt{\mathcal{P}_-(p_\pm)}\sqrt{\mathcal{Q}_+(q_+)}}{\sqrt{q_+ - p_\pm}}(1+\eta)(1-\lambda)\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda+1}{v}\varphi_1^\pm \mp \frac{\eta-1}{m_\pm}\varphi_2]} \\ \langle P\rangle\dfrac{\sqrt{\mathcal{P}_+(p_\pm)}\sqrt{\mathcal{Q}_-(q_+)}}{\sqrt{q_+ - p_\pm}}(1-\eta)(1+\lambda)\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda-1}{v}\varphi_1^\pm \mp \frac{\eta+1}{m_\pm}\varphi_2]} \end{pmatrix}, \tag{3.60}$$

or

$$\sigma = -1: \quad \varepsilon'(q_+, p_\pm) = \frac{c}{4} \begin{pmatrix} -\mathrm{i}\dfrac{\sqrt{\mathcal{P}_-(p_\pm)}\sqrt{\mathcal{Q}_-(q_+)}}{\sqrt{q_+ + p_\pm}}(1\pm\eta)(1+\lambda)\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda-1}{v}\varphi_1^\pm \mp \frac{\pm\eta-1}{m_\pm}\varphi_2]} \\ -\mathrm{i}\dfrac{\sqrt{\mathcal{P}_+(p_\pm)}\sqrt{\mathcal{Q}_+(q_+)}}{\sqrt{q_+ + p_\pm}}(1\mp\eta)(1-\lambda)\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda+1}{v}\varphi_1^\pm \mp \frac{\pm\eta+1}{m_\pm}\varphi_2]} \\ \langle P\rangle\dfrac{\sqrt{\mathcal{P}_-(p_\pm)}\sqrt{\mathcal{Q}_+(q_+)}}{\sqrt{q_+ - p_\pm}}(1\pm\eta)(1-\lambda)\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda+1}{v}\varphi_1^\pm \mp \frac{\pm\eta-1}{m_\pm}\varphi_2]} \\ \langle P\rangle\dfrac{\sqrt{\mathcal{P}_+(p_\pm)}\sqrt{\mathcal{Q}_-(q_+)}}{\sqrt{q_+ - p_\pm}}(1\mp\eta)(1+\lambda)\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda-1}{v}\varphi_1^\pm \mp \frac{\pm\eta+1}{m_\pm}\varphi_2]} \end{pmatrix}. \tag{3.61}$$

It is evident from the above expressions that, for a fixed choice of the signs (η, λ), the single component of the spinor which is non-zero does not contain any phase. Choosing for example the roots q_+ and p_+ such that $\mathcal{P}_+(p_+) = \mathcal{Q}_+(q_+) = 0$, which is realized by fixing $\eta = \lambda = 1$, we see from (3.60) that all the components are zero but the first, which however does not depend any more on (φ_1^+, φ_2). In the twist case this example is equivalent to

$$\varepsilon'(q_+, p_+) = c \begin{pmatrix} -\mathrm{i}\frac{\sqrt{\mathcal{P}_-(p_+)}\sqrt{\mathcal{Q}_-(q_+)}}{\sqrt{q_+ + p}} \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.62}$$

Notice also that these spinors are regular at the location of the bolt $q = q_+$, without moving to its poles. Indeed the $\varphi_1^\pm$ are cancelled in the non-vanishing components by requiring $\mathcal{Q}_\pm(q_+) = 0$. Moreover equations (3.60) and (3.61) show that the spinor has the same chirality at the poles of the spindle in the $\sigma = 1$ case and opposite chiralities when $\sigma = -1$. In turn, this explicitly shows that $(\sigma = -1)$ $\sigma = 1$ corresponds to the (anti-) twist realization of the supersymmetry on the spindle bolt [4].

We end this section by writing the spinor transition function coming from (3.58), obtained by the composition of the transformation on U_+ and the (inverse) transformation on U_-

$$\exp\left[\frac{\mathrm{i}\,\varphi_2}{2}\mathrm{diag}\left\{\chi_\Sigma - \frac{t/v}{m_- m_+}, -\chi_\Sigma + \frac{t/v}{m_- m_+}, \chi_\Sigma + \frac{t/v}{m_- m_+}, -\chi_\Sigma - \frac{t/v}{m_- m_+}\right\}\right], \tag{3.63}$$

where we have used (3.42) and (3.43). Notice that these transition function is valid for both the twist and the anti-twist set-up, as for example in [5, 3], even if only χ_Σ appears.

3.3   Seifert orbifolds with twist and anti-twist

In this section we discuss the global form of the gauge field and Killing spinors at the boundary and then turn to the relationship to rigid supersymmetry for a large class of three dimensional Seifert orbifolds analysed in [7].

We start recalling the explicit form of the metric at the conformal boundary obtained as the leading order term of the four dimensional metric at $q \to +\infty$. In the local 2π-periodic coordinates $(\varphi_1^\pm, \varphi_2^\pm)$ introduced in (3.41), in the two patches $U_\pm$ this reads

$$\begin{aligned}
\mathrm{d}s_{(3)}^2 = &\frac{\mathrm{d}p^2}{-\mathcal{P}(p)} - \frac{\mathcal{P}(p)}{(q_+^2 - p^2)^2 - \mathcal{P}(p)}\frac{\Delta^2}{(2\pi)^2}\mathrm{d}\varphi_2^2 \\
&+ [(q_+^2 - p^2)^2 - \mathcal{P}(p)]\frac{(\Delta\theta_1)^2}{(2\pi)^2}\left\{\mathrm{d}\varphi_1^\pm + \frac{\Delta}{\Delta\theta_1}\frac{(q_+^2 - p^2)(p^2 - p_\pm^2) + \mathcal{P}(p)}{[(q_+^2 - p^2)^2 - \mathcal{P}(p)](q_+^2 - p_\pm^2)}\mathrm{d}\varphi_2\right\}^2,
\end{aligned} \tag{3.64}$$

where the "periodicity" Δ is defined in (3.33). The transition functions for these coordinates are given by (*cf.* equation (3.42))

$$\varphi_1^+ = \varphi_1^- + \frac{t}{m_- m_+}\varphi_2, \quad \varphi_2^+ = \varphi_2^- \equiv \varphi_2. \tag{3.65}$$

Next, let us discuss the gauge field $A_{(3)}$ and a Killing spinor $\chi_{(3)}$, induced at the boundary, obtained as the $q \to +\infty$ expansion of (3.2) and (3.12), respectively. The boundary gauge field $A_{(3)} = pP\mathrm{d}\sigma$, obtained from in the same regular gauge as (3.53) reads

$$U_{\pm}: \quad \left(A_{(3)}\right)'_{\pm} = \frac{\Delta\theta_1}{2\pi}(q_+ Q - pP)\mathrm{d}\varphi_1^{\pm} + \frac{\Delta}{2\pi}\frac{P(p - p_{\pm})}{(q_+^2 - p_{\pm}^2)}\mathrm{d}\varphi_2 \,. \tag{3.66}$$

Using the three-dimensional vierbein induced from the bulk (2.15), namely

$$e_{(3)}^a = \left\{ \frac{\mathrm{d}p}{\sqrt{-\mathcal{P}(p)}}, \sqrt{-\mathcal{P}(p)}\mathrm{d}\sigma, (\mathrm{d}\tau + p^2\mathrm{d}\sigma) \right\} \underset{p\to p_{\pm}}{\simeq} \left\{ \mp\mathrm{d}r_{\pm}, \frac{r_{\pm}}{m_{\pm}}\mathrm{d}\varphi_2, (q_+^2 - p_{\pm}^2)\frac{\Delta\theta_1}{2\pi}\mathrm{d}\varphi_1^{\pm} \right\}, \tag{3.67}$$

the boundary spinor is [16]

$$U_{\pm}: \quad \chi_{(3)} = \begin{pmatrix} \sqrt{\mathcal{P}_-(p)} \\ \sqrt{\mathcal{P}_+(p)} \end{pmatrix} \mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda}{v}\varphi_1^{\pm} \mp \frac{\rho_{\pm}}{m_{\pm}}\varphi_2]} = \begin{pmatrix} \sqrt{\mathcal{P}_-(p)} \\ \sqrt{\mathcal{P}_+(p)} \end{pmatrix} \mathrm{e}^{\frac{\mathrm{i}\eta}{2}[\frac{\kappa}{v}\varphi_1^{\pm} \mp \frac{\sigma_{\pm}}{m_{\pm}}\varphi_2]} \,. \tag{3.68}$$

Here we have used the definitions $\rho_+\rho_- = \sigma$, $\kappa = \eta\lambda$ and introduced the symbol $\sigma_{\pm} = \eta\rho_{\pm}$ which is $\sigma_+ = 1$ and $\sigma_- = \sigma$. We emphasise that since we used the *same regular gauge* for the bulk and boundary gauge field, the phase of the spinor is the same as in (3.54).

Following the steps outlined in (3.54), we can easily show that the transition functions for the gauge field (3.66) across the patches U_+ and U_- are

$$(A_{(3)})_- - (A_{(3)})_+ = \frac{1}{2}\left[\frac{\rho^+ m_- + \rho^- m_+}{m_- m_+} - \lambda\frac{t/v}{m_- m_+}\right]\mathrm{d}\varphi_2 = \frac{\eta}{2}\left[\frac{m_- + \sigma m_+}{m_- m_+} - \kappa\frac{t/v}{m_- m_+}\right]\mathrm{d}\varphi_2 \,. \tag{3.69}$$

This formula shows that the choice of the sign σ, that will determine the type of twist for the graviphoton field in the bulk (see (3.119) below), is captured by the boundary geometry. This is in agreement with the rigid supersymmetry analysis of [7], where this sign was denoted $\mathfrak{s}_{\omega}$. Notice that this formula is perfectly valid also when $t = 0$, for which the boundary is $\Sigma \times S^1$ and contains the spindle as a non-trivial two-cycle. In section 3.5 we will reproduce the same expression integrating $F = \mathrm{d}A$ over the spindle bolt defined in (3.49).

Let us now show that the spinor (3.68) is well defined, in each patch. The boundary frame (3.67) suffers from singularities at the poles $p = p_{\pm}$ exactly as the one in (3.55). We then perform a frame rotation analogous to that in (3.57) on $e_{(3)}^{1,2}$,

$$U_{\pm}: \quad \begin{pmatrix} \hat{e}_{(3)}^1 \\ \hat{e}_{(3)}^2 \\ \hat{e}_{(3)}^3 \end{pmatrix} = \begin{pmatrix} \cos\left[\mp\varphi_2/m_{\pm}\right] & -\sin\left[\mp\varphi_2/m_{\pm}\right] & 0 \\ \sin\left[\mp\varphi_2/m_{\pm}\right] & \cos\left[\mp\varphi_2/m_{\pm}\right] & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e_{(3)}^1 \\ e_{(3)}^2 \\ e_{(3)}^3 \end{pmatrix}, \tag{3.70}$$

with the corresponding action on the spinor[19] given by

$$U_{\pm}: \quad \chi'_{(3)} = \begin{pmatrix} \mathrm{e}^{\pm\frac{\mathrm{i}}{2}\frac{\varphi_2}{m_{\pm}}} & 0 \\ 0 & \mathrm{e}^{\mp\frac{\mathrm{i}}{2}\frac{\varphi_2}{m_{\pm}}} \end{pmatrix} \chi_{(3)} \,, \tag{3.71}$$

---

[19]We are using the standard Pauli matrices σ_a, $a = 1, 2, 3$ as a basis of the three-dimensional Clifford algebra since they are inherited from (2.16).

so that at the two poles we have

$$
\sigma = 1: \quad \chi'_{(3)}(p_\pm) = \frac{1}{2}\begin{pmatrix} \sqrt{\mathcal{P}_-(p)}(1+\eta)\mathrm{e}^{\mp\frac{\mathrm{i}}{2}\frac{\eta-1}{m_\pm}\varphi_2} \\ \sqrt{\mathcal{P}_+(p)}(1-\eta)\mathrm{e}^{\mp\frac{\mathrm{i}}{2}\frac{\eta+1}{m_\pm}\varphi_2} \end{pmatrix} \mathrm{e}^{\frac{\mathrm{i}}{2}\frac{\lambda}{v}\varphi_1^\pm}\,.
$$

$$
\sigma = -1: \quad \chi'_{(3)}(p_\pm) = \frac{1}{2}\begin{pmatrix} \sqrt{\mathcal{P}_-(p)}(1\pm\eta)\mathrm{e}^{\mp\frac{\mathrm{i}}{2}\frac{\pm\eta-1}{m_\pm}\varphi_2} \\ \sqrt{\mathcal{P}_+(p)}(1\mp\eta)\mathrm{e}^{\mp\frac{\mathrm{i}}{2}\frac{\pm\eta+1}{m_\pm}\varphi_2} \end{pmatrix} \mathrm{e}^{\frac{\mathrm{i}}{2}\frac{\lambda}{v}\varphi_1^\pm}
$$

$$(3.72)$$

which are always well defined. We then conclude that the transition function for the boundary spinor coming from (3.71) are given by

$$
\begin{pmatrix} \mathrm{e}^{\frac{\mathrm{i}}{2}\chi_\Sigma\varphi_2} & 0 \\ 0 & \mathrm{e}^{-\frac{\mathrm{i}}{2}\chi_\Sigma\varphi_2} \end{pmatrix}\,,
$$

$$(3.73)$$

which is the same as that for the spinors defined on a spindle [4]! This might be surprising, but it is a direct consequence of the regularity analysis at the bolt. Below we will show that after changing coordinates we recover a standard transition function for spinors on a Seifert manifold (or orbifold). Indeed, from the expressions above it is clear that for $\sigma = 1$ the spinor has the same "transverse chirality"[20] at the poles $p = p_\pm$ of the Seifert orbifold, whilst for $\sigma = -1$ it has opposite chiralities. We therefore refer to the choices $\sigma = \pm 1$ as to *twist and anti-twist for the Seifert orbifold*.

We emphasize that the mechanism by which the bulk "informs" the boundary is through the regularity of the gauge field at the bolt, which then determines a preferred gauge at the boundary[21]. This is the same idea that has proved crucial in the context of supersymmetric black holes, put forward in [42]. In a context analogous to the present one, similar observations were made in [15] and [19]. We expect that, in a fashion analogous to the black hole entropy functions and other gravitational blocks [50, 3, 35], the choice of three-dimensional flat connections will determine the large N limit of the partition functions.

Let us conclude this section by making a comparison with the Seifert orbifolds studied in [7], and for simplicity below we will consider the case for which $t = v = 1$, so that the boundary is topologically a three-sphere. In order to explicitly show that the metric (3.64) is topologically that of smooth S^3 we define new 2π-period coordinates[22] $(\hat{\varphi}_1, \hat{\varphi}_2)$ in each patch

$$
\begin{aligned}
U_-: \quad & \varphi_1^- = \frac{1}{m_-}\hat{\varphi}_1\,, \quad \varphi_2 = m_-\hat{\varphi}_2 - m_+\hat{\varphi}_1\,, \\
U_+: \quad & \varphi_1^+ = \frac{1}{m_+}\hat{\varphi}_2\,, \quad \varphi_2 = m_-\hat{\varphi}_2 - m_+\hat{\varphi}_1
\end{aligned}
$$

$$(3.74)$$

---

[20]The boundary spinor (3.68) is already decomposed in a product of a two dimensional spinor on the base (with coordinates p, φ_2) and a phase on the Seifert fiber $\varphi_1^\pm$. Accordingly, the two dimensional gamma matrices can be taken to be just σ_1 and σ_2, with chirality matrix $\sigma_\star = -\mathrm{i}\sigma_1\sigma_2 = \sigma_3$. Under $\sigma_\star$ the two-dimensional spinor (3.68) is chiral or anti-chiral at the poles of the Seifert orbifold.

[21]In particular, the phase $\mathrm{e}^{\frac{\mathrm{i}}{2}\frac{\lambda}{v}\varphi_1^\pm}$ in the Killing spinor (3.68) is precisely the phase $\mathrm{e}^{\frac{\mathrm{i}}{2}n\psi}$ that appears in the spindle index [8], with $n = \pm 1$ for the accelerating black holes. Indeed as remarked the $\Sigma \times S^1$ geometry is a special case $t = 0$ of the general set-up discussed above. See also footnote (31).

[22]We anticipate that in these coordinates, $(\partial_{\hat{\varphi}_1}, \partial_{\hat{\varphi}_2})$ is a basis for an effective torus action, as it is related by an $SL(2,\mathbb{Z})$ transformation to the basis (E_1, E_2) that we shall introduce in the next section.

so that the metric (3.64) near the poles $p_\pm$ is

$$U_- : \quad \mathrm{d}s_{(3)}^2 \underset{p \to p_-}{\simeq} \mathrm{d}r_-^2 + r_-^2 \mathrm{d}\hat{\varphi}_2^2 + \left(\frac{q_+^2 - p_-^2}{m_-} \frac{\Delta\theta_1}{2\pi} \right)^2 \mathrm{d}\hat{\varphi}_1^2 \, ,$$

$$U_+ : \quad \mathrm{d}s_{(3)}^2 \underset{p \to p_+}{\simeq} \mathrm{d}r_+^2 + r_+^2 \mathrm{d}\hat{\varphi}_1^2 + \left(\frac{q_+^2 - p_+^2}{m_+} \frac{\Delta\theta_1}{2\pi} \right)^2 \mathrm{d}\hat{\varphi}_2^2 \, . \tag{3.75}$$

This shows that near to the north pole ($p = p_-$) the space looks like $\mathbb{R}^2 \times S_{\hat{\varphi}_1}^1$ and near to the south pole ($p = p_+$) it is locally $\mathbb{R}^2 \times S_{\hat{\varphi}_2}^1$.

In order to compare our spinor (3.68) with that given in [7] we can focus on the dependence on the angular coordinates, *i.e.* the phases. Firstly we have to give the map from the coordinates $(\varphi_1^\pm, \varphi_2)$ we have been using thus far in the paper, that are defined in two patches $U_\pm$, with the coordinates $(\hat{\varphi}, \hat{\psi})$ used for example in section 5 of [7], which are in a sense "global", meaning that $(\hat{\varphi}, \hat{\psi})$ are defined in a patch which excludes both $\hat{\theta} = 0$ and $\hat{\theta} = \pi$. In particular we should consider coordinates $\hat{\psi}_{(0)}$, $\hat{\psi}_{(\pi)}$ in [7] which are well-defined in a patch containing $\hat{\theta} = 0, \pi$, respectively. These are defined as[23]

$$\hat{\psi}_{(0)} = \hat{\psi} - \frac{r_+}{m_-}\hat{\varphi} \, , \quad \hat{\psi}_{(\pi)} = \hat{\psi} + \frac{r_-}{m_+}\hat{\varphi} \, , \quad \hat{\psi}_{(\pi)} = \hat{\psi}_{(0)} + \frac{1}{m_- m_+}\hat{\varphi} \, , \tag{3.76}$$

where we have used the definition $m_- r_- + m_+ r_+ = 1$[24]. This transition function is consistent with (3.65) for $t = 1$. The toric coordinates $\hat{\varphi}_1 \equiv m_-\hat{\psi} - r_+\hat{\varphi}$ and $\hat{\varphi}_2 \equiv m_+\hat{\psi} + r_-\hat{\varphi}$ used before in (3.75) and in [7], expressed in the two patches read

$$\hat{\varphi}_1 = m_-\hat{\psi}_{(0)} = m_-\hat{\psi}_{(\pi)} - \frac{1}{m_+}\hat{\varphi} \, , \quad \hat{\varphi}_2 = m_+\hat{\psi}_{(0)} + \frac{1}{m_-}\hat{\varphi} = m_+\hat{\psi}_{(\pi)} \, , \tag{3.77}$$

whilst their inverse expressions are

$$\hat{\psi}_{(0)} = \frac{1}{m_-}\hat{\varphi}_1 \, , \quad \hat{\psi}_{(\pi)} = \frac{1}{m_+}\hat{\varphi}_2 \, , \quad \hat{\varphi} = m_-\hat{\varphi}_2 - m_+\hat{\varphi}_1 \, . \tag{3.78}$$

Using these relations between coordinates, we can now present the transition functions (3.69) and (3.73) in a standard form. Notice that even if $(\hat{\varphi}_1, \hat{\varphi}_2)$ and $(\hat{\varphi}, \hat{\psi})$ are related by an $SL(2, \mathbb{Z})$ transformation, the transition functions in (3.76) (and consequently also the ones in (3.42)) are not $SL(2, \mathbb{Z})$-valued. This is not a problem since $\varphi_i^\pm$ do not give an effectively acting basis for the torus, as we will discuss in the next section. Moreover, a priori the gauge field A^C considered there (*cf.* (5.15) in [7]) should be again written in patches

$$U_- : \quad A_{(0)}^C = \frac{1}{2}\left(\frac{b_s}{f(\hat{\theta})} + \alpha_2^{(0)} \right) \mathrm{d}\hat{\varphi}_1 + \frac{1}{2}\left(\frac{\mathfrak{s}_\omega}{f(\hat{\theta})} + \alpha_3^{(0)} \right) \mathrm{d}\hat{\varphi}_2 \, ,$$

$$U_+ : \quad A_{(\pi)}^C = \frac{1}{2}\left(\frac{b_s}{f(\hat{\theta})} + \alpha_2^{(\pi)} \right) \mathrm{d}\hat{\varphi}_1 + \frac{1}{2}\left(\frac{\mathfrak{s}_\omega}{f(\hat{\theta})} + \alpha_3^{(\pi)} \right) \mathrm{d}\hat{\varphi}_2 \, , \tag{3.79}$$

---

[23] In these coordinates $(\hat{\psi}_{(0)}, \hat{\psi}_{(\pi)})$ the connection which determines the fibration (5.6) in [7] of S^1 over $\Sigma_{[n_1, n_2]}$ is well-defined and regular in each coordinate patch, as our one (see equation (3.25)).

