CHARACTERIZING THE RANGE OF THE COMPLEX MONGE-AMPÈRE OPERATOR

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ABSTRACT. In this article, we solve the complex Monge–Ampère equation for measures with a pluripolar part in compact Kähler manifolds. This result generalizes the classical results obtained by Cegrell in bounded hyperconvex domains. We also discuss the properties of the complex Monge–Ampère operator in some special cases.

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1. INTRODUCTION

Let (X, ω) be a Kähler manifold of complex dimension n. Let $[\alpha] \in H^{1,1}(X, \mathbb{R})$ be another cohomology class. The study of the complex Monge–Ampère type equation

(1.1)
$$(\alpha + dd^c \varphi)^n = \mu$$

has been one of most important topics in Kähler geometry over the past few decades.

When α is positive and μ is a normalized smooth volume form, Yau [25] proved that (1.1) admits a unique smooth solution, solving the famous Calabi conjecture. In this case, the left-hand side of (1.1) is the classic *n*-th wedge product.

When the cohomology class $[\alpha]$ is allowed to be degenerate and the measure μ is allowed to be nonsmooth but non-pluripolar, the solutions of (1.1) are closely related to the singular Kähler–Einstein metric. To solve the corresponding equation, the concept of a non-pluripolar product plays an important role. Related equations have been explored in [6, 7, 16, 17, 18, 20, 21, 22]. In this case, the left-hand side of (1.1) is the non-pluripolar complex Monge–Ampère measure $\langle \alpha_{i\alpha}^n \rangle$.

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For the more general case, when $\alpha := \omega$ and μ is allowed to have a pluripolar part(see Corollary 4.4), we know almost nothing about the related equations. Coman, Guedj, and Zeriahi [15] developed theories to consider related problems; an important concept is the *domain of the definition of the complex Monge–Ampère operator* $DMA(X, \omega)$ (see Definition 2.8). Characterizing the range of the complex Monge–Ampère operator on $DMA(X, \omega)$ is a major open problem.

A special and interesting problem mentioned in [19, Question 12] is that when $X = \operatorname{BL}_p \mathbb{P}^2$, $p \in \mathbb{P}^2$, $0 < \alpha := \omega \in -c_1(K_X)$ and $\mu := [D] \wedge \omega/V$, where D is a smooth anticanonical divisor, [D] is the (1,1)-current integral along D and V is the normalized constant, can one find $\varphi_D \in DMA(X, \omega)$ as the solution of the equation?

Motivated by these related problems, we consider the complex Monge–Ampère equation with a measure that has a pluripolar part in this article. Since we know little about $DMA(X, \omega)$, we consider mainly the Blocki–Cegrell class $\mathcal{D}(X, \theta) \subset DMA(X, \theta)$ (see Definition 2.10; see also Table 1 for the relation between these classes) in this article, where θ is a smooth, closed semi-positive real (1, 1)-form on X whose cohomology class is big. Note that the Kähler case is a special case in our setting.

In the Kähler case, $\alpha := \omega$ is a Kähler form. In a recent Paper [3], Andersson, Witt Nyström and Wulcan defined the finite non-pluripolar energy class $\mathcal{G}(X,\omega)$ (see Definition 2.15). They proved $\mathcal{D}(X,\omega) \subset \mathcal{G}(X,\omega)$; i.e., if $\varphi \in \mathcal{D}(X,\omega)$, then $\varphi \in L^1(\langle \omega_{\varphi}^i \rangle \wedge \omega^{n-i})$, i = 0, 1, ..., n-1, see [3, Theorem 1.10]. One benefit of [3, Theorem 1.10] is that we can express the Monge–Ampère op-

One benefit of [3, Theorem 1.10] is that we can express the Monge–Ampère operator on $\mathcal{D}(X, \omega)$ via the non-pluripolar product; see (3.1). Therefore, we present the corresponding generalized theorem.

Theorem 1.1. (=Theorem 3.2) Let (X, ω) be a compact Kähler manifold of complex dimension n and $\varphi_i \in \mathcal{D}(X, \omega)$, i = 1, ..., n. Then, $\varphi_1 \in L^1(\langle (\omega + dd^c \varphi_2) \land$

... $\wedge (\omega + dd^c \varphi_n) \rangle \wedge \omega).$

In particular, if X is a compact Kähler surface and $\varphi, \psi \in \mathcal{D}(X, \omega)$, then

$$\mathrm{MA}_{\omega}(\varphi,\psi) = \frac{1}{2} \left[(\omega + dd^{c}\varphi) \wedge \langle \omega + dd^{c}\psi \rangle + (\omega + dd^{c}\psi) \wedge \langle \omega + dd^{c}\varphi \rangle \right].$$

The definition of $MA_{\omega}(\cdot, ..., \cdot)$ can be found in Proposition 2.13.