[24] The relation between our variables $(m_-, m_+, \varphi_1^\pm, \varphi_2)$ and the ones defined in [7] is given by $\hat{\psi}_{(\pi)}|_{\text{there}} = \varphi_1^+|_{\text{here}}$, $\hat{\psi}_{(0)}|_{\text{there}} = \varphi_1^-|_{\text{here}}$, $\hat{\varphi}|_{\text{there}} = \varphi_2|_{\text{here}}$, $n_1|_{\text{there}} = m_-|_{\text{here}}$, $n_2|_{\text{there}} = m_+|_{\text{here}}$. Finally, $t_2|_{\text{there}} = r_-|_{\text{here}}$ and $t_1|_{\text{there}} = -r_+|_{\text{here}}$, where $r_\pm$ will be introduced in the next section (see equation (3.92)).

where $b_{s,c}$ are constants representing the squashing of the S^3 and $\mathfrak{s}_\omega = \pm 1$, that we identify as $\mathfrak{s}_\omega = \sigma$. Finally $\alpha_2^{(0,\pi)}$ and $\alpha_3^{(0,\pi)}$ implement gauge transformations in the two patches, and these must be fixed to the values

$$\alpha_3^{(0)} = \sigma, \quad \alpha_2^{(\pi)} = 1, \tag{3.80}$$

for $A_{(0),(\pi)}^C$ to be well defined in the respective patch, where one of the two coordinates $\hat\varphi_1$ and $\hat\varphi_2$ becomes ill-defined at a pole. Instead, $\alpha_3^{(\pi)}$ and $\alpha_2^{(0)}$ remain undetermined. The transition function is then

$$A_{(0)}^C - A_{(\pi)}^C = \frac{1}{2}\left(\alpha_2^{(0)} - 1\right)\mathrm{d}\hat\varphi_1 + \frac{1}{2}\left(\sigma - \alpha_3^{(\pi)}\right)\mathrm{d}\hat\varphi_2, \tag{3.81}$$

and it can be compared to (3.69), with $\varphi_2 = \hat\varphi$ expressed in terms of $\hat\varphi_i$ from (3.78) (with $t = v = 1$)

$$(A_{(3)})_- - (A_{(3)})_+ = \frac{\eta}{2}\left[\left(\frac{\kappa - \sigma m_+}{m_-}\right) - 1\right]\mathrm{d}\hat\varphi_1 + \frac{\eta}{2}\left[\sigma - \left(\frac{\kappa - m_-}{m_+}\right)\right]\mathrm{d}\hat\varphi_2. \tag{3.82}$$

Upon identifying $A^C = \eta A_{(3)}$, the transition functions are the same for a specific choice of the remaining gauge freedom, given by

$$\alpha_2^{(0)} = \frac{\kappa - \sigma m_+}{m_-}, \quad \alpha_3^{(\pi)} = \frac{\kappa - m_-}{m_+}. \tag{3.83}$$

Notice that whilst in [7] the gauge was fixed to $\alpha_2^{(0)} \equiv \alpha_2^{(\pi)} = 1$ and $\alpha_3^{(\pi)} \equiv \alpha_3^{(0)} = \sigma$ (such that $A_{(0)}^C - A_{(\pi)}^C = 0$), we do not have this freedom in our set-up because we must use the preferred gauge (3.53), ensuring that the spinor and gauge field are regular in bulk. Correspondingly, the spinor (3.68) becomes

$$U_-: \quad \chi_{(3)} = \begin{pmatrix} \sqrt{\mathcal{P}_-(p)} \\ \sqrt{\mathcal{P}_+(p)} \end{pmatrix} e^{\frac{i\eta}{2}[\alpha_2^{(0)}\hat\varphi_1 + \sigma\hat\varphi_2]}, \quad U_+: \quad \chi_{(3)} = \begin{pmatrix} \sqrt{\mathcal{P}_-(p)} \\ \sqrt{\mathcal{P}_+(p)} \end{pmatrix} e^{\frac{i\eta}{2}[\hat\varphi_1 + \alpha_3^{(\pi)}\hat\varphi_2]}. \tag{3.84}$$

As usual, we can finally check that the spinor is smooth in $p = p_\pm$ by rotating *differently* the original frame (3.67) in the two patches as

$$
\begin{aligned}
U_-: \quad & \begin{pmatrix} \tilde{e}_{(3)}^1 \\ \tilde{e}_{(3)}^2 \\ \tilde{e}_{(3)}^3 \end{pmatrix} = \begin{pmatrix} \cos\hat\varphi_2 & -\sin\hat\varphi_2 & 0 \\ \sin\hat\varphi_2 & \cos\hat\varphi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e_{(3)}^1 \\ e_{(3)}^2 \\ e_{(3)}^3 \end{pmatrix}, \\
U_+: \quad & \begin{pmatrix} \tilde{e}_{(3)}^1 \\ \tilde{e}_{(3)}^2 \\ \tilde{e}_{(3)}^3 \end{pmatrix} = \begin{pmatrix} \cos\hat\varphi_1 & -\sin\hat\varphi_1 & 0 \\ \sin\hat\varphi_1 & \cos\hat\varphi_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e_{(3)}^1 \\ e_{(3)}^2 \\ e_{(3)}^3 \end{pmatrix}.
\end{aligned}
\tag{3.85}
$$

For example, in the twist case, the resulting action on the spinor near the poles gives

$$
\sigma = +1: \quad
\begin{aligned}
U_-: \quad & \chi'_{(3)}(p_-) = \begin{pmatrix} \sqrt{\mathcal{P}_-(p_-)}(1+\eta)e^{-\frac{i(1-\eta)}{2}\hat\varphi_2} \\ \sqrt{\mathcal{P}_+(p_-)}(1-\eta)e^{\frac{i(1+\eta)}{2}\hat\varphi_2} \end{pmatrix} e^{\frac{i\eta}{2}\alpha_2^{(0)}\hat\varphi_1}, \\
U_+: \quad & \chi'_{(3)}(p_+) = \begin{pmatrix} \sqrt{\mathcal{P}_-(p_+)}(1+\eta)e^{-\frac{i(1-\eta)}{2}\hat\varphi_1} \\ \sqrt{\mathcal{P}_-(p_+)}(1-\eta)e^{\frac{i(1+\eta)}{2}\hat\varphi_1} \end{pmatrix} e^{\frac{i\eta}{2}\alpha_3^{(\pi)}\hat\varphi_2},
\end{aligned}
\tag{3.86}
$$

showing again that for a choice of $\eta = \pm 1$ the spinor is well-defined near $p_\pm$. The anti-twist case works similarly.

We come to the conclusion that the main significant difference with the analysis in [7] is a distinct choice of gauge field, which is imposed on us from the global analysis in the bulk.

3.4 Toric data

To conclude the presentation of our solution (3.1), we will provide a description of the orbifold $\mathbb{C}/\mathbb{Z}_v \hookrightarrow \mathcal{O}(-t) \to \mathbb{\Sigma}_{[m_-,m_+]}$ in the framework of toric geometry, albeit the metric has not a symplectic structure. Indeed the space it describes is toric with a $U(1)^2$ action, corresponding to the $U(1)^2$ isometry of (3.1). Since in the following sections we will be primarily interested in the topological properties of our solution, we now construct a *labelled polytope* [51] in the spirit of [35, 36]. We start defining the loci

$$\mathcal{L}_1 = \{p = p_+\}\,, \qquad \mathcal{L}_2 = \{q = q_+\}\,, \qquad \mathcal{L}_3 = \{p = p_-\}\,, \tag{3.87}$$

where $\mathcal{L}_{1,3}$ are non compact whilst $\mathcal{L}_2$ defines the bolt, and it is indeed compact. These meet at the two fixed points

$$p_1 = \{p_+, q_+\}\,, \quad p_2 = \{p_-, q_+\}\,, \tag{3.88}$$

where by convention we take $\mathcal{L}_a$ to be the one that joins p_{a-1} and p_a. We can extract the toric data, following the prescription of [35] (see also [52]). Indeed, given a generic base (e_1, e_2) of the torus action (not necessarily effective), and a set of Killing vectors ξ_a degenerating on $\mathcal{L}_a$, we use the relation

$$\xi_a = \vec{v}_a \cdot (e_1, e_2)\,, \quad \left.\frac{\partial_\mu |\xi_a|^2 \partial^\mu |\xi_a|^2}{4|\xi_a|^2}\right|_{\mathcal{L}_a} = 1\,, \tag{3.89}$$

to extract the vectors of the fan $\vec{v}_a \in \mathbb{Z}^2$. These in general will not be primitive, and the greatest common divisor of the entries of each vector will represent a label m_a. Such a number is the order of the transverse singularity at *every point* of the toric divisor $\mathcal{L}_a$, meaning that the structure group $\mathbb{Z}_{m_a}$ is acting on the complex direction $\mathbb{C}$ normal to $\mathcal{L}_a$. Notice that we normalized ξ_a to have unit surface gravity. In the following, without loss of generality, we choose the patch U_- in which the normalized Killing vectors read (see (3.39) and (3.42))

$$U_- : \quad \begin{aligned} \xi_1 &= \frac{2}{-\mathcal{P}'(p_+)}(p_+^2 \partial_{\tau^-} - \partial_{\sigma^-}) = m_+ \left[\partial_{\varphi_2^-} - \frac{t}{m_+ m_-}\partial_{\varphi_1^-}\right]\,, \\ \xi_2 &= \frac{2}{\mathcal{Q}'(q_+)}(q_+^2 \partial_{\tau^-} - \partial_{\sigma^-}) = v\partial_{\varphi_1^-}\,, \\ \xi_3 &= \frac{2}{\mathcal{P}'(p_-)}(p_-^2 \partial_{\tau^-} - \partial_{\sigma^-}) = m_-\partial_{\varphi_2^-}\,. \end{aligned} \tag{3.90}$$

Taking inspiration from [35], we take as effective toric basis[25]

$$U_- : \quad E_1 = \partial_{\varphi_1^-} \,, \quad E_2 = \partial_{\varphi_2^-} - \frac{r_+}{m_-} \partial_{\varphi_1^-} \,, \tag{3.91}$$

where $r_\pm$ are integers satisfying the Bézout's lemma

$$r_- m_- + r_+ m_+ = t \,, \tag{3.92}$$

which always exist for coprime $m_\pm$. In this way, we have

$$U_- : \quad \begin{aligned} \xi_1 &= m_+ E_2 + E_1(m_+ r_+ - t)/m_- = m_+ E_2 - r_- E_1 \,, \\ \xi_2 &= v E_1 \,, \\ \xi_3 &= m_- E_2 + r_+ E_1 \,, \end{aligned} \tag{3.93}$$

From the relation (3.89) we extract the following non-primitive vectors

$$\vec{v}_1 = (r_-, -m_+) \,, \quad \vec{v}_2 = (v, 0) \,, \quad \vec{v}_3 = (r_+, m_-) \,, \tag{3.94}$$

which generate the non-compact polytope in figure 2a. This polytope is the same described in [53] as the "blow-up" of $\mathbb{C}/\mathbb{Z}_t$, and from a supergravity point of view it is the non compact version of the solution presented in [35], with the first facet therein is sent to infinity. As anticipated, v is a label for $\mathcal{L}_2$, whilst $\mathcal{L}_{1,3}$ are standard toric divisors. As pointed out in [36], one can create singularities along $\mathcal{L}_{1,3}$ as well by tuning the value of the parameter t. Indeed it can be shown that for the "symmetric" case $t = \bar{t} \, k_+ \, k_-$ with $m_\pm = k_\pm \overline{m}_\pm$, the labels are $m_a = (k_-, v, k_+)$ for $k_\pm \in \mathbb{Z}$. The signs of the vectors $\vec{v}_a$ are chosen requiring them to be ordered counter-clockwise, in our conventions. It should be clear that the computation performed in U_+ leads to the same polytope. Whilst here we do not have a symplectic closed two-form $\omega_{(2)}$, in appendix B we show that for a Ricci-flat version of this construction it is possible to construct the labelled polytope in a standard way, and the result is consistent with the prescription (3.89).

For future reference in figure 2 we have drawn also the weights of the toric action, *i.e.* the outward-pointing vectors $\vec{\mu}_i^{(A)} \equiv \vec{\mu}_{A_i}^{(A)}$ from each vertex $A = 1, 2$ of the polytope. Here $i = 1, 2$ indicates the number of facets that meet at p_A. Given two vectors $\vec{v}_{1,2}$, the respective weights can be extracted as

$$\begin{aligned} \vec{\mu}_1 &= \pm \frac{(\vec{v}_1 \cdot \vec{v}_2) \, \vec{v}_1 - |\vec{v}_1|^2 \, \vec{v}_2}{d_{12}} \,, \\ \vec{\mu}_2 &= \mp \frac{|\vec{v}_2|^2 \, \vec{v}_1 - (\vec{v}_1 \cdot \vec{v}_2) \, \vec{v}_2}{d_{12}} \,, \end{aligned} \tag{3.95}$$

where the sign ambiguities can be treated as before, and for (3.94) they read

$$\begin{aligned} \vec{\mu}_1^{(1)} &= (-m_+, -r_-) \,, & \vec{\mu}_2^{(1)} &= (0, -v) \,, \\ \vec{\mu}_1^{(2)} &= (0, v) \,, & \vec{\mu}_2^{(2)} &= (-m_-, r_+) \,, \end{aligned} \tag{3.96}$$

---

[25]In the patch U_+, the basis is equivalently given by $E_1 = \partial_{\varphi_1^+}$ and $E_2 = \partial_{\varphi_2^+} + \frac{r_-}{m_+} \partial_{\varphi_1^+}$. Moreover, in terms of the coordinates $(\hat{\varphi}_1, \hat{\varphi}_2)$ introduced in (3.74), we have that $E_1 = m_- \partial_{\hat{\varphi}_1} + m_+ \partial_{\hat{\varphi}_2}$ and $E_2 = -r_+ \partial_{\hat{\varphi}_1} + r_- \partial_{\hat{\varphi}_2}$. This transformation, being in $SL(2, \mathbb{Z})$ for $t = 1$, shows that (E_1, E_2) and $(\hat{\varphi}_1, \hat{\varphi}_2)$ are both effective basis for the torus.

We have normalized them such that

$$|\vec{\mu}_1^{(A+1)}|^2 = |\vec{\mu}_2^{(A)}|^2 = |\vec{v}_{A+1}|^2\,, \quad \vec{\mu}_1^{(A)} \cdot \mu_2^{(A)} = \vec{v}_A \cdot \vec{v}_{A+1}\,, \tag{3.97}$$

and

$$\det(\vec{\mu}_1^{(A)}, \vec{\mu}_2^{(A)}) = \det(\vec{v}_A, \vec{v}_{A+1}) = d_A\,. \tag{3.98}$$

where d_A is the order of the singularity at the A-th fixed-point.

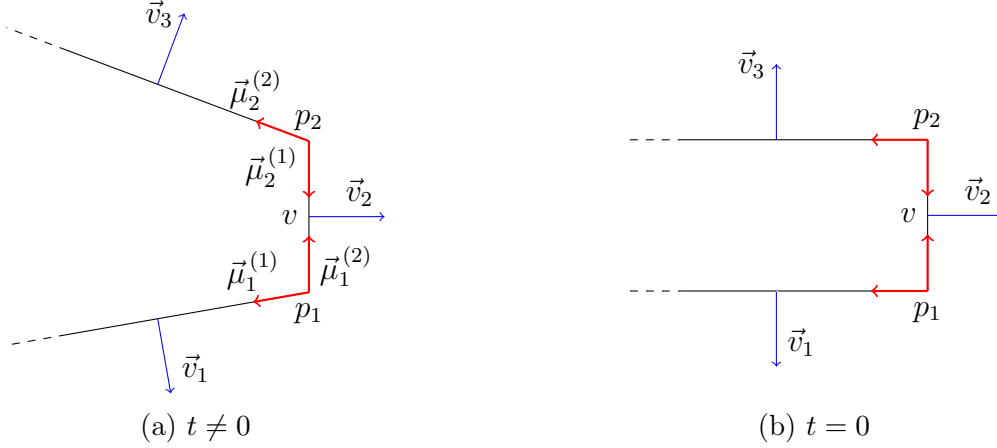(a) $t \neq 0$ (b) $t = 0$

Figure 2: Labelled polytope associated to normal vectors (3.94). The normal singularity $\mathbb{C}/\mathbb{Z}_v$ shows up as a label in the polytope. figure 2a is valid for $t \neq 0$, whilst figure 2b is for $t = 0$.

We conclude this section by considering the special case $t = 0$. From (3.90) we see immediately that the polytope is an open rectangle as in figure 2b. Notice that when $t = 0$ the integers $r_\pm$ in (3.92) loose their meaning. In particular, all the previous formulas are applicable by simply setting $r_\pm = 0$.

3.5 Quantization conditions

In this section we complete the prove of the existence of the solution, *i.e.* we solve the "quantization conditions" (3.34), (3.36) and (3.48) showing that (3.1) and (3.2) describe a supersymmetric solution with topology $\mathbb{C}/\mathbb{Z}_v \hookrightarrow \mathcal{O}(-t) \to \Sigma_{[m_-,m_+]}$. It will follow that the field strength associated to (3.2) is correctly quantized. To this end, it proves useful to express the parameters of the solution (E, N, M, P, Q, α) and $(\Delta\theta_i^\pm)$ in terms of the integers $(t, m_\pm, v)$. Recall also that supersymmetry (3.5) reduces the former down to four independent, namely (N, P, Q, α). In turn, (N, Q, α) are exchanged for the real roots of the metric functions $(p_\pm, q_+)$ and three signs (η, σ, κ) in (3.13)-(3.14). These relevant signs are related to the ones introduced around (3.13) via

$$\eta = \mathrm{sign}\,\mathrm{Re}(P)\,, \quad \sigma = \eta\delta\,, \quad \kappa = \eta\lambda\,, \tag{3.99}$$

where the value of σ will reflect the twist or anti-twist on the spindle bolt. We consider first the generic case with $N \neq 0$, and later we devote a section to $N = 0$. Since, as already observed, the twist case is always simpler, we start the discussion with $\sigma = 1$, turning then to $\sigma = -1$.

3.5.1 Twist

Using the expressions (3.13) and (3.14) with $\sigma = 1$, it is easy to take the ratio of (3.34). Defining the rescaled parameter

$$\tilde{P} = \frac{P}{p_+ + p_-}\,, \tag{3.100}$$

the ratio of (3.34) takes the simple form

$$\frac{m_+}{m_-} = \frac{q_+^2 - p_+^2}{q_+^2 - p_-^2}\frac{p_- + \eta\tilde{P}}{p_+ + \eta\tilde{P}}\,. \tag{3.101}$$

We can parametrize the roots as

$$p_\pm = w(1 \pm x)\,, \quad q_+ = \frac{p_+ + p_-}{2}\tilde{q}_+ = w\tilde{q}_+\,, \tag{3.102}$$

with

$$0 < w < q_+\,, \quad x > 0\,, \quad \tilde{q}_+ > 1 + x\,, \tag{3.103}$$

where these constraints on w and x come from $p_+ \pm p_- > 0$ and $q_+ > p_+$. From (3.101) we get

$$w = -\eta\tilde{P}\frac{\left[\tilde{q}_+^2 - (1-x)^2\right]m_+ - \left[\tilde{q}_+^2 - (1+x)^2\right]m_-}{(1+x)\left[\tilde{q}_+^2 - (1-x)^2\right]m_+ - \left[\tilde{q}_+^2 - (1+x)^2\right](1-x)m_-}\,. \tag{3.104}$$

Then, from both (3.34) and (3.48) we obtain

$$\begin{aligned}
\frac{\Delta}{2\pi} &= \frac{1}{8\eta\tilde{P}}\frac{(1+x)\left[\tilde{q}_+^2 - (1-x)^2\right]m_+ - \left[\tilde{q}_+^2 - (1+x)^2\right](1-x)m_-}{x^2 m_+ m_-}\,, \\
\frac{\Delta\theta_1}{2\pi} &= \frac{1}{2v(\tilde{q}_+ - \kappa)}\frac{1}{w^2\left[2\eta\tilde{P} + w(\tilde{q}_+^2 + 1 - x^2)\right]}\,.
\end{aligned} \tag{3.105}$$

Finally (3.36) is now simply

$$t = \frac{v(\tilde{q}_+ - \kappa)}{x}(m_- - m_+)\,, \tag{3.106}$$

which can be easily inverted to

$$x = (\tilde{q}_+ - \kappa)\frac{(m_- - m_+)v}{t}\,. \tag{3.107}$$

Notice that not all the values of t are admissible from (3.106), due to the constraint $\tilde{q}_+ > 1 + x > 1$ and $N \neq 0$. In particular, we show in appendix C that $t \geq 2$.

We are now in the position to show that the field strength is correctly quantized. We start performing a gauge transformation

$$U_\pm: \quad A'_\pm = A + q_+ Q(\mathrm{d}\theta_1^\pm - \mathrm{d}\theta_2^\pm)\,, \tag{3.108}$$

such that on $\mathcal{L}_2 = \{q = q_+\}$ the gauge field reads simply

$$U_\pm : \quad A'\big|_{\mathcal{L}_2} = -\frac{(q_+^2 - p_\pm^2)(Qq_+ - Pp)}{q_+^2 - p^2}\mathrm{d}\theta_2^\pm . \tag{3.109}$$

It is then possible, using (3.13), (3.14) and the previous quantized quantities to see that

$$\frac{1}{2\pi}\int_{\mathcal{L}_2} F = \frac{\eta}{2}\left(\chi_\Sigma - \kappa\frac{t/v}{m_- m_+}\right), \tag{3.110}$$

meaning that $2A$ is a connection one-form on the line bundle $\mathcal{O}(m_- + m_+)$ over $\Sigma_{[m_-,m_+]}^{q_+}$, and, in turn, that we have a twist.

The additional term $t/(vm_- m_+)$ in (3.110) comes from the non trivial fibration on the spindle bolt in (3.22), more precisely from the normal bundle to $\mathcal{L}_2$. To see this, we focus on $q = q_+$ in (3.22) making the change $q = q_+ + \frac{\mathcal{Q}'(q_+)}{4}r^2$. The result is

$$U_\pm : \quad \mathrm{d}s_4^2 \underset{q_+}{\simeq} \mathrm{d}s_{\Sigma_{q_+}}^2 + (q_+^2 - p^2)\left[\mathrm{d}r^2 + \frac{r^2}{v^2}\left(\mathrm{d}\varphi_1^\pm + \mu_f^\pm\right)^2\right], \tag{3.111}$$

where $\mu_f^\pm$ is the fibration one-form, given by

$$U_\pm : \quad \mu_f^\pm = v\frac{2q_+(q_+^2 - p_\pm^2)\mathcal{P}(p) + (p^2 - p_\pm^2)(q_+^2 - p^2)\mathcal{Q}'(q_+)}{2(q_+^2 - p^2)^2}\mathrm{d}\theta_2^\pm . \tag{3.112}$$

Integrating it we get immediately

$$\frac{1}{2\pi}\int_{\mathcal{L}_2}\mathrm{d}\mu_f^\pm = \frac{t}{m_- m_+}, \tag{3.113}$$

which gives correctly the same value as in (3.35).

In summary, we started from six quantities $(N, P, Q, \alpha, \Delta, \Delta\theta_1)$ subject to four topological constraint for $(t, m_\pm, v)$. We are left with a solution with *two free continuous parameters*, that we choose to be $(\tilde{P}, \tilde{q}_+) \in \mathbb{R}$.

3.5.2 Anti-twist

We now focus on the other case, when $\sigma = -1$. The ratio of (3.34) is much more complicated

$$\frac{m_+}{m_-} = \frac{q_+^2 - p_+^2}{q_+^2 - p_-^2}\frac{\left[2P^2 - (p_+^2 - p_-^2)\eta P - p_-(p_+ + p_-)^3\right]}{\left[\eta P - p_+(p_+ + p_-)\right]\left[2\eta P + (p_+ + p_-)^2\right]}, \tag{3.114}$$

but it can be simplified redefining

$$\eta P = (p_+ + p_-)\eta\tilde{P} = (p_+ + p_-)\frac{p_+ - p_- + \sqrt{(p_+ - p_-)^2 - 8(p_+ + p_-)\eta\overline{P}}}{4}. \tag{3.115}$$

With the parametrization (3.102), we obtain formally the same expression for w as in (3.104) but with $\overline{P}$ instead of $\tilde{P}$, specifically

$$w = -\eta\overline{P}\frac{\left[\tilde{q}_+^2 - (1-x)^2\right]m_+ - \left[\tilde{q}_+^2 - (1+x)^2\right]m_-}{(1+x)\left[\tilde{q}_+^2 - (1-x)^2\right]m_+ - \left[\tilde{q}_+^2 - (1+x)^2\right](1-x)m_-}. \tag{3.116}$$

The other parameters $(\Delta\theta_1, \Delta, t)$ can be obtained similarly, but the expressions do not have a compact form. In particular, $t = t(x)$ is not invertible. For future reference, we report it here

$$
\begin{aligned}
t = {}& v \frac{\left[\tilde{q}_+^2 - (x+1)^2\right]m_- - \left[\tilde{q}_+^2 - (x-1)^2\right]m_+}{\left[\tilde{q}_+^2 - (x+1)^2\right]\left[\tilde{q}_+^2 - (x-1)^2\right]} \times \\
& \frac{\tilde{q}_+ w\left[w(\tilde{q}_+^2 + x^2 - 1) - 2x\eta\tilde{P}\right] + \kappa\left[-2\tilde{P}^2 x - w^2 x(\tilde{q}_+^2 + x^2 - 1) + \eta\tilde{P}w(\tilde{q}_+^2 + 3x^2 - 1)\right]}{w^2 x},
\end{aligned}
\tag{3.117}
$$

where w and $\tilde{P}$ are given by (3.116) and (3.115), respectively. Notice that (3.117) is not written in a completely explicit form. However, when expanded, it does not depend on $\overline{P}$, a fact that will play a role later on. Moreover, here as before, the parameter t can not take any value, and in particular it can not happen $t = 1$. From now on the analysis continues as before, but it is more involved. Nevertheless, it can be proved that

$$
\frac{1}{2\pi}\int_{\mathcal{L}_2} F = \frac{\eta}{2}\left(\frac{m_- - m_+}{m_- m_+} - \kappa\frac{t/v}{m_- m_+}\right). \tag{3.118}
$$

Even though we can not obtain x from (3.117), we regard to this implicit expression as a constraint, which leaves us again a solution with *two free continuous parameters* $(\overline{P}, \tilde{q}_+) \in \mathbb{R}$.

The main result of this section is the following formula which encapsulates simultaneously (3.110) and (3.118)

$$
\frac{1}{2\pi}\int_{\mathcal{L}_2} F = \frac{\eta}{2}\left(\frac{m_- + \sigma m_+}{m_- m_+} - \kappa\frac{t/v}{m_- m_+}\right), \tag{3.119}
$$

and state that $2A$ is a connection on the bundle $\mathcal{O}(m_- + \sigma m_+)$ over the spindle bolt $\Sigma_{[m_-, m_+]}^{q_+}$. It is interesting to notice that both type of twist can be realized in minimal gauged supergravity, as opposite to the black hole solution [5] with a spindly horizon where only the anti-twist is realized. In conclusion, we have two different globally-defined solutions, both describing $\mathbb{C}/\mathbb{Z}_v \hookrightarrow \mathcal{O}(-t) \to \Sigma_{[m_-, m_+]}^{q_+}$, but with a different type of twist on the spindle bolt according to the value of $\sigma = \pm 1$. Moreover, both depend on *two free continuous parameters*, that are $(\tilde{P}, \tilde{q}_+) \in \mathbb{R}$ for the twist case, and $(\overline{P}, \tilde{q}_+) \in \mathbb{R}$ for the anti-twist case. Notice that (3.119) is exactly the same as the one obtained in the boundary by the analysis of the gauge field transition functions (*cf.* (3.69)). From this formula we can deduce some constraints on the possible manifolds on which we can uplift our solutions to eleven dimensions. Following [15], a brief study of this issue is reported in appendix D.

3.5.3 $N = 0$

As explained at the end of section 3.2.1, the case $N = 0$ is of particular interest, because we can have a boundary topology $\Sigma \times S^1$ only if $N = 0$. It is interesting to notice that, when $\mathbf{A} = 0$, $N = 0$ is is equivalent to turning off the NUT parameter. In particular we require [23]

$$
E = P^2 - \alpha - \omega^2, \quad N = 0, \tag{3.120}
$$

where as usual we fix $\omega = 1$ for convenience. This situation is excluded by the general analysis, which assumes that all the parameters in the solution are different from zero. When (3.120) holds, supersymmetry (2.17) can be realized in two distinct ways

$$
\begin{aligned}
&\text{(i)}: \quad P = 0\,, \quad Q = \pm \frac{M}{1 + \sigma\sqrt{\alpha}}\,, \\
&\text{(ii)}: \quad M = 0\,, \quad P = \pm(1 + \sigma\sqrt{\alpha})\,,
\end{aligned}
\tag{3.121}
$$

where $\sigma = \pm 1$ as usual. These supersymmetry conditions are consistent (in Lorentzian signature) with[26] [21]. Gathering these, the metric functions (3.3) result into

$$
\begin{aligned}
&\text{(i)}: \quad \mathcal{P}(p) = (p^2 - 1)(p^2 - \alpha)\,, \quad \mathcal{Q}(q) = (q^2 - 1)(q^2 - \alpha) - 2Mq - \frac{M^2}{(1 + \sigma\sqrt{\alpha})^2}\,, \\
&\text{(ii)}: \quad \mathcal{P}(p) = (p^2 - 1)(p^2 + 1 + 2\sigma\sqrt{\alpha})\,, \quad \mathcal{Q}(q) = (q^2 + \sigma\sqrt{\alpha})^2 - Q^2\,.
\end{aligned}
\tag{3.122}
$$

Notice that we do not have to take $\alpha > 0$, as long as all the relevant roots are positive. We now analyze separately these cases, which have different features.

(i)

Let us start by considering the case[27] $\alpha > 1$, for which

$$
p_- = 1\,, \quad p_+ = \sqrt{\alpha}\,.
\tag{3.123}
$$

From the quantization conditions (3.26) one obtains

$$
q_+ = \alpha^{1/4} \frac{\sqrt{\sqrt{\alpha}m_- - m_+}}{\sqrt{m_- - \sqrt{\alpha}m_+}}\,, \quad \frac{\Delta}{2\pi} = \frac{1}{m_- - \sqrt{\alpha}m_+}\,,
\tag{3.124}
$$

as well as

$$
M = -(q_+ + \delta)(\delta q_+ + \sigma\sqrt{\alpha})(1 + \sigma\sqrt{\alpha})\,,
\tag{3.125}
$$

with $\delta = \pm 1$ and from (3.48)

$$
\frac{\Delta\theta_1}{2\pi} = \frac{1}{v\left[2q_+^3 - q_+(\alpha + 1) - M\right]}\,.
\tag{3.126}
$$

---

[26]In comparing with the Kerr-Newman-Ads section in [21] attention must be paid to the Wick rotation, which can not be performed without introducing the twisting parameter ω_L before. The correct definitions in [21] are

$$
\varepsilon_L = \omega_L^2 + a^2\,, \quad a^2 \equiv \hat{\alpha}_L - P_L^2\,, \quad \Delta(r) = (r^2 + a^2)(\omega_L^2 + r^2) - 2mr + Q_L^2 + P_L^2\,,
$$

and

$$
\mathcal{B}_\pm^L = m^2 - \left[\omega_L^2 + a^2 \pm 2\sqrt{P_L^2(1 - \omega_L^2) + \hat{\alpha}_L\omega_L^2}\right](P_L^2 + Q_L^2)\,,
$$

where a is the Kerr rotation parameter and in particular $\hat{\alpha}_L > 0$. The Wick rotation to Euclidean signature is accomplished by requiring

$$
(\varepsilon, P, m)_L = (\varepsilon, P, M)_E\,, \quad (\omega, Q)_L = \mathrm{i}(\omega, Q)_E\,, \quad \hat{\alpha}_L = 2P_E^2 - \alpha_E\,.
$$

[27]Both $\alpha = 1$ and $\alpha = 0$ are degenerate cases. For $\alpha = 1$ the function $\mathcal{P}(p)$ is never negative, and for $\alpha = 0$ there is a double root in $p = 0$

Finally, the condition (3.36) reads

$$
t = v \frac{(\sigma + \sqrt{\alpha})(m_- - \sigma m_+)}{(\alpha - 1)(\sqrt{m_- - \sqrt{\alpha} m_+})} \left[2\alpha^{1/4} \sqrt{\sqrt{\alpha} m_- - m_+} + \delta(1 + \sigma \sqrt{\alpha}) \sqrt{m_- - \sqrt{\alpha} m_+} \right] .
\tag{3.127}
$$

Gathering all we then obtain

$$
\frac{1}{2\pi} \int_{\mathcal{L}_2} \mathrm{d}A = \frac{\pm \sigma}{2} \left[\frac{m_- + \sigma m_+}{m_- m_+} + \delta \sigma \frac{t/v}{m_- m_+} \right] ,
\tag{3.128}
$$

that means both twist and anti-twist, again. The case $0 < \alpha < 1$ is completely analogous, with only $p_- \leftrightarrow p_+$. Notice that all the parameters appearing here are real when $\alpha > 1$, so that the metric is real as well. This solution does not have a Lorentzian interpretation, since when $P = 0$ we have from the footnote (26) $0 < \hat{\alpha}_L = -\alpha_E$, but here α_E is necessarily positive (and different from one).

When $\alpha < 0$, the roots are

$$
p_- = -1 , \quad p_+ = 1 ,
\tag{3.129}
$$

so that $t = 0$. Moreover it can be seen now that (3.26) gives

$$
m_- = m_+ \equiv 1 , \quad \frac{\Delta}{2\pi} = \frac{q_+^2 - 1}{(1 - \alpha)} ,
\tag{3.130}
$$

and M and $\Delta\theta_1$ take the same values as before. Notice that $M \in \mathbb{C}$, but this parameter does not appear in the spindle metric (3.49); we are then in the general set-up described at the end of section 3.1. It follows that

$$
\frac{1}{2\pi} \int_{\mathcal{L}_2} \mathrm{d}A = 0 ,
\tag{3.131}
$$

which is the no-twist condition. Notice that here σ does not represent the value of the twist, contrarily to the $\alpha > 1$ case. This solution with no-twist is the Euclidean counterpart of the Kerr-Newman-AdS black hole [21], for which the action is given in [43]. We will confirm this statement at the end of section 4.1, where the on-shell action for the CP solution will be computed.

(ii)

We start by requiring $\mathcal{P}(p) \in \mathbb{R}$, for which we need to take $\alpha > 0$. Notice that when $M = 0$ the Lorentzian solutions always possess a naked singularity [39, 21], as it can be seen from the footnote (26). Then, the following cases have significance only in the Euclidean setting and have not been examined before.

If $\sigma = 1$, then the roots are $p_+ = -p_- = 1$ and hence $t = 0$. From (3.26) and (3.48) we get

$$
m_- = m_+ \equiv 1 , \quad \frac{\Delta}{2\pi} = \frac{q_+^2 - 1}{2(1 + \sqrt{\alpha})} , \quad Q = \delta(q_+^2 + \sqrt{\alpha}) , \quad \frac{\Delta\theta_1}{2\pi} = \frac{1}{2v q_+(q_+^2 + \sqrt{\alpha})} ,
\tag{3.132}
$$

so that

$$\frac{1}{2\pi} \int_{\mathcal{L}_2} \mathrm{d}A = \pm 1 \,. \tag{3.133}$$

This means that we have the topological twist, *i.e.* the twist when the spindle is replaced by S^2. Notice that for $\alpha > 0$ all the parameters are real and this twist solution is consequently real. This solution can be interpreted as the Euclidean counterpart of a Kerr-Newman black hole with topological twist and rotation. In Lorentzian signature it has no meaning, and indeed in minimal gauged supergravity a black hole with twist can have only horizons with the topology of a Riemann surface of genus g > 1 [54]. We shall refer to this solution as to Euclidean "Topological Kerr-Newman-AdS".

When $\sigma = -1$, we can take without loss of generality $\alpha > 1$ so that

$$p_- = 1 \,, \quad p_+ = \sqrt{-1 + 2\sqrt{\alpha}} \,, \tag{3.134}$$

and the quantization conditions (3.26) give

$$q_+ = \frac{\sqrt{m_-(-1 + 2\sqrt{\alpha}) - \sqrt{-1 + 2\sqrt{\alpha}}m_+}}{\sqrt{m_- - \sqrt{-1 + 2\sqrt{\alpha}}m_+}} \,, \quad \frac{\Delta}{2\pi} = \frac{1}{m_- - \sqrt{-1 + 2\sqrt{\alpha}}m_+} \,. \tag{3.135}$$

The other conditions (3.48) and (3.36) are solved by

$$Q = \delta(q_+^2 - \sqrt{\alpha}) \,, \quad \frac{\Delta\theta_1}{2\pi} = \frac{1}{2q_+ v(q_+^2 - \sqrt{\alpha})} \,, \quad t = vq_+ \frac{m_- + \sqrt{-1 + 2\sqrt{\alpha}}m_+}{\sqrt{-1 + 2\sqrt{\alpha}}} \,, \tag{3.136}$$

so that

$$\frac{1}{2\pi} \int_{\mathcal{L}_2} \mathrm{d}A = \mp \frac{1}{2} \left[\frac{m_- - m_+}{m_- m_+} \pm \delta \frac{t/v}{m_- m_+} \right] \,. \tag{3.137}$$

For $\alpha > 0$ the solution is again real. Notice that in principle it seems possible to define an "extremal limit" of this solution by imposing $Q = 0$, such that $\mathcal{Q}(q) = (q_+^2 - \sqrt{\alpha})^2$ and $q_+ = \alpha^{1/4}$. However this constraint can be solved only if $\alpha = 1$, which is excluded by the requirement $p_+ \neq p_-$.

Summarizing this section, we have that both $t = 0$ and $t \neq 0$ are possible for the values of the parameters in both (i) and (ii), Moreover the solutions are always real, with the only exception of the no-twist case in (ii), where $t = 0$ and the spindle is replaced by S^2. In appendix C we show that, when $t \neq 0$, only $t \geq 4$ is possible. Notice also that the case (ii) contains the self-dual solution [13]. Indeed $N = M = 0$, and requiring $P = Q$ implies $\sqrt{\alpha} = q_+ = 1$, which is degenerate because the bolt disappears and the topology is the one of AdS$_4$ [13].

3.6 Limits to old solutions

As anticipated, our solution (3.1)-(3.3) generalizes in various ways the ones presented in [15, 16]. In this section we study the limits which reproduce those results.

$U(1) \times U(1)$-**invariant solution of [16]**

Let us start from the case [16], which is simpler to be recovered since the local form of the solution is exactly the same. The main difference from this reference is that therein the regularity led to $P = Q$, $N = M$ and $q_+ = p_+$ for which the boundary is topologically a three-sphere and the bulk contain a nut-type singularity in $q_+ = p_+$. Moreover, exactly when $N = M$ and $P = Q$ the Weyl tensor is self-dual and F is an instanton. The condition $q_+ = p_+$, in terms of the parametrization (3.102), reads

$$q_+ \to p_+ \iff \tilde{q}_+ \to 1 + x \,. \tag{3.138}$$

Moreover, in this limit q_+ and p_+ must be zeros of the same type of function (3.6), which means

$$\eta = \lambda \implies \kappa = 1 \,, \tag{3.139}$$

Finally, since the spindle bolt disappears, we require

$$m_- - m_+ \to 0 \,. \tag{3.140}$$

Notice that in this degenerate situation, contrarily to the general case, $\text{Re}(\eta P) < 0$ due to (3.27) supplemented with (3.15). With these requirements, it can be seen that $N - M \propto P - Q \to 0$, as expected. For future reference, we write also the value of the fibration parameter

$$\begin{aligned} \sigma = 1 : & \quad t \to 0 \,, \\ \sigma = -1 : & \quad t \to \frac{-2}{x} v \,. \end{aligned} \tag{3.141}$$

Clearly in this limit the interpretation of t breaks down (and the negative sign is an artefact due to $\text{Re}(\eta P)$ not being positive anymore). In particular, the Type I and Type II solutions of [16] have boundary topology of S^3, for which more properly $t = 1$ in both cases.

$SU(2) \times U(1)$-**invariant solution of [15]**

Now we move to the $SU(2) \times U(1)$ invariant solution presented in [15] and summarized in our conventions in appendix A. In the spirit of [30] (see appendix C therein), we construct a scaling limit under which (3.1) and (3.2) become (A.1) and (A.2), at least locally. Notice that in our case the solution is not (anti) self-dual, so that [30] must be extended. The limit is obtained for $\epsilon \to 0^+$ on the coordinates

$$p = s(1 - \epsilon x \cos \hat{\theta}) \,, \quad \tau = s \left[2\hat{\psi} + \frac{\hat{\varphi}}{\epsilon x} \right] \,, \quad \sigma = \frac{\hat{\varphi}}{s} \frac{1}{\epsilon x} \,, \tag{3.142}$$

as well as on the parameters

$$N = -s(4s^2 - 1) \,, \quad E = 1 - 6s^2 \,, \quad \alpha = \hat{P}^2 - s^2 \left[3s^2 - 1 + \epsilon^2 x^2 \right] \,, \tag{3.143}$$

where we employed the parametrization (3.102) for the roots and $\hat{\theta} \in [0, \pi]$. Moreover, we identify

$$q = r \,, \quad (M, P, Q) = (\hat{M}, \hat{P}, \hat{Q}) \,, \quad w = s \,, \tag{3.144}$$

where s is the squashing parameter as in (A.5). In the limit, the BPS conditions in (3.5) boil down to (A.6), as expected. Notice that under this scaling limit we get $p_+ \to p_-$, which is in a certain sense the complementary situation to $q_+ \to p_+$ of the previous subsection. As $p_+ \to p_-$, the spindle degenerates and the parametrization variable w is unconstrained, since $m_+/m_- = 1 + O(\epsilon)$. However, the limit procedure instructs us that $w = s + O(\epsilon)$, and applying this on (3.104) one obtains

$$\sigma = 1: \quad w = s + O(\epsilon) \iff m_\pm = \frac{1}{2(2s^2 + \eta\tilde{P})} \mp \frac{s^2(r_0^2 + s^2 + \eta\tilde{P})}{(r_0^2 - s^2)(2s^2 + \eta\tilde{P})^2}\,\epsilon x + O(\epsilon^2)\,. \tag{3.145}$$

Even if it looks odd, using the 1/4-BPS conditions (A.6) the first term is exactly 1, as expected for a spindle which becomes a sphere. This expression makes (3.105) infinite

$$\Delta = \frac{2\pi(r_0^2 - s^2)}{s}\frac{1}{\epsilon x} + O(\epsilon)\,, \tag{3.146}$$

in the precise way such that $\mathrm{vol}(\Sigma_{q_+}) < \infty$ and (3.111) coincides with (4.7) in [15]. Plugging the values (3.145) in (3.106) and inverting the parametrization (3.102) to

$$\sigma = 1: \quad \tilde{P} = \frac{P}{2s}\,, \quad \tilde{q}_+ = \frac{r_0}{s}\,, \tag{3.147}$$

we obtain correctly that (3.106) boils down to $t = v\hat{t} + O(\epsilon)$, respectively, for the appropriate class of regularity conditions (A.8) with $\kappa = \pm 1$. The procedure can be now repeated also for the anti-twist case, for which

$$\sigma = -1: \quad m_\pm = 1 \mp \frac{2s\big[s + (r_0^2 - s^2)(2s \mp \sqrt{4s^2 - 1})\big]}{r_0^2 - s^2}\,\epsilon x + O(\epsilon^2)\,, \tag{3.148}$$

where we already plugged (A.6), and with

$$\sigma = -1: \quad \overline{P} = -\frac{P(-2s^2\epsilon x + \eta P)}{4s^3}\,, \quad \tilde{q}_+ = \frac{r_0}{s}\,, \tag{3.149}$$

one gets again $w = s + O(\epsilon)$ and $t = v\hat{t} + O(\epsilon)$.

Finally, let us comment on the relation with the supersymmetric nuts of [15]. These must correspond to the even more degenerate case in which p_-, p_+ and q_+ coincide, thus finishing all the possibilities. We shall refrain to construct explicitly this limit, since it was already observed in [16] that the supersymmetric 1/2 and 1/4-BPS nuts of [15] are subcases of the nuts in [16]. In addition, the nuts of [15] must come from a limit in which $r_0 \to s$ on the Bolts solutions, so that the function $\Omega(r)$ in (A.3) has a double root in $r_0 = s$ and the nut singularity becomes "naked". Therefore, one can think to perform the aforementioned limit in two steps, moving q_+ to p_+ before, and collapsing p_- on p_+ subsequently. In the next section we will compute the on-shell action for our solution, showing later that it reproduces the Type I and 1/4-BPS on-shell actions for $\sigma = 1$, and Type II and 1/2-BPS for $\sigma = -1$.

4 On-shell action

In this section we compute the renormalized on-shell action for the solutions (3.1)-(3.3). At this point it should be clear that the twist case $\sigma = 1$ discussed in section 3.5 is simpler, and indeed we will be able to express the on-shell action in a compact form, depending only on $(t, v, m_\pm)$. Even if the solution has two free continuous parameters $(\tilde{P}, \tilde{q}_+) \in \mathbb{R}$, these do not appear in the action. It is the same phenomenon already observed in the Type I solutions of [16] and in the 1/4-BPS Bolt solutions of [15], which we summarized in section 2.2. Different is the case for the anti-twist $\sigma = -1$, where the action will be a function of one (of the two) free real parameters, $\tilde{q}_+$. Again, this is familiar from the old Type II [16], 1/2-BPS [15] and black hole [6] solutions. Using the results of section 3.6, we will show that the on-shell action computed here reduces to (2.24) and (2.26) in the appropriate limit. We will then demonstrate how to reproduce our result starting from the localization of the action [31, 32], thus furnishing a non-trivial check of the validity of the method. We will conclude this section with an intriguing observation about the extremization of the action.