Now assume that $\alpha := \theta$ is a smooth, closed semi-positive real (1, 1)-form on X whose cohomology class is big, we consider the complex Monge–Ampère equation with a normalized measure μ . It follows from the Cegrell–Lebesgue decomposition, Corollary 4.4, that we have the unique decomposition $\mu = \mu_r + \mu_s$, where μ_r is the non-pluripolar measure and $\mu_s = \mathbf{1}_{\{u=-\infty\}}\mu$ for some $u \in \text{PSH}(X, \omega)$. Considering the range of the complex Monge–Ampère operator on $\mathcal{D}(X, \theta)$, a natural idea is to perform the same decomposition for $\text{MA}_{\theta}(\varphi)$, $\varphi \in \mathcal{D}(X, \theta)$ and then couple the regular part and the singular part. To make this idea feasible, we provide a key theorem as follows:

Theorem 1.2. (=Theorem 4.5) Let (X, ω) be a compact Kähler manifold of complex dimension n. Let $\varphi_1, ..., \varphi_n \in \mathcal{D}(X, \theta)$, where θ is a smooth, closed semi-positive real (1,1)-form on X whose cohomology class is big. Then, we have

$$\mathrm{MA}_{\theta}(\varphi_1, ..., \varphi_n)_r = \langle \theta_{\varphi_1} \wedge ... \wedge \theta_{\varphi_n} \rangle$$

and

$$\mathrm{MA}_{\theta}(\varphi_1, ..., \varphi_n)_s = \mathbf{1}_{\bigcup_j \{\varphi_j = -\infty\}} \mathrm{MA}_{\theta}(\varphi_1, ..., \varphi_n).$$

In particular, if $\varphi \in \mathcal{D}(X, \theta)$, then we have

$$\mathrm{MA}_{\theta}(\varphi)_r = \langle \theta_{\varphi}^n \rangle$$
 and $\mathrm{MA}_{\theta}(\varphi)_s = \mathbf{1}_{\{\varphi = -\infty\}} \mathrm{MA}_{\theta}(\varphi).$

This generalizes the local result by Benelkourchi, Guedj, and Zeriah, [8], and Åhag, Czyż, and Lu, Rashkovskii, [2, Page 7-8] to the global setting(see also [15, Corollary 1.8] for the Kähler case). This theorem combined with the recent work of Darvas, Di Nezza, and Lu, [16, 17, 18], named relative pluripotential theory, we can solve the complex Monge–Ampère equation in a particular case:

Theorem 1.3. (=Theorem 4.9) Let (X, ω) be a compact Kähler manifold of complex dimension n, and let θ be a normalized, smooth closed semi-positive real (1,1)-form on X whose cohomology class is big such that $\int_X \theta^n = 1$. Let μ_s be a pluripolar measure on X supported on some pluripolar set. If there exists $\varphi \in \mathcal{D}(X, \theta)$ such that $MA_{\theta}(\varphi)_s = \mu_s$, then for all non-pluripolar measures ν such that $\int_X \nu = 1 - \int_X \mu_s$ and $\nu = f\omega^n$, for some $f \in L^p(\omega^n)$, p > 1, there exists a $\psi \in PSH(X, \theta)$ that is the solution of the equation

$$\operatorname{MA}_{\theta}(\psi) = \nu + \mu_s, \ \psi \in \mathcal{D}(X, \theta).$$

1.0.1. Comparing the classical result in the local setting. Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain. In [1, 2, 9, 10, 12, 13], a powerful theory of the Dirichlet problem of the complex Monge–Ampère equation on Ω was established by several authors. In particular, the author was inspired by [12, 13] to consider the corresponding global result.

Setting $\mathcal{F}(\Omega) := \{ \varphi \in \mathrm{PSH}^{-}(\Omega) : \exists \varphi_j \in \mathcal{E}_0(\Omega), \varphi_j \searrow \varphi \text{ and } \sup_j \int_{\Omega} (dd^c \varphi_j)^n < 0 \}$ $+\infty$. Clearly, we have $\mathcal{F}(\Omega) \subset \mathcal{D}(\Omega)$ (the definition of $\mathcal{E}_0, \mathcal{D}$ can be found in Section 2.2). Let μ be a positive Radon measure. Cegrell proved the following:

Theorem 1.4. ([13, Theorem 6.2]) Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain and $\mu = f(dd^c \varphi_0)^n + \mu_s$, where $\varphi_0 \in \mathcal{E}_0(\Omega)$ and $f \in L^1((dd^c \varphi_0)^n)$. If there exists $\psi \in \mathcal{F}(\Omega)$ such that $MA(\psi) = \mu_s$, then there exists $\varphi \in \mathcal{F}(\Omega)$ such that

$$MA(\varphi) = \mu.$$

In the global setting, we replace the condition of the pluripolar part of μ with the condition that there exists $\varphi \in \mathcal{D}(X, \theta)$ such that $MA_{\theta}(\varphi)_s = \mu_s$ and require that $\mu_r = f\omega^n$ for some $f \in L^p(\omega^n), p > 1$.