4.1 Holographic renormalization

The computation is a standard procedure in literature [55, 56, 57] so we will be as brief as possible. It consists in adding the standard boundary term and the counter-terms to remove the singularities that appears as $q \to \infty$. In particular we consider the renormalized action

$$S_{\text{ren}} = S_{E\Lambda} + S_F + S_{GH} + S_{\text{ct}} \,, \tag{4.1}$$

evaluated on the solution (3.1) and (3.2), where the bulk contributions are

$$S_{E\Lambda} = \frac{-1}{16\pi G_4} \int \mathrm{d}^4 x \sqrt{g} \left(R^{(g)} + 6 \right) , \quad S_F = \frac{-1}{16\pi G_4} \int \mathrm{d}^4 x \sqrt{g} \left(F^2 \right) , \tag{4.2}$$

whilst the boundary ones are

$$S_{GH} = \frac{-1}{8\pi G_4} \int_{q=q_0} \mathrm{d}^3 x \sqrt{\gamma} \left(\gamma^{\mu\nu} K^{(\gamma)}_{\mu\nu} \right) , \quad S_{ct} = \frac{1}{8\pi G_4} \int_{q=q_0} \mathrm{d}^3 x \sqrt{\gamma} \left(2 + \frac{1}{2} R^{(\gamma)} \right) . \tag{4.3}$$

The hypersurface $S_0 = \{q = q_0\}$ with induced metric γ_{ij} acts as a cut-off, and in the limit $q_0 \to +\infty$ the total action should be finite. Here we have introduced the second fundamental form $2K_{\mu\nu} \equiv (\mathcal{L}_n^{(g)} \gamma)_{\mu\nu}$, with n^μ a unit vector normal to S_0 and such that $g_{\mu\nu} = n_\mu n_\nu + \gamma_{\mu\nu}$. In the case at hand, we have

$$n = \sqrt{\frac{\mathcal{Q}(q)}{q^2 - p^2}} \, \partial_q \,, \quad K^{(\gamma)} = \nabla_\mu^{(g)} n^\mu = \frac{2q\mathcal{Q}(q) + (q^2 - p^2)\mathcal{Q}'(q)}{2(q^2 - p^2)^{\frac{3}{2}} \sqrt{\mathcal{Q}(q)}} \,. \tag{4.4}$$

Whilst the contribution of the gauge field in (4.2) is already finite

$$S_F = (16\pi G_4)\Delta\tau\Delta\sigma(p_+ - p_-)q_+ \left[\frac{(P-Q)^2}{(q_+ - p_+)(q_+ - p_-)} + \frac{(P+Q)^2}{(q_+ + p_+)(q_+ + p_-)} \right] , \tag{4.5}$$

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the Einstein$-\Lambda$ term is indeed divergent in the limit $q_0 \to \infty$

$$S_{E\Lambda} = (32\pi G_4)\Delta\tau\Delta\sigma \int_{p_-}^{p_+} \mathrm{d}p \left[q_0^3 - q_+^3 + 3(q_+ - q_0)p^2 \right].$$ (4.6)

Moreover

$$S_{GH} + S_{ct} = (32\pi G_4)\Delta\tau\Delta\sigma \int_{p_-}^{p_+} \mathrm{d}p \left[-q_0^3 + 3q_0 p^2 + M \right],$$ (4.7)

so that the terms containing q_0 cancel. Then the renormalized on-shell action does not display infinities and reads

$$S_{\mathrm{ren}} = \frac{\Delta\tau\Delta\sigma(p_+ - p_-)q_+}{8\pi G_4}\left[p_+^2 + p_+ p_- + p_-^2 + \frac{M}{q_+} - q_+^2 \right.$$
$$\left. + \frac{(P-Q)^2}{2(q_+ - p_+)(q_+ - p_-)} + \frac{(P+Q)^2}{2(q_+ + p_+)(q_+ + p_-)} \right],$$ (4.8)

where recall that the product $\Delta\tau\Delta\sigma$ is equal in the patches $U_\pm$ and is given by (3.37). Let us anticipate that in the limit $P \to Q$ and $q_+ \to p_+$ the action (4.8) returns to that of [16] (see (5.7) therein), thus confirming our expectations.

For the twist case ($\sigma = 1$), employing all the results of section 3.5, we can express the renormalized action simply as

$$\text{twist}: \quad S_{\mathrm{ren}} = \frac{\pi}{8G_4 v}\left[2\chi_\Sigma - \kappa\frac{t/v}{m_- m_+} - \kappa\frac{v(m_- - m_+)^2}{t\, m_- m_+} \right], \quad t \geq 2.$$ (4.9)

In particular, notice that the on-shell action is fixed in terms of the integers $(t, v, m_\pm)$ and $(\tilde{P}, \tilde{q}_+)$ do not appear. From this expression we can easily recover the Type I (2.24) and 1/4-BPS (2.26) renormalized actions using with the limits explained in section 3.6. For the Type I case, $m_\pm \to 1$ and $t \to 0$ from (3.141), for which only the first contribution in (4.9) survives. For the other case $m_\pm \to 1 + O(\epsilon)$ from (3.145) but $t = v\hat{t} + O(\epsilon)$ remains non-zero. Thus only the last term in (4.9) vanishes. From this point of view it is not a case that the two actions in the first lines of (2.24), (2.26) coincide for $\hat{t} = 0$. Notice that it is still left a factor v^{-1} in front of the action, which generalizes the old results by the presence of a transverse singularity to the bolt.

The $\sigma = -1$ is, as usual, more involved. Looking explicitly at (4.8), it proves useful to define

$$\sigma = -1: \quad \mathfrak{q} = \frac{q_+^2 - p_+^2}{p_+^2 - p_-^2}\frac{(p_+ - p_-)\left[\eta P + p_-(p_+ + p_-)\right]}{\left[(p_+ - p_-)\eta P - (p_+ + p_-)(q_+^2 - p_+ p_-)\right]}$$
$$= \frac{\left[\tilde{q}_+^2 - (1+x)^2\right]\left[w(x-1) - \eta\tilde{P}\right]}{2w(\tilde{q}_+^2 + x^2 - 1) - 4x\eta\tilde{P}}.$$ (4.10)

Notice that (4.10) seems to depend also on $\overline{P}$, but if one solves for $\tilde{P}$ and w through (3.115) and (3.116), $\overline{P}$ disappears. Recall also that $x = x(\tilde{q}_+)$ is fixed (non-explicitly) via (3.117) which in this context reads

$$\sigma = -1: \quad t = v\left[(\tilde{q}_+^2 - (x+1)^2)m_- - (\tilde{q}_+^2 - (x-1)^2)m_+\right] \times$$
$$\frac{-\tilde{q}_+^3 + \tilde{q}_+\left[1 + 2(1 + 2\mathfrak{q})x + x^2\right] + \kappa\left[\tilde{q}_+^2 - (1+x)^2 + 2\mathfrak{q}(-1 + \tilde{q}_+^2 + x^2)\right]}{x\left[1 - \tilde{q}_+^2 + 2(1 + 2\mathfrak{q})x + x^2\right]^2}.$$ (4.11)

Therefore, as $x(\tilde{q}_+)$, also $\mathfrak{q} = \mathfrak{q}(\tilde{q}_+)$. The renormalized action for $\sigma = -1$ can then be written as a function of $\tilde{q}_+$ only

$$\sigma = -1: \quad S_{\text{ren}}(\tilde{q}_+) = \frac{\pi\left[(\tilde{q}_+^2 - (x+1)^2)m_- - (\tilde{q}_+^2 - (x-1)^2)m_+\right]^2}{8G_4 t\, m_- m_+} \times$$
$$\frac{2\tilde{q}_+(1+x+2\mathfrak{q})x - \kappa\left[\tilde{q}_+^2 + (1+x)^2 + 2\mathfrak{q}(1+\tilde{q}_+^2 + x^2)\right]}{x^2\left[1 - \tilde{q}_+^2 + 2(1+2\mathfrak{q})x + x^2\right]^2}. \tag{4.12}$$

We see that in the anti-twist case S_{ren} not only depends on the choice of the integers $(t, v, m_\pm)$, but contains a real degree of freedom $\tilde{q}_+$ as well. Again, we can obtain the Type II (2.24) and 1/2-BPS (2.26) with the limits of section 3.6. With $t = -2v/x$ from (3.141) and $m_\pm \to 1$, we get

$$\text{Type II}: \quad S_{\text{ren}} = \frac{\pi}{2G_4} \frac{1}{1 - 4[Q/(p_+ + p_-)^2]^2}. \tag{4.13}$$

This is equivalent to the one presented in (2.24) when $p_+ + p_- = 1$, a condition fixed by the authors of [16] taking advantage of the scaling symmetry (2.10). Here we have restored that factor, since $p_+ + p_- \neq 1$ for us. The limit to the 1/2-BPS action in more complicated, since in this limit $\mathfrak{q}$ is infinite. However, $\mathfrak{q}/m_\pm$ is finite and using (3.148) and $t = v\hat{t} + O(\epsilon)$ we obtain indeed the second line of (2.26).

We point out that the 1/2-BPS and 1/4-BPS nuts of [15] are subcases of the ones in [16]. The action of the 1/4-BPS nut in (2.28) is already equal to the one of Type I in (2.24). For 1/2-BPS, we can rewrite the Type II action (4.13) using the parametrization (3.102) as

$$\text{Type II}: \quad S_{\text{ren}} = \frac{2\pi}{G_4} \frac{w^4}{4w^4 - P^2}, \tag{4.14}$$

where recall that $Q = P$ for Type I and Type II. Then, since $w = s$ and $P = P(s)$ from (3.144) and (A.6), we reproduce exactly the second line of (2.28).

We conclude by writing (some) of the on-shell actions for the solutions with $N = 0$ in section 3.5.3. We have

$$\text{(i) with twist}: \quad S_{\text{ren}} = \frac{\pi}{8G_4 v}\left[2\chi_\Sigma + \delta\frac{t/v}{m_- m_+} + \delta\frac{v(m_- - m_+)^2}{t\, m_- m_+}\right], \qquad t \geq 4,$$

$$\text{(i) with anti-twist}: \quad S_{\text{ren}} = \frac{\pi(1 + \delta q_+)^2}{G_4 v\left[1 + 2\delta q_+(1 - \sigma\sqrt{\alpha}) - \alpha\right]}, \qquad t = 0,$$

$$\text{(ii) with twist} \quad S_{\text{ren}} = \frac{\pi}{2G_4 v}, \qquad t = 0,$$
$$\tag{4.15}$$

The anti-twist actions in (i) and (ii) with $t \neq 0$ are as usual very complicated, and we do not report them here. Some comments are now in order. As anticipated, the action in the second line refers to the Euclidean counterpart of the Kerr-Newman-AdS black hole [21, 43]. Indeed it can be rewritten in the form

$$S_{\text{ren}} = \pm\frac{1}{2G_4 v}\frac{\varphi^2}{z}, \quad \varphi - \frac{1}{2}z = \pm\pi, \tag{4.16}$$

where we have introduced

$$\varphi = \pm 2\pi \frac{(1 + \delta q_+)}{1 - \sigma\sqrt{\alpha}}, \quad z = \pm 2\pi \frac{\left[1 + 2\delta q_+(1 - \sigma\sqrt{\alpha}) - \alpha\right]}{(1 - \sigma\sqrt{\alpha})^2}. \tag{4.17}$$

This is a particular case of the general action for the accelerating black hole[28] (2.31). When $m_- = m_+$ the magnetic charge of the spherical black hole is zero, *i.e.* $Q_m = 0$, and $\chi(S^2) = 2$, so that the second term in (2.31) is absent. Finally, the action of the Euclidean "Topological Kerr-Newman-AdS" solution in the last line of (4.15) agrees with both the Type I and 1/4-BPS actions (when $\hat{t} = 0$ and $v = 1$) in (2.24), (2.26). Moreover, it is also compatible with the g = 0 case of a topologically twisted black hole with a Σ_g horizon [54]. In Lorentzian signature the solution with g = 0 does not exist, as explained at the end of section 3.5.3, but it is perfectly well-defined in Euclidean.

4.2 Equivariant localization

We now want to recover these values of the on-shell actions using the results of [31, 32], with reference to figure 2a, normal vectors (3.94) and toric weights (3.96). To this end, we elaborate briefly on the general setting, rewriting the main result of [31] in a way which is more useful for our purposes. Moreover, we will make a connection with the gravitational blocks for the supersymmetric black holes proposed in [3], which correspond to figure 2b and $t = 0$.

In [31] the action is computed on-shell by using the fact that for supersymmetric solutions with a supersymmetric Killing vector $\vec{\epsilon} = (\epsilon_1, \epsilon_2)$ the bulk metric can be written as a $U(1)$ circle fibration generated by $\vec{\epsilon}$ over a non trivial three-dimensional base. The action reduced on this base generalizes [10] by the presence of a gauge field and is then evaluated explicitly employing supersymmetry and the bilinears constructed on top of it. The final formula turns out to be dependent only on quantities at the fixed points of $\vec{\epsilon}$, which resembles the results of equivariant localization, *i.e.* the Duistermaat-Heckman, Berline-Vergne or Atiyah-Bott theorems [58, 59, 60]. In fact, the procedure presented in [32, 61] is based on the equivariant integration of a certain equivariant closed poly-form Φ with respect to the supersymmetric Killing vector[29]

$$\epsilon = \varepsilon^\dagger \Gamma^\mu \Gamma_\star \varepsilon \, \partial_\mu, \tag{4.18}$$

that is $(\mathrm{d} - i_{\vec{\epsilon}})\Phi = 0$, and depends on the chirality of $\vec{\epsilon}$ at the fixed points as well. In particular, the "off-shell" action (*i.e.* the action for a generic "Reeb" vector $\vec{\epsilon}$) is computed as

$$I_{\text{off-shell}}(c_A, \vec{\epsilon}) = \frac{\pi}{2G_4} \left\{ \sum_{A=1}^{2} -c_A \frac{\left[b_1^{(A)} - c_A b_2^{(A)}\right]^2}{4 \, d_A \, b_1^{(A)} b_2^{(A)}} \right\}, \tag{4.19}$$

where c_A are signs linked to the chirality of the spinor at the fixed point p_A (3.88), on which the sum runs, and $b_i^{(A)}$ are the weights of $\vec{\epsilon}$ on each copy of $\mathbb{C}_i \subset T_{p_A}\mathcal{M}_4$. Since, more precisely, the tangent space at the fixed points is not just $\mathbb{C} \oplus \mathbb{C}$ but there are orbifolds

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[28]Notice the absence of the i factors with respect to (2.31) or [43]. This is due to the fact that here we are working in Euclidean signature.

[29]We use the the notation $\vec{\epsilon}$ when the supersymmetric Killing vector is written in an effective toric basis $\vec{E}_i$, so that in this basis $\epsilon = \vec{\epsilon} \cdot \vec{E}_i$.

singularities, a factor of d_A (the order of the singularity at p_A) must be included [33], so that now (4.19) holds also for orbifold solutions. For the toric case, the weights of $\vec{\epsilon}$ can be simply expressed in terms of the toric weights (3.96) as [31]

$$b_{1,2}^{(A)} \equiv \vec{\epsilon} \cdot \vec{\mu}_{1,2}^{(A)} , \qquad (4.20)$$

where for our case they are given by (3.96).

A word of caution is needed here: our solutions are "bolt" in the sense that K_{q_+} introduced in section 3.2.2 has a two-dimensional fixed point set $S_+ = \{q = q_+\}$, but are "two-nuts" solutions with respect to the supersymmetric Killing vector which degenerates only in $(q_+, p_\pm)$ as we will see. Therefore we use the "nuts" result in [32], which is indeed (4.19). Moreover (4.19) is exactly the result found previously in [30] in a different way for (anti-)self-dual solutions with the topology of the four-ball, *i.e.* with a single nut. There it is shown that the bulk is regular only if $\epsilon_2/\epsilon_1 > 0$ or $\epsilon_2 = -\epsilon_1$, and these two distinct behaviours will play a role later on. We note that at the moment it is not known if, for a generic choice of $\vec{\epsilon}$, there exists a supergravity solution with such a specific vector. However, if it exists, its renormalized on-shell action should be given by (4.19). We will come back to this observation in 4.4.

In [32] it is showed that (4.19), when computed on the supersymmetric Killing vector $\vec{\epsilon}_*$, reproduces the supergravity computation for the renormalized action, that is

$$I_{\text{on-shell}} \equiv I_{\text{off-shell}}(c_A, \vec{\epsilon}_*) = S_{\text{ren}} , \qquad (4.21)$$

in our language. To show this explicitly in our case, let us present a more convenient expression for $I_{\text{off-shell}}$ in the toric case, when (4.20) holds. Since for a generic set-up of two normal vectors $\vec{v}_{1,2}$, the weights can be written as in (3.95), it turns out that (4.19) can be rewritten in a symmetric form as

$$I_{\text{off-shell}}(c_A, \mathrm{Q}) = \frac{\pi}{8G_4} \left[2 \left[\frac{1}{d_{1,2}} + \frac{1}{d_{2,3}} \right] - \frac{c_1}{d_{1,2}} \left[\mathrm{Q}^2 + \frac{1}{\mathrm{Q}^2} \right] - \frac{c_2}{d_{2,3}} \left[\tilde{\mathrm{Q}}^2 + \frac{1}{\tilde{\mathrm{Q}}^2} \right] \right] , \qquad (4.22)$$

where $d_{a,b} = \det(\vec{v}_a, \vec{v}_b)$ and we defined the "off-shell" quantities

$$\mathrm{Q}^2 = -\frac{\det(\vec{\epsilon}, \vec{v}_1)}{\det(\vec{\epsilon}, \vec{v}_2)} , \quad \tilde{\mathrm{Q}}^2 = -\frac{\det(\vec{\epsilon}, \vec{v}_3)}{\det(\vec{\epsilon}, \vec{v}_2)} . \qquad (4.23)$$

Notice that (4.22) can be also written as

$$I_{\text{off-shell}}(c_A, \mathrm{Q}) = -\frac{\pi}{8G_4} \left[\frac{c_1}{d_{1,2}} \left(\mathrm{Q} - \frac{c_1}{\mathrm{Q}} \right)^2 + \frac{c_2}{d_{2,3}} \left(\tilde{\mathrm{Q}} - \frac{c_2}{\tilde{\mathrm{Q}}} \right)^2 \right] . \qquad (4.24)$$

and that in general the action can be complex. The formula for the localized action (4.19) is invariant under a generic $SL(2, \mathbb{Z})$ transformation of the basis of the torus. This is evident from (4.22), which depends only on the $SL(2, \mathbb{Z})$-invariant determinants contained in (4.23). It is also independent on the overall normalization of the vector $\vec{\epsilon}$. Moreover these quantities are not independent, since

$$d_{2,3} \mathrm{Q}^2 + d_{1,2} \tilde{\mathrm{Q}}^2 = -d_{1,3} , \qquad (4.25)$$

where we used that for a generic vector $\vec{v}$ the following four-vectors identity holds

$$d_{1,2}d_{v,3} + d_{2,3}d_{v,1} = d_{1,3}d_{v,2} \,. \qquad (4.26)$$

Specifically, it is clear that in general the on-shell action depends on the supersymmetric Killing vector $\vec{\epsilon} = (\epsilon_1, \epsilon_2)$ only through the combination $z \equiv \epsilon_2/\epsilon_1$. To make this dependence explicit it is useful to insert the toric data (3.94). After defining the (a priori complex) variables

$$\beta_+^2 \equiv \frac{1}{v}\left(\frac{m_+}{z} + r_-\right), \qquad \beta_-^2 \equiv \frac{1}{v}\left(\frac{m_-}{z} - r_+\right), \qquad (4.27)$$

the action takes the suggestive form

$$I_{\text{off-shell}}(\sigma, z) = -\frac{c_2\pi}{8G_4 v}\left[\frac{1}{m_-}\left(\beta_- - \frac{c_2}{\beta_-}\right)^2 - \frac{\sigma}{m_+}\left(\beta_+ + \frac{\sigma c_2}{\beta_+}\right)^2\right], \qquad (4.28)$$

where $c_1 c_2 = \sigma$ and

$$m_-\beta_+^2 - m_+\beta_-^2 = \frac{t}{v} \,. \qquad (4.29)$$

In the next section we will indeed show that the on-shell actions of our solutions, for twist or anti-twist cases, take precisely this form. In addition, in (4.28) the dependence on z is particularly simple and we expect that this form should be suitable for comparisons with large N calculations in the dual field theories. Next, let us consider the case $t = 0$, that is relevant for supersymmetric black holes. This corresponds to the rectangular polytope in figure 2b, and may be obtained from (4.28) simply setting $r_\pm = 0 = t$. The resulting expressions for the twist and anti-twist read

$$\text{anti-twist}: \quad I_{\text{off-shell}}^{\text{anti-twist}}(z) = -\frac{c_2\pi}{8G_4}\left[\left(\frac{\sqrt{z}}{m_-} - \frac{c_2}{v\sqrt{z}}\right)^2 + \left(\frac{\sqrt{z}}{m_+} - \frac{c_2}{v\sqrt{z}}\right)^2\right], \quad t = 0, \quad (4.30)$$

$$\text{twist}: \quad I_{\text{off-shell}}^{\text{twist}}(z) = \frac{\pi\chi_\Sigma}{8G_4}\left[\frac{2}{v} + c_2\left(\frac{1}{m_+} - \frac{1}{m_-}\right)z\right], \qquad t = 0. \quad (4.31)$$

It is then straightforward to see that the two expressions above reproduce the gravitational block form of the black hole entropy functions for minimal $d = 4$ gauged supergravity, proposed[30] in [3]. Indeed, defining

$$\mathfrak{n} = \frac{m_+ + \sigma m_-}{4m_- m_+}, \qquad 2\varphi - \frac{m_+ - \sigma m_-}{2m_- m_+}z = \frac{1}{v}, \qquad (4.32)$$

we have[31]

$$I_{\text{off-shell}}^{\text{anti-twist}}(z) = F^+(\varphi, z; \mathfrak{n}) = \frac{\pi}{G_4}\left[\frac{\varphi^2}{z} + \mathfrak{n}^2 z\right], \quad I_{\text{off-shell}}^{\text{twist}}(z) = F^-(\varphi, z; \mathfrak{n}) = \frac{\pi\mathfrak{n}\varphi}{G_4}2, \quad (4.33)$$

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[30]These expressions have been proved in supergravity using different methods, considering near-horizon geometries with a spindle factor [62, 63, 53, 64]. See also [65] for a discussion in the context of higher derivative supergravity. In the dual field theories these are reproduced by the large N limit of the spindle index [9].

[31]We have adapted the overall normalization and inserted a v factor, to compare to our results. Moreover in both cases $c_2 = -1$.

with

$$F^{-\sigma}(\varphi, z; \mathfrak{n}) = \frac{1}{z}\left[\mathcal{F}_3(\varphi + \mathfrak{n}z) - \sigma\mathcal{F}_3(\varphi - \mathfrak{n}z)\right],\tag{4.34}$$

where $\mathcal{F}_3$ is proportional to the S^3 off-shell free energy of the ABJM theory (*cf.* Table 2 in [3]) and $\sigma = \pm 1$ is as usual for the twist or the anti-twist, respectively. Notice that for the twist on the sphere, with $c_2 = c_1$ and $m_- = m_+ = 1$, the on-shell action does not depend any more on the supersymmetric Killing vector, and the result is simply $I = \pi/(2vG_4)$.

4.3 Comparison

We will now compare the general results obtained from localization with the gravitational on-shell actions (4.9) and (4.12) computed performing holographic renormalization. Again we will refer to the polytopes of figure 2, normal vectors (3.94) and toric weights (3.96). From the explicit Killing spinors (3.12) we obtain the supersymmetric Killing vector[32]

$$\epsilon_* = \sqrt{\alpha}\partial_\tau + \partial_\sigma\,.\tag{4.35}$$

Notice that this degenerates only at the points $(q_+, p_\pm)$ (as visible from (3.19)), so that the fixed points p_A are "nuts" for ϵ_*. From (3.60) and (3.61) we recall that

$$\sigma = 1: \quad \begin{aligned} \Gamma_\star\varepsilon &= \kappa\varepsilon \text{ at } p_1\,, \\ \Gamma_\star\varepsilon &= \kappa\varepsilon \text{ at } p_2\,, \end{aligned}\tag{4.36}$$

and

$$\sigma = -1: \quad \begin{aligned} \Gamma_\star\varepsilon &= -\kappa\varepsilon \quad \text{at } p_1\,, \\ \Gamma_\star\varepsilon &= \kappa\varepsilon \quad \text{at } p_2\,, \end{aligned}\tag{4.37}$$

where the chirality matrix has been defined in (2.16). The chirality itself is fixed by the value of $\kappa = \pm 1$, so that the correct assignment for the signs c_A is

$$\begin{aligned} \sigma = 1: &\quad c_2 = c_1 = \kappa\,, \\ \sigma = -1: &\quad c_2 = -c_1 = \kappa\,. \end{aligned}\tag{4.38}$$

That the spinor must be chiral at the fixed points comes from general arguments, in particular as a consequence of the bilinear $\varepsilon^\dagger\varepsilon$ being nowhere vanishing [4, 32]. In the coordinates (3.91), which recall are written in the patch U_-, the supersymmetric Killing vector (4.35) reads

$$\vec{\epsilon}_* = \frac{1}{q_+^2 - p_-^2}\left\{\left[(p_-^2 + \sqrt{\alpha})\frac{2\pi}{\Delta\theta_1} + \frac{r_+}{m_-}(q_+^2 + \sqrt{\alpha})\frac{2\pi}{\Delta\theta_2^-}\right]E_1 + (q_+^2 + \sqrt{\alpha})\frac{2\pi}{\Delta\theta_2^-}E_2\right\}.\tag{4.39}$$

It is now a matter of algebra to plug explicitly the vector $\vec{\epsilon}_*$ (4.39) in (4.22) by using (4.23). After some work, using the Bézout's lemma (3.92) and eliminating everywhere

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[32]We have normalized differently the spinor ε than before, but the overall normalization will not be important since the action (4.28) depends only on z.

$\Delta\theta_1$ from (4.39) in favour of the fibration parameter t using (3.36), the result takes the form

$$I_{\text{on-shell}}(c_A, \mathsf{q}_*) = \frac{\pi}{8G_4 v}\left[2\chi_\Sigma - \frac{c_1}{m_+}\left[\frac{\mathsf{q}_*^2 t}{vm_-} + \frac{vm_-}{\mathsf{q}_*^2 t}\right] - \frac{c_2}{m_-}\left[\frac{\tilde{\mathsf{q}}_*^2 t}{vm_+} + \frac{vm_+}{\tilde{\mathsf{q}}_*^2 t}\right]\right], \qquad (4.40)$$

where we have defined the on-shell reduced quantities

$$\tilde{\mathsf{q}}_*^2 = \frac{q_+^2 - p_+^2}{p_+^2 - p_-^2}\frac{p_-^2 + \sqrt{\alpha}}{q_+^2 + \sqrt{\alpha}}, \quad \mathsf{q}_*^2 = -\frac{q_+^2 - p_-^2}{p_+^2 - p_-^2}\frac{p_+^2 + \sqrt{\alpha}}{q_+^2 + \sqrt{\alpha}}, \quad \tilde{\mathsf{q}}_*^2 + \mathsf{q}_*^2 = -1. \qquad (4.41)$$

such that $(Q_*^2, \tilde{Q}_*^2) = (t/vm_- m_+)(m_+\mathsf{q}_*^2, m_-\tilde{\mathsf{q}}_*^2)$. As a consequence of the reality discussion at the end of section 3.1, the parameter $\sqrt{\alpha}$ here can be complex and in turn the action itself can be complex. It should be no surprise at this point that the equal-signs case $(c_2 = c_1 = \kappa)$ is particularly easy to handle. From the point of view of the equivariant localization, this simplification comes from $\tilde{\mathsf{q}}_*^2 + \mathsf{q}_*^2 = -1$ which enters directly in (4.40). Indeed we have

$$\text{twist}: \quad (\mathsf{q}_*^2, \tilde{\mathsf{q}}_*^2) = \frac{1}{m_- - m_+}(m_-, -m_+), \qquad (4.42)$$

and as a consequence

$$\text{twist}: \quad I_{\text{on-shell}}(c_2, \mathsf{q}_*) = \frac{\pi}{8G_4 v}\left[2\chi_\Sigma - c_2\frac{t/v}{m_- m_+} - c_2\frac{v(m_- - m_+)^2}{t\, m_- m_+}\right], \quad t \geq 2, \quad (4.43)$$

which reproduces precisely (4.9) for $c_2 = \kappa$ as evident from (4.36) and (4.38). Correspondingly, the Reeb vector (4.39) is fixed

$$\text{twist}: \quad \vec{\epsilon}_* = f(\tilde{P}, \tilde{q}_+)\left[(r_+ + r_-)E_1 + (m_- - m_+)E_2\right], \qquad (4.44)$$

where f is a complicated but unimportant function of the free real parameters of the solution $(\tilde{P}, \tilde{q}_+)$.

When $c_2 = -c_1 = \kappa$, using the parametrization (3.13) with $\sigma = -1$, it is very easy to recognize that $\mathsf{q} = \mathsf{q}_*$ in (4.10). Then, gathering this fact, the expression (4.11) and the non-trivial identity

$$\sigma = -1: \quad \frac{m_+}{m_-} = -\frac{\tilde{\mathsf{q}}_\star^2\left[2\tilde{\mathsf{q}}_\star^2(-1 + \tilde{q}_+^2 - 2x + x^2) - (2-x)(\tilde{q}_+^2 - (x+1)^2)\right]}{(1 - \tilde{\mathsf{q}}_\star^2)\left[2\tilde{\mathsf{q}}_\star^2(-1 + \tilde{q}_+^2 + 2x + x^2) + x(\tilde{q}_+^2 - (x+1)^2)\right]}, \qquad (4.45)$$

it is possible to show that indeed (4.40) matches (4.12). In this case the ratio of the entries of the Reeb vector (4.39) is not fixed, and depends on $\tilde{q}_+$ but not on $\overline{P}$, as the action (4.12). This is then a highly non-trivial check of the validity of the equivariant procedure of [32]. We have also checked that the on-shell actions for $N = 0$ case in (4.15) are also correctly reproduced by the expressions obtained with equivariant localization.

4.4 Extremization

Since we have an *off-shell* action as a function of the equivariant parameter $z \equiv \epsilon_2/\epsilon_1$, it is natural to wonder whether the on-shell actions corresponding to the explicit solutions

extremize this. In fact, we will show that the extremization of (4.28) with respect z reproduces the gravitational results for the renormalized on-shell action computed previously *only in the twist case* $\sigma = 1$. Choosing again $c_2 = c_1 = \kappa$ for the twist, there are two values of z that extremize I, namely

$$z_\pm^* = \frac{\pm m_- - m_+}{\pm r_+ + r_-} \,. \tag{4.46}$$

It is then straightforward to compute the action at the extremizing values, using Bézout's lemma (3.92). At the saddle z_+^* we have

$$I_{\text{off-shell}}(c_A, z_+^*) = S_{\text{ren}}^{(\sigma=1)} \,, \tag{4.47}$$

where the latter is given for example in (4.9). Indeed, the value of z_+^* is constant and compatible with (4.44). Notice that in the limit $m_- \to m_+$, $z_+^* \to v$ from (3.106) with $\tilde{q}_+ = 1 + x$ and $\kappa = 1$ as discussed in 3.6. For $v = 1$, this is the expected result for the 1/4-BPS spherical bolt (see (2.27)). For the other extremum, with the same choice of signs, the action reads

$$I_-^*(m_\pm, v) \equiv I(m_\pm, v, z_-^*) = \frac{\pi}{8G_4 v}\left[2\chi_\Sigma - \kappa\frac{t/v}{m_- m_+} - \kappa\frac{v(m_- + m_+)^2}{t\,m_- m_+}\right] \,. \tag{4.48}$$

This equation does not reproduce any known result for $m_- = m_+ = 1$. Moreover, whilst the vector $\vec{\epsilon}_+ = -(1, z_+^*)$ always lies inside the polytope, that is $\vec{\epsilon}_+ \cdot \vec{\mu}_{1,2}^{(A)} > 0$, $\vec{\epsilon}_- = -(1, z_-^*)$ is always outside it.

On the other hand, in the anti-twist case, the on-shell actions match the off-shell one for generic values of z, depending on continuous parameters, and therefore there is *no extremization* taking place in this case. At present we do not have an understanding of the reason underlying this difference[33]. In the case $t = 0$, with anti-twist, we noted that the on-shell action reduces to the black hole entropy functions (4.33) which indeed matches with the complexified black hole off-shell action for generic values of the parameter z. However, in that case it is known that a Legendre transform yields the extremal BPS black hole. Assuming that in the twist case solutions must extremize the off-shell action, we then conclude from (4.33) that in the minimal gauged supergravity there cannot be supersymmetric black holes with twist. This is consistent with the fact that supersymmetric accelerating black holes with twist have been found only the STU supergravity [66, 4, 67]. However, for $m_+ = m_- = 1$ the off-shell action becomes a constant and indeed we noted that the Topological Kerr-Newman-AdS solution realizes this possibility. This may then be interpreted as the gravitational saddle dual to the large N limit of the topologically twisted index [68] for the sphere.

It is interest to note that the equivariant volume introduced in [53] reproduces the action (4.19) *only* in the twist case. To see this, we write the fixed-point formula for the equivariant volume presented in [53] in our conventions (already specialized to the planar

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[33]One first observation is that this is consistent with the behaviour observed in [30]. Even if there the focus was on solutions with a single nut in the bulk geometry, the authors observed that the solutions were everywhere regular only if the components of the Reeb vector satisfy $\epsilon_2/\epsilon_1 > 0$ or $\epsilon_2 = -\epsilon_1$ (with some conventional choice on the sign of one between ϵ_i).

case) as[34]

$$\mathbb{V}(\lambda_a, \vec{\epsilon}) = \frac{\pi}{4G_4} \sum_{A=1}^{2} \frac{1}{d_A} \prod_{i=1}^{2} \frac{e^{\lambda_{a_i}^A b_i^{(A)}}}{b_i^{(A)}} , \qquad (4.49)$$

where the $b_i^{(A)}$ are defined in (4.20), the Kähler parameters for each divisor are identified as $\lambda_1^1 \equiv \lambda_1$, $\lambda_2^1 = \lambda_1^2 \equiv \lambda_2$, $\lambda_2^2 \equiv \lambda_3$ and the coefficient $(\pi/4G_4)$ has here been tuned *a posteriori* to have agreement with the supergravity computations. We now Taylor expand this formula around $\lambda_a \sim 0$ and pick the second order term

$$\frac{4G_4}{\pi} \mathbb{V}(\lambda_A, \vec{\epsilon})\Big|_{\lambda_a^2} = \sum_{A=1}^{2} \frac{1}{d_A} \left[\lambda_1^A \lambda_2^A + \frac{1}{2} \frac{b_1^A \lambda_1^A}{b_2^A} + \frac{1}{2} \frac{b_2^A \lambda_2^A}{b_1^A} \right]$$

$$= \frac{1}{d_1} \left[\lambda_1 \lambda_2 + \frac{1}{2} \frac{b_1^{(1)} \lambda_1}{b_1^{(1)}} + \frac{1}{2} \frac{b_2^{(1)} \lambda_2}{b_1^{(1)}} \right] + \frac{1}{d_2} \left[\lambda_2 \lambda_3 + \frac{1}{2} \frac{b_1^{(2)} \lambda_2}{b_2^{(2)}} + \frac{1}{2} \frac{b_2^{(2)} \lambda_3}{b_1^{(2)}} \right] .$$

$$\qquad (4.50)$$

To write each term in square brackets in the form of (4.19), we need $\lambda_1 = \lambda_2 = \pm 1$ but also $\lambda_2 = \lambda_3 = \pm 1$. But then $\lambda_1 = \lambda_2 = \lambda_3 \equiv -\kappa = \pm 1$ and the (second order of the) equivariant volume becomes

$$\mathbb{V}(\kappa, \vec{\epsilon})\Big|_{\lambda^2} = \frac{\pi}{2G_4} \sum_{A=2}^{3} \frac{1}{d_A} \left[-\kappa \frac{\left[b_1^{(A)} - \kappa \, b_2^{(A)} \right]^2}{4 \, b_1^{(A)} b_2^{(A)}} \right] . \qquad (4.51)$$

This is indeed (4.19) for $c_A = \kappa$, which is the twist case (4.38).

5 Accelerating solutions

We now come back to the study of the general class of Plebański-Demianski solutions (2.5)-(3.3). Since the analysis will be considerably more difficult, but also similar, to sections 3.2 and 3.4, we will recap only the key ingredients and results. Before studying the regularity of the boundary and the bulk metrics, we will compute the holographically renormalized on-shell action of the general non supersymmetric PD solutions. Indeed, during the procedure we will obtain the correct boundary metric, which will be analyzed subsequently. Although we will not attempt here to solve explicitly the supersymmetry equation (2.4), we will be able to infer the supersymmetric Killing vector indirectly by comparing the solution at the boundary with the general analysis of [8, 7].

5.1 General on-shell action

We start by computing the general on-shell action for the non-supersymmetric solutions. Such general result is not present in the literature, although it has been computed in numerous sub-cases (see *e.g.* [69, 16, 70, 6]). We consider again the renormalized action, which is obtained by the procedure explained in section 4. For convenience, we recap here

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[34]No sum over A in the exponent is intended.

the various terms

$$S_{E\Lambda} + S_F + S_{GH} + S_{ct} = -\frac{1}{16\pi G_4} \int d^4x \sqrt{g} \left[R^{(g)} + 6 - F^2 \right]$$
$$+ \frac{1}{8\pi G_4} \int d^3x \sqrt{\gamma} \left[2 + \frac{1}{2} R^{(\gamma)} - K^{(\gamma)} \right], \tag{5.1}$$

where the second integral must be computed on the boundary, which is now located at $q = (\mathbf{A}p)^{-1}$ instead of at $q = +\infty$. As usual, the contribution of the gauge field from S_F is already finite, and we can anticipate the result

$$\frac{(16\pi G_4)S_F}{(\mathbf{p}_+ - \mathbf{p}_-)\Delta\tau\Delta\sigma} = \mathbf{q}_+ \left[\frac{(P-Q)^2}{(\mathbf{q}_+ - \omega\mathbf{p}_+)(\mathbf{q}_+ - \omega\mathbf{p}_-)} + \frac{(P+Q)^2}{(\mathbf{q}_+ + \omega\mathbf{p}_+)(\mathbf{q}_+ + \omega\mathbf{p}_-)} \right],$$
$$+ \mathbf{A}(\mathbf{p}_+ - \mathbf{p}_-)\frac{2\mathbf{A}\omega PQ(\mathbf{p}_+^2 + \mathbf{p}_-^2) - (P^2 + Q^2)(1 + \mathbf{A}^2\omega^2\mathbf{p}_+^2\mathbf{p}_-^2)}{(1 - \mathbf{A}^2\omega^2\mathbf{p}_+^4)(1 - \mathbf{A}^2\omega^2\mathbf{p}_-^4)}. \tag{5.2}$$

Here we integrated simply q between $\mathbf{q}_+$ and $(\mathbf{A}p)^{-1}$ and p between $\mathbf{p}_-$ and $\mathbf{p}_+$. Notice that the first line, which is non-vanishing for $\mathbf{A} \to 0$, coincides with (4.5). To compute correctly the on-shell action, we need to bring the metric in the Fefferman-Graham form [71], following standard literature (see for example [69, 70]), which is

$$ds^2 \underset{r\to 0^+}{\simeq} \frac{dr^2}{r^2} + \frac{ds_3^2}{r^2}, \tag{5.3}$$

where

$$ds_3^2 = ds_{(0)}^2 + ds_{(1)}^2 r + ds_{(2)}^2 r^2 + ds_{(3)}^2 r^3 + O(r^4), \tag{5.4}$$

and the leading contribution is (a conformal representative of) the boundary metric, $ds_{(0)}^2 \equiv ds_\gamma^2$. To this end we make the following change of coordinates[35]

$$q = \frac{1}{\mathbf{A}x} - \sum_{i=1}^4 f_i(x)r^i, \quad p = x + \sum_{i=1}^4 g_i(x)r^i, \tag{5.5}$$

where all the functions except for $f_1(x)$ are determined by requiring the absence of mixed $(dr\, dx)$ terms in the metric[36] and that the coefficient of dr^2 is 1. As an example, we get

$$g_1(x) = \frac{\mathbf{P}(x)}{\mathbf{A}x^2\mathbf{Q}\left(\frac{1}{\mathbf{A}x}\right)} f_1(x), \tag{5.6}$$

where the following identity holds

$$\mathbf{Q}(\mathbf{A}^{-1}p^{-1}) = \frac{\mathbf{P}(p)}{\mathbf{A}^2p^4} + \frac{p^{-4} - \mathbf{A}^2\omega^2}{\mathbf{A}^4}. \tag{5.7}$$

In passing, let us notice that $f_1(x)$ defines a conformal class for the boundary metric, and we expect that the finite terms are independent of it. For future reference, let us define a specific boundary metric ds_b^2 given by

$$ds_{(0)}^2 = \frac{\mathbf{Q}\left(\frac{1}{\mathbf{A}x}\right)}{f_1(x)^2(1 - \omega^2\mathbf{A}^2x^4)^2} ds_b^2, \tag{5.8}$$

---

[35]By inspection, we noticed that $f_5(x)$ and $g_5(x)$ do not enter in $ds_{(3)}^2$.

[36]In particular, for the procedure to be correct, we have to expand the metric until $ds_{(5)}^2$ and require that the Ricci scalar of the metric in (r,x) coordinates is $R^{(g)} = -12 + \mathcal{O}(r^4)$.

where

$$ds_b^2 = \frac{(1-\omega^2\mathbf{A}^2x^4)^2\mathrm{d}x^2}{-\mathbf{P}(x)} - \frac{\mathbf{A}^8x^8\mathbf{Q}\left(\frac{1}{\mathbf{A}x}\right)^2\mathbf{P}(x)}{\left[1+\mathbf{A}^2\mathbf{P}(x)\right]}\mathrm{d}\sigma^2$$
$$+ \mathbf{A}^4x^4\left[1+\mathbf{A}^2\mathbf{P}(x)\right]\mathbf{Q}\left(\frac{1}{\mathbf{A}x}\right)\left[\mathrm{d}\tau + \frac{\omega x^2}{1+\mathbf{A}^2\mathbf{P}(x)}\mathrm{d}\sigma\right]^2. \tag{5.9}$$

The strategy now is to perform a mixed analysis, where the bulk integrals will be computed in the original coordinate system (p,q), whilst the boundary ones will be in (x,r) coordinates, taking due care to the Jacobian for (5.5). The bulk contribution is

$$S_{E\Lambda} = -\frac{\Delta\tau\Delta\sigma}{16\pi G_4}\int_{\mathbf{p}_-}^{\mathbf{p}_+}\mathrm{d}p\int_{\mathbf{q}_+}^{q_\epsilon(p)}\mathrm{d}q\sqrt{g}\left(-12+6\right), \tag{5.10}$$

where $q_\epsilon(p)$ is determined as follows. The expansion for p in (5.5) can be inverted order by order to give

$$x(p,r) = p + \sum_{i=1}^{4}\tilde{g}_i(p)r^i + \mathcal{O}(r^5) = p - g_1(p)r + \left[g_1'(p)g_1(p) - g_2(p)\right]r^2 + \ldots + O(r^5), \tag{5.11}$$

where the derivatives are taken with respect to p. From this mixed expression for $x(p,r)$ one easily gets $q = q(p,r)$ by inserting (5.11) in the q expansion (5.5). Finally, we define the limit of integration to be $q_\epsilon(p) \equiv q(p,r)|_{r=\epsilon}$. From this computation we get divergent terms, proportional to $\epsilon^{-3}, \epsilon^{-2}, \epsilon^{-1}$, and a finite contribution. The infinite parts should be cancelled from the boundary, which gives

$$S_{ct} + S_{GH} = \frac{\Delta\tau\Delta\sigma}{8\pi G_4}\int_{\mathbf{p}_-}^{\mathbf{p}_+}\mathrm{d}p|\partial_p x(p,\epsilon)|\sqrt{\gamma}\left[2 + \frac{1}{2}R^{(\gamma)} - K^{(\gamma)}\right]\Bigg|_{x=x(p,\epsilon)}, \tag{5.12}$$

where $\partial_p x(p,\epsilon)$ can be computed from (5.11). The terms that diverge as ϵ^{-3} for $\epsilon \to 0^+$ are cancelled in the sum before computing the integral in dp. Differently, both the ϵ^{-2} terms are (separately) integrated to zero, because they both can be written as (proportional to)

$$\partial_p\left[\frac{\mathbf{A}^8p^4\mathbf{P}(p)\mathbf{Q}^2(\mathbf{A}^{-1}p^{-1})}{(1-\omega^2\mathbf{A}^2p^4)^2f_1(p)^2}\right]\frac{1}{\epsilon^2}, \tag{5.13}$$

Since $\mathbf{P}(\mathbf{p}_\pm) = 0$ by definition, they do not contribute to the final answer. Finally, the terms proportional to ϵ^{-1}, after being summed up, are again of the form $\partial_p\left[\mathbf{P}(p)F(p)\right]$ for some function $F(p)$. After some work we can integrate the finite part of the computation, finding

$$S_{\mathrm{ren}} = \frac{\Delta\tau\Delta\sigma}{\mathbf{A}16\pi G_4}\Bigg[\frac{-\mathbf{q}_+^2}{(1-\mathbf{A}p\mathbf{q}_+)^2} - \frac{\omega^2(1-2\mathbf{A}p\mathbf{q}_+)}{\mathbf{A}^2\mathbf{q}_+^2(1-\mathbf{A}p\mathbf{q}_+)^2} + \frac{\omega^2p^2(1+16\omega^2\mathbf{A}^2p^4)}{1-\omega^2\mathbf{A}^2p^4}$$
$$- \frac{\mathbf{A}^2\left[64\omega^4p^6 + 4\omega^2p^3\mathbf{P}'(p) - [\mathbf{P}'(p)]^2\right]}{4(1-\omega^2\mathbf{A}^2p^4)} - \frac{\mathbf{P}''(p)}{6}\Bigg]_{\mathbf{p}_-}^{\mathbf{p}_+} + S_F, \tag{5.14}$$

with S_F as in (5.2). Despite appearances, after the result is made explicit by substituting $\mathbf{P}(p)$, the expression is finite for $\mathbf{A} \to 0$ as expected. Indeed, in this limit we get

$$
S_{\text{ren}} = \frac{\Delta\tau\Delta\sigma(\mathbf{p}_+ - \mathbf{p}_-)\mathbf{q}_+}{8\pi G_4}\left[\omega^2\big[\mathbf{p}_+^2 + \mathbf{p}_+\mathbf{p}_- + \mathbf{p}_-^2\big] + \frac{M}{\mathbf{q}_+} - \mathbf{q}_+^2\right.
$$
$$
\left. + \frac{(P-Q)^2}{2(\mathbf{q}_+ - \omega\mathbf{p}_+)(\mathbf{q}_+ - \omega\mathbf{p}_-)} + \frac{(P+Q)^2}{2(\mathbf{q}_+ + \omega\mathbf{q}_+)(\mathbf{q}_+ + \omega\mathbf{p}_-)}\right],
$$
$$
(5.15)
$$

which coincides with (4.8) for $\omega = 1$.