1.0.2. Notation and Conventions. In this article, unless stated otherwise, we always assume that

- In the local setting, we consider the domain Ω in \mathbb{C}^n . In the global setting, we consider the compact Kähler manifolds (X, ω) .
- In the global setting, let $[\theta] \in H^{1,1}(X, \mathbb{R})$ be a semi-positive and big cohomology class; that is, θ is a smooth closed real (1,1)-form such that $\theta \geq 0$ and $\int_X \theta^n > 0$. A local potential of θ means that locally, $\theta := dd^c f$, with f being a smooth psh function. • The operator $dd^c := \frac{\sqrt{-1}}{\pi} \partial \bar{\partial}$ is the normalized $\sqrt{-1} \partial \bar{\partial}$ operator.

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- In the global setting, let $PSH(X, \theta)$ be the set of upper semicontinuous functions φ such that $dd^c \varphi + \theta \ge 0$.
- Plurisubharmonic functions are abbreviated as psh functions. When $\varphi \in PSH(X, \theta)$, we abbreviate this by saying that φ is a θ -psh function.
- In any setting, we always assume that a measure is a positive Borel measure.
- Let X be a manifold and {U_i} be an open covering of X. The partition of unity of {U_i} is a family of smooth functions {χ_i} on X such that 0 ≤ χ_i ≤ 1, ∑_i χ_i ≡ 1 and suppχ_i ∈ U_i.

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2. Preliminaries

In this section, we assume that (X, ω) is a compact Kähler manifold of complex dimension n.

In pluripotential theory, the definition of the complex Monge-Ampère operator is important. In the local setting, let $u_1, ..., u_k$ be a locally bounded psh function. Following the construction of Bedford–Taylor [9, 10, 11] (see also [22, Chapter 3]), where the current $dd^c u_1 \wedge ... \wedge dd^c u_k$ is always well defined.

This is also true in the global setting. Let $\varphi^i \in \text{PSH}(X, \theta^i) \cap L^{\infty}(X)$, i = 1, ..., p, where θ^i , i = 1, ..., p are smooth, closed semi-positive real (1, 1)-forms on X whose cohomology classes are big. Then, $(\theta^1 + dd^c \varphi_1) \wedge ... \wedge (\theta^p + dd^c \varphi_p)$ is well-defined due to Bedford–Taylor.

2.1. Non-pluripolar products. In this section, we consider mainly compact Kähler manifolds (X, ω) . Given the positive (1,1)-currents $\theta^1 + dd^c \varphi_1, ..., \theta^p + dd^c \varphi_p$, where θ^i , i = 1, ..., p are smooth, closed semi-positive real (1, 1)-forms on X whose cohomology classes are big and $\varphi_i \in \text{PSH}(X, \theta^i)$. Following the construction of Bedford–Taylor [11] in the local setting, it was shown in [7] (when $\theta^i \ge 0$, $V_{\theta^i} \equiv 0$) that the sequence of currents

 $\mathbf{1}_{\bigcap_{i=1}^{p} \{\varphi_i > -k\}} \left((\theta^1 + dd^c \max\{\varphi_1, -k\}) \land \dots \land (\theta^p + dd^c \max\{\varphi_p, -k\}) \right)$

is non-decreasing in k and converges weakly to the so-called *non-pluripolar product*

$$\langle \theta^1_{\varphi_1} \wedge ... \wedge \theta^p_{\varphi_n} \rangle.$$

For a θ -psh function φ , the non-pluripolar complex Monge-Ampère measure of φ is

$$\langle \theta_{\varphi}^n \rangle := \langle (\theta_{\varphi})^n \rangle.$$

Let X be a complex manifold. A set $A \subset X$ is said to be *pluripolar* in X if, for all $z \in A$, there exist a holomorphic coordinate chart U near z in X and a psh function $\varphi \in \text{PSH}(U)$ such that $A \cap U \subset \{\varphi = -\infty\}$. A positive measure μ of X is said to be a *non-pluripolar measure* if $\int_A \mu = 0$ for all pluripolar sets $A \subset X$.

In particular. Let (X, ω) be a compact Kähler manifold. In the above argument, for all pluripolar sets $A \subset X$ and $\varphi_i \in \text{PSH}(X, \theta^i)$, i = 1, ..., n, we have $\int_A \langle \theta^1_{\varphi_1} \wedge ... \wedge \theta^n_{\varphi_n} \rangle = 0$; i.e., $\langle \theta^1_{\varphi_1} \wedge ... \wedge \theta^n_{\varphi_n} \rangle$ is a non-pluripolar measure.