It is also interesting to compare our result (5.14) to the on-shell action of [6]. Such an accelerating black hole is obtained from the general (Lorentzian) PD solution by turning off the NUT parameter. In our parametrization of the solution we should impose (see *e.g.* [25])

$$
N = -\omega\mathbf{A}M\,.
$$
$$
E = \frac{N\omega}{\mathbf{A}M} + \frac{\mathbf{A}}{\omega MN}\big[(M^2 + N^2)(-P^2 + \alpha)\omega^2 + N^2(P^2 - Q^2)\big]\,,
$$
$$
(5.16)
$$

With these conditions imposed we have

$$
\mathbf{P}(p) = (p^2 - 1)\Big\{P^2 - \alpha - 2\mathbf{A}Mp + \Big[\mathbf{A}^2\big[P^2 - Q^2 - (P^2 - \alpha)\omega^2\big] + \omega^2\Big]p^2\Big\}\,, \quad (5.17)
$$

so that the roots are taken to be $\mathbf{p}_\pm = \pm 1$ [23, 5]. Moreover, to compare with [6] we identify

$$
\mathbf{q}_+ = r_+\,, \quad \mathbf{A} = \alpha\,, \quad Q = -\mathrm{i}e\,, \quad P = g\,, \quad M = m\,, \quad \omega = -\mathrm{i}a\,, \quad \alpha = P^2 - 1\,, \quad (5.18)
$$

where the i factors are due to the analytic continuation from our Euclidean solution. Upon these identifications, (5.14) matches perfectly the one presented in [6].

Summarizing, (5.14) is the most general on-shell action for the PD solution, it is very general and is valid independently on the way in which the roots of $\mathbf{P}(p)$ are chosen.

5.2 Supersymmetric Killing vector

Even if we do not want to solve the supersymmetry equation (2.4), we will be interested in the supersymmetric Killing vector: from this, as in section 4.2, we will recover our on-shell action (5.14), once evaluated on some specific cases in the next sections. In particular we can obtain the supersymmetric Killing vector using the boundary metric (5.9) and the boundary gauge field

$$
A_b = \frac{\mathbf{A}p(P\omega\mathbf{A}p^2 - Q)\mathrm{d}\tau + (pP - Q\omega\mathbf{A}p^3)\mathrm{d}\sigma}{1 - \omega^2\mathbf{A}^2p^4}\,, \quad (5.19)
$$

obtained from (2.6) evaluated at the boundary $q = (\mathbf{A}p)^{-1}$. Following [8, 7], we write the boundary fields as[37]

$$\mathrm{d}s_b^2 = f(p)^2 \mathrm{d}p^2 + h_{11}(p)\mathrm{d}\sigma^2 + 2h_{12}\mathrm{d}\sigma\mathrm{d}\tau + h_{22}\mathrm{d}\tau^2 \,,$$

$$A_b = \frac{-v(p)^3}{4f(p)\sqrt{h(p)}}\left[\frac{1}{\mathrm{w}}\partial_p\left[\frac{h_{11}(p)}{v(p)^2}\right]\mathrm{d}\sigma - \partial_p\left[\frac{h_{22}(p)}{v(p)^2}\right]\mathrm{d}\tau\right] + \frac{MP - NQ}{2\mathbf{A}(P^2 - Q^2)}\mathrm{d}\sigma + \frac{NP - MQ}{2(P^2 - Q^2)}\mathrm{d}\tau \,,$$
$$(5.20)$$

where $h(p) \equiv \det(h_{ij})$ and

$$v(p)^2 = h_{11}(p) + 2\mathrm{w}h_{12}(p) + \mathrm{w}^2 h_{22}(p)^2 \,. \tag{5.21}$$

In this formalism, the supersymmetric killing vector reads[38]

$$\epsilon_* \propto \partial_\sigma + \mathrm{w}\partial_\tau \,, \tag{5.22}$$

and by comparison we obtain the relative weight w as

$$\mathrm{w} = \frac{\mathbf{A}\rho}{\omega(MQ - NP)}\frac{\alpha\omega(2\mathbf{A}MN\rho^2 + \omega\mu^2) - E\rho(N^2\rho + \omega^2 P^2\mu)}{2\mathbf{A}NP\rho^2 + \omega(MP - NQ)\mu + 2\omega^2\mathbf{A}P^2(MQ - NP)\rho} \,, \tag{5.23}$$

where for convenience we defined $\rho = P^2 - Q^2$ and $\mu = M^2 - N^2$. This formula still needs to be evaluated on the supersymmetric relations (2.21) or (2.22). In the simpler case of (2.21), we get

$$\mathrm{w} = \frac{1}{\mathbf{A}}\frac{MP - NQ}{NP - MQ} \,, \tag{5.24}$$

which is ill-defined in the limit $\mathbf{A} \to 0$. Using instead (2.22), we obtain a complicated formula which reduces to

$$\mathrm{w} \underset{\mathbf{A}\to 0}{\to} \sqrt{\alpha} \,, \tag{5.25}$$

as expected from the direct computation in the non-accelerating case (4.35).

5.3 Global analysis and toric data

For the regularity analysis, we follow the same procedure of section 3.2. In particular, we introduce again two patches $U_\pm$ on which we change coordinates

$$U_\pm : \quad \tau^\pm = \mathbf{q}_+^2\theta_1^\pm - \omega\mathbf{p}_\pm^2\theta_2^\pm \,, \quad \sigma^\pm = -\omega\theta_1^\pm + b^\pm\theta_2^\pm \,, \tag{5.26}$$

for which $(\Delta\tau\Delta\sigma)^\pm = (\mathbf{q}_+^2 - \omega^2\mathbf{p}_\pm^2)\Delta\theta_1\Delta\theta_2^\pm$. Upon this change, the boundary metric (5.9) becomes[39]

$$U_\pm : \quad \mathrm{d}s_b^2 = \frac{(1 - \omega^2\mathbf{A}^2 p^4)^2}{-\mathbf{P}(p)}\mathrm{d}p^2 - \frac{(\mathbf{q}_+^2 - \omega^2\mathbf{p}_\pm^2)^2\mathbf{P}(p)\left[1 - \omega^2\mathbf{A}^2 p^4 + \mathbf{A}^2\mathbf{P}(p)\right]^2}{\left[(\mathbf{q}_+^2 - \omega^2 p^2)^2 + (\mathbf{A}^2\mathbf{q}_+^4 - \omega^2)\mathbf{P}(p)\right]}(\mathrm{d}\theta_2^\pm)^2$$
$$+ \left[1 - \omega^2\mathbf{A}^2 p^4 + \mathbf{A}^2\mathbf{P}(p)\right]\left[(\mathbf{q}_+^2 - \omega^2 p^2)^2 + (\mathbf{A}^2\mathbf{q}_+^2 - \omega^2)\mathbf{P}(p)\right]\left[\mathrm{d}\theta_1^\pm + \mu_f^\pm\right]^2 \,,$$
$$(5.27)$$

---

[37]Notice an overall minus sign with respect to [8]. Moreover, we have performed a specific gauge transformation to compare with (5.19).

[38]The overall normalization does not play any role in the following discussions.

[39]Since at the leading order $p = x + \mathcal{O}(r)$ from (5.11), we can take the boundary metric in (p, q) as written in (5.9) with $x = p$.

where $\mu_f^\pm$ is a fibration one-form and reads

$$U_\pm: \quad \mu^{(\pm)} = \omega \frac{(\mathbf{q}_+^2 - \omega^2 p^2)(p^2 - \mathbf{p}_\pm^2) + (1 - \mathbf{A}^2 \mathbf{p}_\pm^2 \mathbf{q}_+^2)\mathbf{P}(p)}{(\mathbf{q}_+^2 - \omega^2 p^2)^2 + (\mathbf{A}^2 \mathbf{q}_+^4 - \omega^2)\mathbf{P}(p)} \, \mathrm{d}\theta_2^\pm \, . \tag{5.28}$$

Not surprisingly, this is the accelerating version of the boundary metric (3.23). As a consequence, we define again

$$\Delta = (\mathbf{q}_+^2 - \omega^2 \mathbf{p}_+^2)\Delta\theta_2^+ = (\mathbf{q}_+^2 - \omega^2 \mathbf{p}_-^2)\Delta\theta_2^- \, , \tag{5.29}$$

and impose

$$U_\pm: \quad \pm \frac{\mathbf{P}'(\mathbf{p}_\pm)}{2(\mathbf{q}_+^2 - \omega^2 \mathbf{p}_\pm^2)}\Delta = \frac{2\pi}{m_\pm} \, , \quad t = \omega \, m_- m_+ \frac{\mathbf{p}_+^2 - \mathbf{p}_-^2}{(\mathbf{q}_+^2 - \omega^2 \mathbf{p}_+^2)(\mathbf{q}_+^2 - \omega^2 \mathbf{p}_-^2)}\frac{\Delta}{\Delta\theta_1} \, . \tag{5.30}$$

The regularity condition from the bulk is identical to (3.48)

$$\frac{\mathbf{Q}'(\mathbf{q}_+)}{2}\Delta\theta_1 = \frac{2\pi}{v} \, , \tag{5.31}$$

and collecting all these expressions we can write

$$\Delta\tau^\pm \Delta\sigma^\pm = \Delta\theta_1 \Delta = \omega \frac{m_- m_+}{t}\Delta^2 \frac{\mathbf{p}_+^2 - \mathbf{p}_-^2}{(\mathbf{q}_+^2 - \omega^2 \mathbf{p}_-^2)(\mathbf{p}_+^2 - \omega^2 \mathbf{p}_-^2)} \, , \tag{5.32}$$

similarly to (3.37). Once everything is imposed, the topology is again $\mathbb{C}/\mathbb{Z}_v \hookrightarrow \mathcal{O}(-t) \to \Sigma_{[m_-, m_+]}$. Consequently, the polytope describing this accelerating Bolt solution is the same as in figure 2, with normal singularity to $\mathcal{L}_2 = \{q = \mathbf{q}_+\}$ given by v. In particular we can use the same basis (3.91), and all the other expressions (3.90)-(3.93) remain unchanged. The only (small) difference lies in the definition of the Killing vectors, that should be taken as

$$U_\pm: \quad K_{\mathbf{q}_+} = (\mathbf{q}_+^2 \partial_{\tau^\pm} - \omega \partial_{\sigma^\pm}) = \partial_{\theta_1^\pm} \, , \quad K_{\mathbf{p}_\pm} = -(\omega \mathbf{p}_\pm^2 \partial_{\tau^\pm} - \partial_{\sigma^\pm}) = \partial_{\theta_2^\pm} \, . \tag{5.33}$$

5.4 A special class

As already observed, the accelerating solution is richer but also tremendously more complicated. For this reason, we will impose a specific (but arbitrary) constraint between the parameters with the sole purpose of simplifying the computations. Without turning to zero any parameter we remain in the generic situation, and indeed we will find that both the twist and the anti-twist are admitted. It is worth noting that requiring (5.16) and $\alpha = P^2 - 1$ as in (5.18) we recover the complex Euclidean version of the accelerating black hole. However, since this is discussed extensively in [6], we will not consider this further. On the other hand, to our knowledge no example of this special class has been analyzed in the literature before, therefore in this section we will not recover any previous solutions.

The special class of accelerating solutions that we will consider is obtained by setting[40]

$$\alpha = P^2 \, , \quad \mathbf{A} = \frac{\omega}{Q} \, . \tag{5.34}$$

---

[40]In the following equations we are taking a specific branch for the roots, such that $\sqrt{P^2} = P$ and $\sqrt{Q^2} = Q$. Recall also that the parameter ω, as introduced for example in [23], is a scaling parameter. Thus we are only imposing a single simplifying condition here.

In this case the supersymmetry constraints (2.22) collapse to

$$E = \frac{N^2}{Q^2} + 2P \,, \quad M = \frac{NP}{Q} \,, \qquad (5.35)$$

which are formally very similar to the non-accelerating ones (3.5). The strong simplification is due to the fact that the square root in the first line of (2.22) becomes a perfect square. Further, the metric functions simplify to

$$\mathbf{P}(p) = \frac{1}{\omega^2 Q^2} \mathbf{P}_+(p) \mathbf{P}_-(p) \,, \qquad \mathbf{P}_\pm(p) = \omega^2 P p^2 - \omega N p + Q^2 (1 \pm 1) \,,$$
$$\mathbf{Q}(q) = \frac{1}{P^2 Q^2} \mathbf{Q}_+(q) \mathbf{Q}_-(q) \,, \qquad \mathbf{Q}_\pm(q) = P Q q^2 - N P q + P Q (P \pm Q) \,, \qquad (5.36)$$

so that the four roots of $\mathbf{P}(p)$ are

$$\mathbf{P}_+(p): \quad r_1 = \frac{N - \sqrt{N^2 - 8PQ^2}}{2\omega P} \,, \qquad r_2 = \frac{N + \sqrt{N^2 - 8PQ^2}}{2\omega P} \,,$$
$$\mathbf{P}_-(p): \quad r_3 = 0 \,, \qquad\qquad\qquad\quad r_4 = \frac{N}{\omega P} \,. \qquad (5.37)$$

From these roots we can generate instances of solutions with the twist or the anti-twist realization of supersymmetry, according to the paring we choose. In particular, as in section 3.5, we will end up with a twist if $\mathbf{p}_\pm$ are roots of the same parabola, i.e. $\mathbf{P}_+(\mathbf{p}_\pm) = 0$ or $\mathbf{P}_-(\mathbf{p}_\pm) = 0$; otherwise we will get an anti-twist. However, we can not have a case with $t = 0$ because it is not possible to choose the parameters such that $\mathbf{p}_- = -\mathbf{p}_+$. Even if the analysis can be carried out (in principle) in full generality, meaning by parametrizing the various parameters in terms of the roots $\mathbf{p}_\pm$ and $\mathbf{q}_+$ as in section 3.1, we take a more direct approach. More specifically, we choose the parameters of the solution such that the largest roots of $\mathbf{P}(p)$ are given by

$$\mathbf{p}_- = r_3 = 0 \,, \quad \mathbf{p}_+ = r_4 = \frac{N}{\omega P} \,, \qquad (5.38)$$

and as a consequence we will have a twist. The case $\mathbf{p}_- = r_4$ and $\mathbf{p}_+ = r_3$ is as well possible, and can be obtained by changing the sign of the parameters. Then from the first two conditions in (5.30) we have

$$N = \frac{P \mathbf{q}_+ \sqrt{m_- - m_+}}{\sqrt{m_-}} \,, \quad \frac{\Delta}{2\pi} = \frac{\omega \mathbf{q}_+}{P \sqrt{m_-} \sqrt{m_- - m_+}} \,, \qquad (5.39)$$

and from $\mathbf{Q}_\lambda(\mathbf{q}_+) = 0$ (with $\lambda = \pm 1$ determining if $\mathbf{q}_+$ is solution of $\mathbf{Q}_+(q)$ or $\mathbf{Q}_-(q)$) we obtain

$$\mathbf{q}_+ = \frac{\sqrt{Q} \sqrt{P + \lambda Q} \, m_-^{1/4}}{\sqrt{P \sqrt{m_- - m_+} - \sqrt{m_-} Q}} \,. \qquad (5.40)$$

For $\mathbf{p}_+ > \mathbf{p}_- = 0$ and $\Delta > 0$ we should require (N, P, ω) to be positive. Moreover, for $\mathbf{p}_\pm$ to be the largest roots we must have then that $N^2 - 8PQ^2 < 0$, so that $r_{1,2} \in \mathbb{C}$. Finally, from the the second condition in (5.30) and (5.31) we get

$$\frac{\Delta\theta_1}{2\pi} = \frac{\lambda}{v(N - 2Q\mathbf{q}_+)} \,, \quad t = \frac{\lambda \sqrt{m_- - m_+} \left[P \sqrt{m_- - m_+} - 2Q\sqrt{m_-} \right] v}{P} \,. \qquad (5.41)$$

Here $P\sqrt{m_- - m_+} > 2Q\sqrt{m_-}$ strictly, due to the condition $\mathbf{Q}'(\mathbf{q}_+) > 0$. Consequently, as anticipated, $t \neq 0$ in every case[41]. Moreover in appendix C we show that for this solution $t \geq 1$, so that the boundary can have the topology of the (squashed) S^3. The last can be easily inverted to give, for example, $Q = Q(P, t)$. With these results, it is immediate to show that

$$\frac{1}{2\pi} \int_{\mathcal{L}_2} \mathrm{d}A = \frac{1}{2}\left[\frac{m_- + m_+}{m_- m_+} + \lambda \frac{t/v}{m_- m_+}\right], \tag{5.42}$$

confirming that $2A$ is a connection on the twisted line bundle $\mathcal{O}(m_- + m_+)$ over $\Sigma_{[m_-, m_+]}$. Notice that in principle one could have taken the parameters $(N, P, \omega) \in \mathbb{C}$ such that $\mathbf{p}_+ \in \mathbb{R}_+$. However, requiring $(\mathbf{q}_+, \Delta, \Delta\theta_1, t) \in \mathbb{R}_+$ implies $(N, P, \omega, Q) \in \mathbb{R}$. This is consistent with the statement that the solutions with twist are real. Moreover we can now also insert all the "quantized" expressions in the on-shell action (5.14). Using explicitly these results and (5.32), it is immediate to show that

$$\text{twist}: \quad S_{\text{ren}} = \frac{\pi}{8G_4 v}\left[2\chi_\Sigma + \lambda \frac{t/v}{m_- m_+} + \lambda \frac{v(m_- - m_+)^2}{t\, m_- m_+}\right], \tag{5.43}$$

which is equal, for example, to (4.9). This result is expected, since the polytope of the solution is the same as for the non-accelerating case (see figure 2a), and the solution is of the twist type. Consequently, as observed in section 4.4, the extremization of the action (4.19) should give the same result, independently of the specific form of the solution. We can provide further confirmation of this by plugging explicitly the supersymmetric vector (5.22), (5.23) in the action (4.19). However, since (5.35) makes w infinite in (5.24), we can not use (4.40), (4.41) directly. Indeed this means that for the specific sub-case under examination the supersymmetric Killing vector is aligned only with the τ direction, $\epsilon_* \propto \partial_\tau$ (see also [21]). As a consequence we need to write the single Killing vectors ∂_τ and ∂_σ in the toric basis (3.91)

$$U_- : \quad \begin{aligned} \partial_{\tau^-} &= \left[\frac{1}{\mathbf{q}_+^2 - \omega^2 \mathbf{p}_-^2}\frac{2\pi}{\Delta\theta_1} + \omega\frac{r_+}{m_-}\frac{2\pi}{\Delta}\right]E_1 + \omega\frac{2\pi}{\Delta}E_2\,, \\ \partial_{\sigma^-} &= \left[\frac{\omega\mathbf{p}_-^2}{\mathbf{q}_+^2 - \omega^2 \mathbf{p}_-^2}\frac{2\pi}{\Delta\theta_1} + \mathbf{q}_+^2\frac{r_+}{m_-}\frac{2\pi}{\Delta}\right]E_1 + \mathbf{q}_+^2\frac{2\pi}{\Delta}E_2\,. \end{aligned} \tag{5.44}$$

Inserting the first line in (4.19) with $c_2 = c_1 = \lambda$ and using (5.38)-(5.41), we find again (5.43). The situation here is completely analogous to that of section 4.1. Indeed in the CP solution we started with six parameters plus the periodicities $(\Delta\theta_1, \Delta)$, minus two constraints from supersymmetry and four regularity conditions, remaining with two free parameters $(\tilde{q}_+, \tilde{P})$. In passing from CP to PD we add two parameters $(\omega, \mathbf{A})$, but we also fixed two simplifying relations (5.34). Then we are similarly left with *two free continuous parameters* (ω, P), that however do not appear in the on-shell action.

Certainly other choices of roots are possible, but the remaining twist case is excluded because there are no parameter values for which r_1 and r_2 can be the largest roots.

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[41]The case $m_- = m_+$ is excluded because it would imply $N = 0$ from (5.39), but in turn this means $\mathbf{p}_+ = \mathbf{p}_-$ from (5.38), which is degenerate because of the presence of a double root. Moreover, when $N = 0$ the relevant roots should not be r_3 and r_4, but for example $\mathbf{p}_- = r_3 = r_4 = 0$ and $\mathbf{p}_+ = r_2$, for which $t \neq 0$ again. It follows that this class of solutions is not directly related with the twist case with $t = 0$ in section 3.5.3.

Instead all the alternative combinations of roots which give the anti-twist are admissible. In these cases we have checked that the gravitational on-shell action (5.14) is reproduced by (4.28) with $c_2 = -c_1$, as expected. However we will not present an explicit expression for the anti-twist action, since as in section 4.1 it is very complicated and not particularly illuminating.

We conclude our general analysis noticing that since we have imposed the simplifying constraint (5.34), from this special class of solutions it is impossible to reach smoothly the non-accelerating CP families. However, since in all cases we discussed the form of the on-shell action is always consistent with the predictions of equivariant localization, combined with observations we made on its extremization, we believe that in principle the general PD family should give rise to *two* distinct families of solutions, corresponding to twist and anti-twist, each depending on three continuous parameters, in addition to the topological data $(m_{\pm}, v, t)$, that contain all the classes discussed in this paper. Moreover, in principle there could be an even larger class of solutions, depending on a larger number of continuous parameters, containing the entire PD family.

6 Discussion

In this paper we constructed several new supersymmetric solutions of four-dimensional minimal gauged supergravity. These have been obtained starting with the family of local Plebański-Demianski solutions, which comprises the most general Petrov type D metric of the Maxwell-Einstein-Λ theory and possesses an $U(1) \times U(1)$ isometry. Despite these local solutions have been known for almost fifty years, many different global completions, either in Lorentzian, or in Euclidean, gave rise to a plethora of solutions over the years, with very different physical interpretations. For example, the local Kähler-Einstein foliations used in the construction of the $L^{a,b,c}$ Sasaki-Einstein manifolds [72] were obtained from a scaling limit of the Carter-Plebański sub-family [73]. Moreover, the various supersymmetric NUTs and Bolts solutions constructed in the series of papers [13, 14, 15, 16] arise as special cases or limits of this sub-family. Significant supersymmetric solutions lying in the more general Plebański-Demianski family are the accelerating black hole [5] and its complex non-extremal deformation [6]. The *global solutions* that we have constructed in this paper generalize all these solutions and include them as special cases or limits. A compendium of the old and new solutions is given in Table 1.

All the solutions that we have discussed comprise a metric on some orbifold line bundle over the spindle, with the latter appearing (in the bulk) as the zero section, which is usually called a "bolt" [10]. The topology of the conformal boundary can be that of a smooth lens space, but generically it displays orbifold singularities, which we refer to as branched lens spaces. As particular cases we can also have the topology of the three-sphere ($t = 1$), as well as that of the direct product $\mathbb{\Sigma} \times S^1$ ($t = 0$). The various classes of solutions display both types of twist for the graviphoton gauge field, characterised by the flux/curvature integrated over the spindle bolt, see *e.g.* (3.119). As the solutions have $U(1) \times U(1)$ isometry, the underlying orbifolds may be described in terms of toric geometry[42] employing the approach described in [36]. In particular, we have derived the toric data for all the solutions, consisting in two-dimensional labelled polyhedral cones,

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[42]Our solutions are in fact explicit examples of the general framework introduced in [34].

Parameters			Twist	Bolt	Notable cases
$\mathbf{A} = 0$	$N \neq 0$		$\sigma = +1$	Σ	Type I [16], 1/4-BPS bolts [15]
			$\sigma = -1$	Σ	Type II [16], 1/2-BPS bolts [15]
	$N = 0$	$P = 0$	$\sigma = \pm 1$	Σ	-
			$\sigma = -1$	S^2	Kerr-Newman-AdS [74, 43] ($t = 0$)
		$M = 0$	$\sigma = +1$	S^2	Topological Kerr-Newman-AdS ($t = 0$)
			$\sigma = -1$	Σ	-
$\mathbf{A} \neq 0$	$\alpha = P^2, \omega = \mathbf{A}Q$		$\sigma = +1$	Σ	-
			$\sigma = -1$	Σ	-
	(5.16)		$\sigma = -1$	Σ	Accelerating black hole [5, 6] ($t = 0$)

Table 1: A summarizing table of the CP and PD solutions of sections 3 and 5, respectively. Since $N = 0$ is the only case for which $t = 0$, we have split the table according to the value of N. Further, for $N = 0$ supersymmetry can be realized in two different ways, with $P = 0$ or $M = 0$, respectively (see section 3.5.3). Only the spherical bolts and the accelerating black hole have $t = 0$. The last row is dedicated to the PD solution, with $\mathbf{A} \neq 0$. The second column shows the admissible realization of supersymmetry on the spindle (or the sphere). Finally, the last column indicates notable cases, which include some old solutions and the Euclidean Topological Kerr-Newman-AdS, which was missed in the literature.

as depicted in figure 2. Incidentally, these may be obtained from the compact polytopes (quadrilaterals) described in [36] by removing one of the four edges and indeed, the local form of the co-homogeneity two metrics discussed in [36] is remarkably similar (but not diffeomorphic) do the metrics of Plebański-Demianski, that we discussed presently.

Obtaining explicit solutions corresponding to non-compact toric orbifolds with more than two fixed points would be impressive, but we believe that with present days technology, this is improbable. Thus our solutions are likely to remain the state-of-the-art of explicit examples of toric gravitational instantons of minimal gauged supergravity for some time. The regularity analysis of the Plebański-Demianski solutions with generic values of the parameters appears technically cumbersome. However, it may be possible that with some canny reparametrization and/or changes of variables (see *e.g.* [25, 26, 75]) one can make analytic progress. We leave the problem of improving our global analysis of the Plebański-Demianski solutions to someone else starting afresh. It might also be interesting to investigate spindle Bolt solutions of the type we discussed here in five or higher dimensions, extending the few known examples of topological solitons in dimensions higher than four [76, 77, 78].

Although the main motivation for this paper was holography, and therefore we focussed on asymptotically hyperbolic non-compact solutions, from a mathematical viewpoint it may be interesting to perform a systematic analysis analogous to our one in the compact setting. As we noted around equation (2.3), solutions to four dimensional Einstein-Maxwell-Λ theory have been actively investigated in the mathematics literature in the past decade, starting with the work of [38], where it was pointed out the relation of these to constant scalar curvature metrics. Interestingly, in this reference it was also noted that any constant scalar curvature Kähler metric on a compact 4-manifold is actu-

ally a solution to the Einstein-Maxwell-Λ equations, for a specific choice of curvature F. Examples of this correspondence, for smooth compact manifolds as well as for orbifolds, are discussed in [79, 80, 81, 82, 83, 84]. It would be interesting to study how this correspondence works in the sub-set of supersymmetric solutions of minimal four-dimensional supergravity. Moreover, it is also natural to wonder whether existence and uniqueness of the non-compact, asymptotically hyperbolic, solutions may be addressed employing the ideas of volume extremization [84].

Perhaps the most significant general lesson that we drew from our results is that the geometry of the conformal boundary encodes in a subtle way the information about how the solution "fills" this in the bulk. Generically, the three-dimensional space at the boundary is a Seifert orbifold, with the topology of a lens space $L(t, 1)$, possibly with orbifold singularities. In addition, however, the three-dimensional Killing spinors and gauge field are characterised by further data, namely the type of twist ($\sigma = \pm 1$), as well as the the positive integers $m_\pm$. These data then determine the topology of the four-dimensional filling, which correspondingly comprises an orbifold line bundle over a spindle $\mathbb{\Sigma}_{[m_+, m_-]}$, with twist or anti-twist for the graviphoton. We noted that this dichotomy explains the existence of the old classes of 1/4-BPS and 1/2-BPS Bolt solutions of [15], which arise as limits of the two different twists on the spindle. Specifically, the 1/4-BPS Bolt solutions have a topological twist for the graviphoton, while the 1/2-BPS Bolt solutions have no twist. Compare *e.g.* equation (3.119) with $m_+ = m_- = 1$ with the integrated fluxes in (2.26). We have shown that after fixing all the topological data as well as the type of twist, in general there exist multiple solutions which are not diffeomorphic. For example, picking $\sigma = 1$, and setting (m_+, m_-, v, t) to particular values there exist both non-accelerating as well as accelerating solutions (see appendix C). As discussed at the end of section 5 this is an indication that for fixed topological data (m_+, m_-, v, t) and type of twist $\sigma = \pm 1$, there should exist a family with several continuos parameters, encapsulating the different sub-families that we analyzed.

For all our solutions, we have computed the holographically renormalized on-shell action, uncovering an intriguing pattern. In all cases the on-shell action takes the general form (4.19), predicted by equivariant localization [31, 32, 34]. In particular, it can be expressed entirely in terms of the data at the two fixed points of the orbifolds (the poles of the spindle) *and* the choice of supersymmetric Killing vector $\vec{\epsilon}$. However, we have found that the form of this Killing vector for the explicit solutions depends in an interesting way on the type of twist for the graviphoton field of the specific solution, leading to two markedly different types of expressions for the on-shell action in the twist and anti-twist cases. Specifically, we uncovered that in the presence of twist, the supersymmetric Killing vector field of the explicit solutions coincides with the value determined by *extremizing* the "off-shell action" with respect to the vector field itself. Thus, the "critical" vector field does depend on the topological data of the underlying space – see (4.46). This is completely analogous to the Sasaki-Einstein [28] and GK [85] extremization problems and leads to an universal expression for the on-shell action of all the solutions with twist, see *e.g.* equations (4.9) and (5.43). On the other hand, in all the anti-twist cases the supersymmetric Killing vector field is arbitrary, and may depend on the continuous parameters of the solutions, so it is not determined by extremizing anything. This is consistent with the behaviour of the supersymmetric non-extremal deformations of the AdS$_4$ black holes [43, 6], that indeed correspond to anti-twist cases and have generic values of the

supersymmetric Killing vectors[43]. In the light of our findings, the 1/4-BPS and 1/2-BPS spherical Bolt solutions of [15] are earlier examples of this pattern[44]. Understanding the origin of the different behaviour of twist and anti-twist is perhaps the most intriguing open problem stemming from our work. This is likely to be related to two different types of boundary conditions in the supergravity variational problem: in the anti-twist case the supersymmetric Killing vector should be a fixed boundary datum of the problem, while in the twist case this is not fixed and as a result the off-shell action must be extremized with respect to different choices of the supersymmetric Killing vector.

Via holography, this problem can be reformulated in terms of properties of the dual three-dimensional supersymmetric field theories placed on the geometry of the boundary [7]. In particular, we have pointed out that for a given topology of the three-dimensional boundary, *e.g.* a lens space $L(t, 1)$, there exist *two choices*, labelled by a sign $\sigma = \pm 1$, that lead to two different behaviours of the localized partition functions. We refer to these two choices as to twist and anti-twist, despite the fact that there are no two-cycles in the three dimensional geometry. Moreover, we expect that the additional choices of flat connections depending on $m_\pm \in \mathbb{N}$, will determine the large N limit of the partition function, through a mechanism analogous to that governing the large N limit of the spindle index [9], which corresponds to the case $t = 0$ of the general set-up that we described.

It will be very interesting to reproduce our predictions from the large N limit of the three dimensional localized partition functions.

Acknowledgments

A   Old bolt solutions

In this section we review briefly some aspects of the spherical Bolt solutions of [15], since these are sub-cases of (3.1)-(3.2). Throughout this section we shall employ hatted quantities to distinguish them from the parameters in the main body of the paper. Locally, the solution is given by

$$\mathrm{d}s_4^2 = \frac{r^2 - s^2}{\Omega(r)}\mathrm{d}r^2 + (r^2 - s^2)(\sigma_1^2 + \sigma_2^2) + \frac{4s^2\Omega(r)}{r^2 - s^2}\sigma_3 \,, \tag{A.1}$$

with graviphoton

$$A = \left[\hat{P}\frac{r^2 + s^2}{r^2 - s^2} - \hat{Q}\frac{2rs}{r^2 - s^2}\right]\sigma_3 \,. \tag{A.2}$$

---

[43]In these cases the extremal solutions correspond to extremizing the "off-shell entropy", namely performing the Legendre transform of the action.

[44]In the self-dual subcases, a similarly different behaviour of the 1/4-BPS and 1/2-BPS NUT solutions of [15] was observed in [30], and it was related to fact that generically there exist two complementary fillings the unit sphere in $\mathbb{C}^2$.

The metric function is

$$\Omega(r) = (r^2 - s^2)^2 + (1 - 4s^2)(r^2 + s^2) - 2\hat{M}r + \hat{P}^2 - \hat{Q}^2\,, \tag{A.3}$$

and the σ_i are the $SU(2)$ left-invariant one-forms, that for definiteness we take to be

$$\sigma_1 + \mathrm{i}\sigma_2 = e^{-\mathrm{i}\hat{\psi}}(\mathrm{d}\hat{\theta} + \mathrm{i}\sin\hat{\theta}\mathrm{d}\hat{\varphi})\,, \quad \sigma_3 = \mathrm{d}\hat{\psi} + \cos\hat{\theta}\mathrm{d}\hat{\varphi}\,. \tag{A.4}$$

As demonstrated in [15], this is the most general local form of the supersymmetric solutions with $SU(2) \times U(1)$ symmetry, which is enhanced with respect to the $U(1) \times U(1)$ symmetry of the spindle bolt. For large r, the metric becomes

$$\mathrm{d}s_4^2 = \frac{\mathrm{d}r^2}{r^2} + r^2(\sigma_1^2 + \sigma_2^2 + 4s^2\sigma_3^2)\,, \tag{A.5}$$

so that the space is asymptotically locally $\mathbb{H}^4$ and s is a squashing parameter. The BPS conditions then split this solution into two classes[45]

$$\begin{aligned}
\frac{1}{2}\text{-BPS}: \quad &\hat{M} = \eta\hat{Q}\sqrt{4s^2 - 1}\,, \qquad \hat{P} = -\eta s\sqrt{4s^2 - 1}\,, \\
\frac{1}{4}\text{-BPS}: \quad &\hat{M} = 2\eta s\hat{Q}\,, \qquad\qquad \hat{P} = -\eta\frac{4s^2 - 1}{2}\,,
\end{aligned} \tag{A.6}$$

for which, in both cases, the topology can be the one of a non-trivial fibration on a round bolt $\mathcal{M}_4 = \mathcal{O}(-\hat{t}) \to S^2$, or a nut $\mathbb{R}^4/\mathbb{Z}_{\hat{t}}$, with the origin identified by $r = s$. From symmetry, one can always take the squashing parameter to be positive $s > 0$, and the round S^3 case is obtained for $s = 1/2$. The range of the radial coordinate is taken to be $s \le r_0 \le r < +\infty$, where r_0 is the largest root of $\Omega(r)$, with $r_0 > s$ for the bolt case. The regularity analysis of the metric in the two classes leads to the following conditions

$$\frac{1}{2}\text{-BPS}: \quad \hat{Q} = \eta\frac{(128s^4 - 16s^2 - \hat{t}^2)\kappa}{64s^2}\,, \quad r_0 = \frac{1}{8}\left[\frac{\hat{t}}{s} - 4\kappa\sqrt{4s^2 - 1}\right]\,, \tag{A.7}$$

and

$$\frac{1}{4}\text{-BPS}: \quad
\begin{aligned}
&\hat{Q} = \eta\frac{\hat{t}^2 \pm (\hat{t} - 16s^2)\sqrt{\mathcal{G}_+(\hat{t}, s)}}{128s^2}\,, \quad r_0 = \frac{\hat{t} \pm \sqrt{\mathcal{G}_+(\hat{t}, s)}}{16s}\,, \quad \kappa = 1\,, \\
&\hat{Q} = -\eta\frac{\hat{t}^2 \mp (\hat{t} + 16s^2)\sqrt{\mathcal{G}_-(\hat{t}, s)}}{128s^2}\,, \quad r_0 = \frac{\hat{t} \mp \sqrt{\mathcal{G}_-(\hat{t}, s)}}{16s}\,, \quad \kappa = -1\,.
\end{aligned} \tag{A.8}$$

where

$$\mathcal{G}_\pm(\hat{t}, s) = (\hat{t} \pm 16s^2)^2 - 128s^2\,. \tag{A.9}$$

Notice that here, contrarily to [15], we considered two branches for both the 1/4-BPS conditions, as pointed out in [46].

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[45]Notice the insertion of the sign $\eta = \pm 1$ here, which accounts in the two possible choices explained in [15].

B A spindle Calabi-Yau metric

We devote this appendix to the investigation of a Ricci-flat version of the set-up considered in section 3. The local form of the metric can be obtained as the Ricci-flat limit of the one presented in [86, 73], which is generalized to arbitrary dimensions in [87, 48], although the four-dimensional case (corresponding to $n = 0$ in [48]) is not discussed in the latter. See also [88]. In this simpler setting, which is toric in the standard mathematical language, we are able to explicitly write down the symplectic 2-form and the corresponding moment maps, enabling the computation of the toric diagram [52]. This fact, alongside with the total space being Calabi-Yau, which will provide a generalization of the computation in section 3.3.2 of [53], serves as main motivation for this appendix.

Explicitly, we introduce the metric

$$ds_4^2 = \frac{\eta - \xi}{2}\left[\frac{d\eta^2}{G(\eta)} + \frac{d\xi^2}{-F(\xi)}\right] + \frac{2}{\eta - \xi}\left[G(\eta)(d\phi + \xi\,d\psi)^2 - F(\xi)(d\phi + \eta\,d\psi)^2\right], \quad \text{(B.1)}$$

where we have defined

$$F(\xi) = E\,\xi^2 - 2N\,\xi - P^2\,, \quad G(\eta) = E\,\eta^2 - 2N\,\eta - Q^2\,. \quad \text{(B.2)}$$

To ensure the correct Euclidean signature we take $\eta > \xi$, $G(\eta) > 0$ and $F(\xi) < 0$ in a compact interval. This is possible only if $E > 0$, so that[46] $\xi_- \leq \xi \leq \xi_+ < \eta_+ \leq \eta$, where we use η_+ to denote the largest real root of $G(\eta)$ and $\xi_\pm$ the two roots of $F(\xi)$. Given the vierbein choice

$$e^1 = \sqrt{\frac{\eta - \xi}{-2F(\xi)}}d\xi\,, \qquad\qquad e^2 = \sqrt{\frac{-2F(\xi)}{\eta - \xi}}(d\phi + \eta\,d\psi)\,,$$

$$e^3 = \sqrt{\frac{2G(\eta)}{\eta - \xi}}(d\phi + \xi\,d\psi)\,, \qquad e^4 = \sqrt{\frac{\eta - \xi}{2G(\eta)}}d\eta\,, \quad \text{(B.3)}$$

we can construct the Kähler 2-form $\omega_{(2)}$ and the holomorphic $(2,0)$-form $\Omega^{(2,0)}$ as

$$\omega_{(2)} = e^1 \wedge e^2 + e^4 \wedge e^3\,,$$
$$\Omega^{(2,0)} = e^{2i(E\phi + N\psi)}(e^1 + ie^2) \wedge (e^4 + ie^3)\,. \quad \text{(B.4)}$$

They satisfy the algebraic conditions of a $SU(2)$-structure, namely $\omega \wedge \Omega = 0$ and $-\frac{1}{2}\omega \wedge \omega = -\frac{1}{4}\Omega \wedge \bar{\Omega} = \text{vol}_4$, and it can be easily checked that they are closed. Moreover, it can be checked that (B.1) is Ricci-flat, $R_{\mu\nu} = 0$.

Regularity

The procedure adopted to ensure regularity and find the quantization conditions is analogous to the one of section 3.2. Likewise, we introduce two patches $U_\pm$ such that $\xi_+ \in U_+$, $\xi_- \in U_-$ and $\mathcal{M}_4 = U_+ \cup U_-$. We then adopt a change of coordinates

$$U_\pm: \quad \phi^\pm = \eta_+\theta_1^\pm - \xi_\pm\theta_2^\pm\,, \quad \psi^\pm = -\theta_1^\pm + \theta_2^\pm\,, \quad \text{(B.5)}$$

---

[46]In a similar fashion as the comment at the beginning of paragraph 3.2.2 we note that $\eta = \eta_+ = \xi_+ = \xi$ is a singular point.

after which the metric reads

$$
U_\pm : \quad \mathrm{d}s_4^2 = \frac{\eta - \xi}{2G(\eta)}\mathrm{d}\eta^2 + \frac{\eta - \xi}{-2F(\xi)}\mathrm{d}\xi^2 - \frac{2(\eta - \xi_\pm)^2 F(\xi)}{\eta - \xi}\mathrm{d}(\theta_2^\pm)^2
$$
$$
- 2(\eta - \eta_+)\frac{\mathrm{d}(\theta_1^\pm)^2(\eta - \eta_+) - 2(\eta - \xi_\pm)\mathrm{d}\theta_1^\pm \mathrm{d}\theta_2^\pm}{\eta - \xi}F(\xi) \tag{B.6}
$$
$$
+ 2\frac{\left[(\xi - \xi_\pm)\mathrm{d}\theta_2^\pm + \mathrm{d}\theta_1^\pm(\eta_+ - \xi)\right]^2}{\eta - \xi}G(\eta) \, .
$$

The boundary metric, defined as

$$
\lim_{\eta \to +\infty} \mathrm{d}s_4^2 = \mathrm{d}\eta^2/(2E\eta) + (2E\eta)\mathrm{d}s_b^2 + O(1/\eta) \, , \tag{B.7}
$$

is found to be

$$
U_\pm : \quad \mathrm{d}s_b^2 = \frac{-\mathrm{d}\xi^2}{4EF(\xi)} - \frac{(\eta_+ - \xi_\pm)^2 F(\xi)}{E(\eta_+ - \xi)^2 - F(\xi)}\mathrm{d}(\theta_2^\pm)^2
$$
$$
+ \frac{[E(\eta_+ - \xi)^2 - F(\xi)]}{E}\left\{\mathrm{d}\theta_1^\pm + \frac{E(\eta_+ - \xi)(\xi - \xi_\pm) + F(\xi)}{[E(\eta_+ - \xi)^2 - F(\xi)]}\mathrm{d}\theta_2^\pm\right\}^2 , \tag{B.8}
$$

where the normalization is such that the Ricci scalar reads $R = 6E^2$. As in (3.24), we can zoom near the zeros of $F(\xi)$, namely $\xi_\pm$, and impose

$$
U_\pm : \quad \pm F'(\xi_\pm)\Delta\theta_2^\pm = \frac{2\pi}{m_\pm} \, , \tag{B.9}
$$

after which the metric (B.8) describes topologically a fibration over a spindle $\mathbb{\Sigma}_{[m_-, m_+]}^\infty$.

Similarly to (3.28), the metric on $\mathbb{\Sigma}_{[m_-, m_+]}^\infty$ is the first line of (B.8) and again we can show that $\Delta\theta_1^+ = \Delta\theta_1^- \equiv \Delta\theta_1$. Moreover, introducing

$$
\Delta \equiv (\eta_+ - \xi_+)\Delta\theta_2^+ = (\eta_+ - \xi_-)\Delta\theta_2^- \, , \tag{B.10}
$$

equation (B.9) and the well-definiteness of the fibration read

$$
U_\pm : \quad \pm \frac{F'(\xi_\pm)}{\eta_+ - \xi_\pm}\Delta = \frac{2\pi}{m_\pm}, \quad t = m_- m_+ \frac{(\xi_+ - \xi_-)}{(\eta_+ - \xi_+)(\eta_+ - \xi_-)}\frac{\Delta}{\Delta\theta_1} \, . \tag{B.11}
$$

Proceeding on the same lines to analyze the bulk we find an additional condition (analogous to (3.48))

$$
G'(\eta_+)\Delta\theta_1 = \frac{2\pi}{v} \, . \tag{B.12}
$$

After imposing (B.11) and (B.12), the topology is indeed $\mathbb{C}/\mathbb{Z}_v \hookrightarrow \mathcal{O}(-t) \to \mathbb{\Sigma}_{[m_-, m_+]}$.

Quantization

We now proceed to solve all the quantization conditions (B.11) and (B.12), using the following expressions for the parameters in terms of the roots[47]

$$E = -\frac{1}{\xi_- \xi_+} P^2,$$

$$N = -\frac{(\xi_- + \xi_+)}{2\,\xi_- \xi_+} P^2,$$

$$Q^2 = -\frac{\eta_+ (\eta_+ - \xi_+ - \xi_-)}{\xi_- \xi_+} P^2.$$

(B.13)

In particular, employing the parametrization[48] $\xi_\pm = w(1 \pm x)$, equation (B.11) implies

$$w = \frac{m_- - m_+}{m_-(1+x) - m_+(1-x)}\,\eta_+, \quad \frac{\Delta}{2\pi} = \frac{(m_- - m_+)(x^2 - 1)}{\big[m_-(1+x) - m_+(1-x)\big]^2}\frac{\eta_+^2}{P^2},$$

(B.14)

and

$$\frac{\Delta\theta_1}{2\pi} = \frac{(m_- - m_+)^2(x^2 - 1)}{2x(m_- + m_+)\big[m_-(1+x) - m_+(1-x)\big]}\frac{\eta_+}{vP^2}.$$

(B.15)

Finally (B.12) results in

$$t = v\,(m_- + m_+).$$

(B.16)

Toric data

Let us now construct the polytope. The symplectic two-form $\omega_{(2)}$ in (B.4), after plugging in the choice of the vierbein (B.3), can be equivalently rewritten as

$$U_\pm: \quad \omega_{(2)} = \mathrm{d}(\eta + \xi) \wedge \mathrm{d}\phi^\pm + \mathrm{d}(\eta\,\xi) \wedge \mathrm{d}\psi^\pm.$$

(B.17)

In the 2π-periodic coordinates

$$\varphi_1^\pm = \frac{2\pi}{\Delta\theta_1}\theta_1^\pm, \quad \varphi_2^\pm = \frac{2\pi}{\Delta\theta_2}\theta_2^\pm,$$

(B.18)

where the coordinates $(\theta_1^\pm, \theta_2^\pm)$ were defined in (B.5), (B.17) reads

$$\omega_{(2)} = \mathrm{d}\bigg\{\frac{\Delta\theta_1}{2\pi}[\eta_+(\eta + \xi) - \eta\,\xi]\bigg\} \wedge \mathrm{d}\varphi_1^\pm + \mathrm{d}\bigg\{-\frac{\Delta\theta_2^\pm}{2\pi}[\xi_\pm(\eta + \xi) - \eta\,\xi]\bigg\} \wedge \mathrm{d}\varphi_2^\pm.$$

(B.19)

Without loss of generality we restrict to the patch U_-. Then, with respect to the toric base

$$E_1 \equiv \partial_{\phi_1} = \partial_{\varphi_1^-}, \quad E_2 \equiv \partial_{\phi_2} = \partial_{\varphi_2^-} - \frac{r_+}{m_-}\partial_{\varphi_1^-},$$

(B.20)

---

[47]Note that E is required to be positive, implying that ξ_- is negative, which in turn means that $\xi_+ > 0$, due to the negative sign in front of P^2 and Q^2 in $G(\eta)$ and $F(\xi)$.

[48]For this parametrization to be effective we should take either $(0 < w < \eta_+,\ x > 1)$ or $(w < 0,\ x < -1)$.

the symplectic form ω becomes

$$
\begin{aligned}
\omega_{(2)} =& \mathrm{d}\left\{\frac{\Delta\theta_1}{2\pi}[\eta_+(\eta+\xi)-\eta\,\xi]\right\} \wedge \mathrm{d}\phi_1+ \\
& \mathrm{d}\left\{-\frac{\Delta\theta_1}{2\pi}[\eta_+(\eta+\xi)-\eta\,\xi]\frac{r_+}{m_-}-\frac{\Delta\theta_2^-}{2\pi}[\xi_-(\eta+\xi)-\eta\,\xi]\right\} \wedge \mathrm{d}\phi_2\,.
\end{aligned}
\tag{B.21}
$$

Therefore, the moment map relative to (B.20) is

$$
\vec{\mu}(\xi,\eta) = \frac{\Delta\theta_1}{2\pi}\left([\eta_+(\eta+\xi)-\eta\,\xi]\,,-[\eta_+(\eta+\xi)-\eta\,\xi]\frac{r_+}{m_-}-\frac{\Delta\theta_2^-}{\Delta\theta_1}[\xi_-(\eta+\xi)-\eta\,\xi]\right).
\tag{B.22}
$$

We define the loci

$$
\mathcal{L}_1 = \{\xi=\xi_+\}\,, \qquad \mathcal{L}_2 = \{\eta=\eta_+\}\,, \qquad \mathcal{L}_3 = \{\xi=\xi_-\}\,,
\tag{B.23}
$$

which intersect at the two fixed points

$$
p_1 = \{\xi=\xi_+,\eta=\eta_+\}\,, \quad p_2 = \{\xi=\xi_-,\eta=\eta_+\}\,.
\tag{B.24}
$$

Formally we can also define

$$
p_3 = \{\xi=\xi_-,\eta=\eta_0>\eta_+\}\,, \quad p_4 = \{\xi=\xi_+,\eta=\eta_0>\eta_+\}\,,
\tag{B.25}
$$

where η_0 is taken to be much greater than η_+.

The Killing vectors K_a degenerating on $\mathcal{L}_a$ and normalized to have unit surface gravity

$$
\begin{aligned}
K_1 &= \frac{1}{-F'(\xi_+)}(\xi_+\partial_{\phi^-}-\partial_{\psi^-}) = m_+\left(\partial_{\varphi_2^-}-\frac{t}{m_+m_-}\partial_{\varphi_1^-}\right), \\
K_2 &= \frac{1}{G'(\eta_+)}(\eta_+\partial_{\phi^-}-\partial_{\psi^-}) = v\,\partial_{\varphi_1^-}\,, \\
K_3 &= \frac{1}{F'(\xi_-)}(\xi_-\partial_{\phi^-}-\partial_{\psi^-}) = m_-\partial_{\varphi_2^-}\,,
\end{aligned}
\tag{B.26}
$$

in this case become

$$
\begin{aligned}
K_1 &= m_+E_2 - r_-E_1\,, \\
K_2 &= v\,E_1\,, \\
K_3 &= m_-E_2 + r_+E_1\,.
\end{aligned}
\tag{B.27}
$$

We can now construct the polytope both from the moment map (B.22) or from the prescription $K_a = \vec{v}_a\cdot(E_1,E_2)$, which allows one to extract the (non-primitive) vectors $\vec{v}_a$ describing the labelled polytope. Using the moment map (B.22) we get

$$
\begin{aligned}
\vec{\mu}(p_4) - \vec{\mu}(p_1) &= \frac{\Delta\theta_1}{2\pi}(\eta_+-\eta_0)(\eta_+-\xi_+)(-1,-r_-/m_+)\,, \\
\vec{\mu}(p_1) - \vec{\mu}(p_2) &= \left(0,\frac{\Delta\theta_1}{2\pi}\frac{t}{m_-m_+}(\eta_+-\xi_+)(\eta_+-\xi_-)\right), \\
\vec{\mu}(p_2) - \vec{\mu}(p_3) &= \frac{\Delta\theta_1}{2\pi}(\eta_+-\eta_0)(\eta_+-\xi_-)(1,-r_+/m_-)\,.
\end{aligned}
\tag{B.28}
$$

The outward-pointing normal vectors coming from (B.27) or (B.28) are

$$\vec{v}_1 = (r_-, -m_+)\,, \quad \vec{v}_2 = (v, 0)\,, \quad \vec{v}_3 = (r_+, m_-)\,, \tag{B.29}$$

The resulting polytope is exactly the same of figure 2a, or also the one in figure 5 of [53] in the CY context. The charges defining the Kähler quotient such that $\sum_a^3 Q^a \vec{v}_a = 0$ can be taken to be

$$Q^a = (v\, m_-, -t, v\, m_+)\,. \tag{B.30}$$

Since t in our explicit solution is given by (B.16), we have that $\sum_a^3 Q^a = 0$ and, in turn, our space is a CY_2[49]. This generalizes the computation in [53], where the horizontal vector was not associated to a label; the topology considered there was $\mathbb{C} \hookrightarrow \mathcal{O}(-t) \to \mathbb{\Sigma}_{[m_-,m_+]}$, while here it is $\mathbb{C}/\mathbb{Z}_v \hookrightarrow \mathcal{O}(-t) \to \mathbb{\Sigma}_{[m_-,m_+]}$.

C Allowed values of the parameters

In this appendix we comment on the allowed lens spaces $L(t, 1)$ at the boundary and their fillings for the cases presented in the paper, showing some explicit examples. We start from the simplest examples, which are the subcases of CP with $N = 0$ in section 3.5.3. Then we consider the general CP solution with twist, concluding finally with the special solution of PD of section 5.4. Recall that in all the cases, when $t = 0$ we have $m_- = m_+ = 1$ and the spindle degenerates to a sphere (see the summarizing table 1). We say that a solution is acceptable if

$$\mathbf{P}(p) \leq 0\,, \ \mathbf{Q}(q) \geq 0\,, \ \mathbf{q}_+^2 - \mathbf{p}_\pm^2 \geq 0\,, \ \mathbf{p}_- < \mathbf{p}_+ < \mathbf{q}_+\,, \ \mathbf{p}_- + \mathbf{p}_+ > 0\,, \ (m_\pm, v, t) \in \mathbb{N}\,. \tag{C.1}$$

The first four requirements come from the fact that the metric must have Euclidean signature, whilst $\mathbf{p}_- + \mathbf{p}_+ > 0$ has been fixed using the scaling symmetry (2.10).

$N = 0$, $P = 0$, $\alpha > 1$ case

We begin by studying the case (i) of 3.5.3, focusing on $\alpha > 1$. In this case t can be read from (3.127), namely

$$t = v \frac{(\sigma + \sqrt{\alpha})(m_- - \sigma m_+)}{(\alpha - 1)(\sqrt{m_- - \sqrt{\alpha} m_+})} \left[2\alpha^{1/4}\sqrt{\sqrt{\alpha} m_- - m_+} + \delta(1 + \sigma\sqrt{\alpha})\sqrt{m_- - \sqrt{\alpha} m_+} \right]\,. \tag{C.2}$$

Recall that this case admits both twist and anti-twist. Moreover, note that we require $\alpha > 1$ and $m_- > \sqrt{\alpha}\, m_+$; these two conditions are sufficient to ensure that the requirements (C.1) are satisfied.

---

[49]Since this is two-dimensional toric, Calabi-Yau metric, it should be diffeomorphic to a two-center Gibbons-Hawking metric [89]. We thank B. Acharya for making this observation.

Anti-twist

In the anti-twist case, which corresponds to $\sigma = -1$, (C.2) reads

$$t = v\frac{(-1+\sqrt{\alpha})(m_- + m_+)}{(\alpha-1)(\sqrt{m_- - \sqrt{\alpha}m_+})}\left[2\alpha^{1/4}\sqrt{\sqrt{\alpha}m_- - m_+} + \delta(1-\sqrt{\alpha})\sqrt{m_- - \sqrt{\alpha}m_+}\right]. \quad \text{(C.3)}$$

In both the cases $\delta = \pm 1$, we are able to prove that

$$\partial_\alpha t(\alpha) > 0, \quad \forall\alpha \in \left(1, \frac{m_-^2}{m_+^2}\right), \quad \text{(C.4)}$$

hence implying that $t(\alpha)$ attains its infimum at $\alpha = 1$, which is $\inf_\alpha t(\alpha) = v(m_- + m_+)$. Given that $(m_-, m_+, v) \in \mathbb{N}$ with $m_- > m_+$, we have $t \geq 4$. Note that the inequality

$$t > v(m_- + m_+) > 3, \quad \text{(C.5)}$$

means that at fixed t we have only a finite set of (m_-, m_+, v) triples which yields an acceptable solution. For completeness, we note that a possible example for $t = 4$ is $(t = 4, v = 1, m_+ = 1, m_- = 2)$ and α fixed from (C.3), namely $\alpha \approx 1.36$ for $\delta = 1$ and $\alpha \approx 1.63$ for $\delta = -1$.

Twist

In the twist case, which corresponds to $\sigma = 1$, t reads

$$t = v\frac{(1+\sqrt{\alpha})(m_- - m_+)}{(\alpha-1)(\sqrt{m_- - \sqrt{\alpha}m_+})}\left[2\alpha^{1/4}\sqrt{\sqrt{\alpha}m_- - m_+} + \delta(1+\sqrt{\alpha})\sqrt{m_- - \sqrt{\alpha}m_+}\right]. \quad \text{(C.6)}$$

When $\delta = -1$, for example, we can prove in a similar fashion that

$$\partial_\alpha t(\alpha) > 0, \quad \forall\alpha \in \left(1, \frac{m_-^2}{m_+^2}\right), \quad \text{(C.7)}$$

resulting again in

$$t > v(m_+ + m_-), \quad \text{(C.8)}$$

which implies $t \geq 4$ and an analogous finite bound on the number of (m_-, m_+, v) triples at fixed t. A possible example is $(t = 4, v = 1, m_+ = 1, m_- = 2)$ and α fixed from (C.6), namely $\alpha \approx 1.81$.

$N = 0$, $M = 0$, $\alpha > 1$ **case**

Moving now to the case (ii) of 3.5.3 with $\alpha > 1$, which is an anti-twist case, t is

$$t = v\frac{\sqrt{-m_+ + m_-\sqrt{-1+2\sqrt{\alpha}}}}{\sqrt{m_- - m_+\sqrt{-1+2\sqrt{\alpha}}}}\frac{m_- + m_+\sqrt{-1+2\sqrt{\alpha}}}{(-1+2\sqrt{\alpha})^{1/4}}. \quad \text{(C.9)}$$

Here we have to ensure that $\alpha > 1$ and $m_- > m_+\sqrt{-1+2\sqrt{\alpha}}$; these conditions are sufficient to meet the requirements in (C.1).

We can show that

$$\partial_\alpha t(\alpha) > 0 \,, \quad \forall \alpha \in \left(1, \frac{(m_-^2 + m_+^2)^2}{4 m_+^4}\right) . \tag{C.10}$$

In this case the main implication is as well

$$t > v(m_+ + m_-), \tag{C.11}$$

resulting in $t \geq 4$ and a finite number of (v, m_-, m_-) triples at fixed t. As usual, we provide an example: an acceptable solution is given by $(t = 4, v = 1, m_+ = 1, m_- = 2)$ and α determined by (C.9), that is $\alpha \approx 1.57$.

Generic non-accelerating solutions

We can try a similar analysis on the solutions presented in section 3.5.1. In the twist case it can be shown easily that $t > 1$. Indeed, the expression (3.106) for t is simply

$$t = \frac{v(\tilde{q}_+ - \kappa)}{x}(m_- - m_+) . \tag{C.12}$$

Since $m_- > m_+$ and $\tilde{q}_+ > 1 + x$ from (3.103), this clearly implies $t > v \geq 1$. There are, however, acceptable configurations for $t = 2$, such as $(t = 2, v = 1, m_- = 2, m_+ = 1)$, given $(\tilde{q}_+ = 4, \kappa = 1, x = 3/2)$. In the anti-twist case a general analysis is much more complicated, due to the form of t in (3.117). We shall refrain to do such an analysis, simply noticing that exist infinite values of the parameters for which all the conditions (C.1) are met. As an example we can take $(t = 10, v = 1, m_- = 4, m_+ = 1)$ as well as $(\tilde{q}_+ = 2, x \approx 0.39, \kappa = -1)$.

Accelerating solutions

For the solution discussed in section 5.4, we can produce various examples, such as

$$\lambda = 1, t = 1, v = 1, m_- = 3, m_+ = 1, Q = 1, P = 2\sqrt{6}, \omega = 1 . \tag{C.13}$$

For CP we can not have $t = 1$, which can occurs only in the $U(1) \times U(1)$-invariant solution of [16]. This means that *only* in the accelerating case we can have the squashed S^3 at the boundary.

En passant, we also note that the two following solutions are admissible:

$$\begin{aligned} \text{CP}: \quad & \lambda = -1, t = 5, m_- = 5, m_+ = 1, v = 1, Q = \frac{9P}{4\sqrt{5}}, P > 0, \omega > \frac{2}{\sqrt{5}} , \\ \text{PD}: \quad & \lambda = \eta = 1, t = 5, m_- = 5, m_+ = 1, v = 1, \tilde{q}_+ = 1 + \frac{5x}{4}, \tilde{P} > 0, x > \frac{8}{9} ; \end{aligned} \tag{C.14}$$

this shows that the same topological data $(m_\pm, v, t)$ can indeed correspond to two different non-diffeomorphic solutions.

D   Uplift to M-theory

In this appendix we comment on the conditions under which our solution to (2.2) and (2.4) uplifts to a M-theory solution. In particular, using the results in [40, 41] one can uplift locally any supersymmetric solution on a seven-dimensional Sasaki-Einstein manifold, but there can be global obstructions.

We begin our analysis recalling the bosonic part of the Euclidean 11d supergravity action[50],[51]

$$
-(16\pi G_{11})S_{11} = \int \left[R \star 1 - \frac{1}{2} \star G_{(4)} \wedge G_{(4)} \right] - \frac{1}{6} \int G_{(4)} \wedge G_{(4)} \wedge C_{(3)}
$$
$$
= \int \mathrm{d}^{11}x \sqrt{g} \left[R - \frac{1}{2}|G_{(4)}|^2 \right] - \frac{\tilde{\epsilon}^{M_1...M_{11}}}{(3!\,2)^4} \int \mathrm{d}^{11}x G_{M_1...M_4} G_{M_5...M_8} C_{M_9...M_{11}},
$$
$$(D.1)$$

whose equations of motion read

$$
R_{MN} = \frac{1}{12} \left[G_{MR_1...R_3} G_N{}^{R_1...R_3} - \frac{1}{12} g_{MN} G_{R_1...R_4} G^{R_1...R_4} \right], \quad \mathrm{d} \star G_{(4)} = -\frac{1}{2} G_{(4)} \wedge G_{(4)},
$$
$$(D.2)$$

where $G_{(4)} = \mathrm{d}C_{(3)}$ and $\star$ is the Hodge star map with respect to the eleven-dimensional metric. It is possible to show (see [40, 41]) that the Ansatz

$$
\mathrm{d}s_{11}^2 = L^2 \left\{ \frac{1}{4} \mathrm{d}s_{4\mathrm{d}}^2 + \left[\eta + \frac{1}{2} A_{4\mathrm{d}} \right]^2 + \mathrm{d}s_T^2 \right\},
$$
$$
G_{(4)} = L^3 \left\{ \frac{3}{8} \star_{4\mathrm{d}} 1 - \frac{1}{4} \star_{4\mathrm{d}} F_{4\mathrm{d}} \wedge \mathrm{d}\eta \right\},
$$
$$(D.3)$$

with $R_{mn}^{(7)} = 6g_{mn}^{(7)}$, indeed leads to a solution of (D.2) for any solution of (2.2). Here we adopt the conventions of [15], where the effective radius L of AdS$_4$ is determined by the quantization of $\star G_{(4)}$ through Y_7, which is a Sasaki-Einstein seven-manifold with contact one-form η and transverse Kähler-Einstein metric $\mathrm{d}s_T^2$.

The uplift of our spindle Bolt solution parallels the one in section 6.3 of [15]. We basically just need to remark that the spindle Bolt solution is topologically $\mathcal{M}_4 \equiv \mathbb{C}/\mathbb{Z}_v \hookrightarrow \mathcal{O}(-t) \to \Sigma_{[m_+,m_-]}^{q_+}$, while the gauge field is quantized according to (3.119), which reads

$$
\frac{1}{2\pi} \int_{\mathcal{L}_2} F_{4\mathrm{d}} = \frac{\eta}{2} \left(\frac{m_- + \sigma m_+}{m_- m_+} - \kappa \frac{t/v}{m_- m_+} \right),
$$
$$(D.4)$$

where $\mathcal{L}_2$ is the surface of the spindle bolt Σ^{q_+}. Let us focus on the case of a regular Sasaki-Einstein internal manifold Y_7. By definition, in this case, Y_7 can be regarded as the $U(1)$ fibration over a six-dimensional Kähler-Einstein base B_6 with positive curvature and metric $\mathrm{d}s_T^2$. In particular, we can write $\eta = \mathrm{d}\psi + \sigma$, where the Ricci form ρ of B_6 is $\rho = 4\,\mathrm{d}\sigma$ and the coordinate ψ is such that the Reeb vector ξ reads $\xi = \partial_\psi$. The period of

---

[50]Here and in what follows we define, for any p-form ω, $\omega^2 = \frac{1}{p!}\omega_{\mu_1..\mu_p}\omega^{\mu_1..\mu_p}$.

[51]We adopt the following (classical) convention for the indices: capital Latin letters $\{M, N, \ldots\}$ are reserved for the eleven-dimensional metric and take values in $\{1, \ldots, 11\}$, small Greek letters $\{\mu, \nu, \ldots\}$ are used for the four-dimensional external space, and small Latin letters $\{m, n, \ldots\}$ are left for the internal seven-dimensional metric.

ξ is canonically $2\pi/4$, which coincides with the smallest period required in order to have that the two Killing spinors are globally defined. Smaller periods would imply non-single valued Killing spinors, but larger ones are indeed admissible. In particular, recall that a Sasaki-Einstein is simply connected if and only if ξ has period $2\pi I/4$, where $I \equiv I(B_6)$ is the Fano index of B_6.

We can then consider discrete quotients of Sasaki-Einstein manifolds at the price of losing their simple connectedness; for example we can consider $S^7/\mathbb{Z}_4$ or $S^7/\mathbb{Z}_2$. This discussion can be summarized saying that ξ has periodicity $\frac{2\pi I}{4k}$, where the integer k divides I.

In the connection part of the uplift metric in (D.3) we see that, since ξ has period $\frac{2\pi I}{4k}$, we must have

$$\frac{4k}{2I} \int_{\mathcal{L}_2} \frac{F_{\text{4d}}}{2\pi} = \frac{m}{v\,m_-m_+}\,, \quad m \in \mathbb{Z}\,, \tag{D.5}$$

in order to have a well-defined $U(1)$ orbibundle.

Comparing (D.4) and (D.5), it is immediate to see that

$$k\eta\,(m_-v + \sigma m_+ v - \kappa t) = mI\,. \tag{D.6}$$

When $k = I$, which amounts to requiring canonical period $2\pi/4$ for ξ, it is easy to see that the condition (D.6) is always satisfied, and then any spindle Bolt solution can be globally uplifted to a M-theory solution on a regular SE manifold. Examples of internal manifolds with such property are, for example, the homogeneous $S^7/\mathbb{Z}_4$, $V_{5,2}/\mathbb{Z}_3$, $Q^{2,2,2} = Q^{1,1,1}/\mathbb{Z}_2$, $M_{3,2}$, $N_{1,1}$.

However, in general, the condition (D.6) leads to restrictions. For example, consider the case of $M_7 = S^7/\mathbb{Z}_2$, which implies $I = 2$, $k = 1$. By (D.6) this means that $m_-v + \sigma m_+ v - \kappa t$ is divisible by 2, which is not satisfied for all values of (m_-, m_+, v, t); in those cases where this condition is not met, the solution cannot be globally uplifted to a M-theory solution on $\text{AdS}_4 \ltimes S^7/\mathbb{Z}_2$.

To conclude the analysis, let us note that, when Y_7 is irregular, the uplift is not possible in general. This is due to the fact that when Y_7 is irregular, ξ cannot be periodically identified on Y_7. Therefore, since $\eta + \frac{1}{2}A_{\text{4d}}$ defines a global one-form only when ξ is periodically identified in $\eta = \mathrm{d}\psi + \sigma$, our Bolt solutions cannot be lifted on irregular Sasaki-Einstein manifolds. The caveat here is that the flux of A_{4d} through the spindle bolt can vanish in some cases: this happens, for example, in the no-twist case in (i) of 3.5.3, or whenever (D.4) vanishes, namely

$$v(m_- + \sigma m_+) - \kappa t = 0\,. \tag{D.7}$$

Note that, when $\sigma = 1$ and $\kappa = 1$, this is precisely the condition (B.16). When (D.7) is met, it is indeed possible to uplift the solution for any choice of Y_7.

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