Proposition 2.1. ([7, Proposition 1.4.c)]) Non-pluripolar products, which are symmetric, are also multilinear in the following sense: if θ^j , j = 0, 1, ..., p are smooth, closed semi-positive real (1,1)-forms on X whose cohomology classes are big, where $\varphi_j \in \text{PSH}(X, \theta^j)$, then

$$\langle (\theta^0_{\varphi_0} + \theta^1_{\varphi_1}) \wedge \theta^2_{\varphi_2} \dots \wedge \theta^p_{\varphi_p} \rangle = \langle \theta^0_{\varphi_0} \wedge \theta^2_{\varphi_2} \dots \wedge \theta^p_{\varphi_p} \rangle + \langle \theta^1_{\varphi_1} \wedge \theta^2_{\varphi_2} \dots \wedge \theta^p_{\varphi_p} \rangle.$$

If φ and ψ are two θ -psh functions on X, then ψ is said to be *less singular* than φ , i.e., $\varphi \leq \psi$, if they satisfy $\varphi \leq \psi + C$ for some $C \in \mathbb{R}$. We say that φ has the same singularity as ψ , i.e., $\varphi \simeq \psi$, if $\varphi \leq \psi$ and $\psi \leq \varphi$. Then, we have

Theorem 2.2. ([16, Theorem 1.1]) Let θ^j , $j \in \{1, ..., n\}$ be a smooth, closed semipositive real (1,1)-form on X whose cohomology classes are big. Let $u_j, v_j \in PSH(X, \theta^j)$ be such that u_j is less singular than v_j , $j \in \{1, ..., n\}$. Then,

$$\int_X \langle \theta^1_{u_1} \wedge \ldots \wedge \theta^n_{u_n} \rangle \geq \int_X \langle \theta^1_{v_1} \wedge \ldots \wedge \theta^n_{v_n} \rangle$$

Note that in [24], Witt Nyström proves a special case of Theorem 2.2. Now, we introduce two important concepts in pluripotential theory.

Envelopes. Let f be a function on X such that $f : X \to \mathbb{R} \cup \{\infty\}$. We define the envelope of f in the class $PSH(X, \theta)$ as

$$P_{\theta}(f) := (\sup\{\varphi \in \mathrm{PSH}(X, \theta) : \varphi \leq f\})^*.$$

Where $h^*(x) := \limsup_{y \to x} h(y)$ and h^* is automatically an upper semicontinuous function. Observe that $P_{\theta}(f) \in \text{PSH}(X, \theta)$ iff there exists some $\varphi \in \text{PSH}(X, \theta)$ lying below f. In the particular case where $f := \min(\psi, \phi)$, we denote the envelope as $P_{\theta}(\psi, \phi) := P_{\theta}(\min(\psi, \phi))$.

In our study of the complex Monge–Ampère equation, the following envelope construction is essential:

Definition 2.3. Given $\psi, \varphi \in PSH(X, \theta)$, the envelope with respect to singularity type $P_{\theta}[\psi](\varphi)$ is defined by

$$P_{\theta}[\psi](\varphi) := \left(\lim_{C \to +\infty} P_{\theta}(\psi + C, \varphi)\right)^*.$$

When $\varphi = 0$, we simply write $P_{\theta}[\psi] := P_{\theta}[\psi](0)$.

We summarize [18, Remark 3.4, Theorem 3.14] as follows.

Proposition 2.4. Let $\varphi \in \text{PSH}(X, \theta)$. Then, we have (i). $\int_X \langle \theta_{\varphi}^n \rangle = \int_X \langle \theta_{P_{\theta}[\varphi]}^n \rangle$, (ii). Set $F_{\varphi} := \{v \in \text{PSH}^-(X, \theta) : \varphi \leq v \leq 0 \text{ and } \int_X \langle \theta_v^n \rangle = \int_X \langle \theta_{\varphi}^n \rangle \}$. Then we have $P_{\theta}[\varphi] = \sup F_{\varphi}$. In particular, $P_{\theta}[\varphi] \succeq \varphi$ and $P_{\theta}[\varphi] = P_{\theta}[P_{\theta}[\varphi]]$.

A θ -psh function φ is said to be a model potential if $P_{\theta}[\varphi] = \varphi$ and $\int_{X} \langle \theta_{\varphi}^{n} \rangle > 0$. From Proposition 2.4, $P_{\theta}[\varphi]$ is a model potential for all $\varphi \in \text{PSH}(X, \theta)$ such that $\int_{X} \langle \theta_{\varphi}^{n} \rangle > 0$.

Relative full mass class.

Definition 2.5. Given a potential $\phi \in PSH(X, \theta)$ such that $\int_X \theta_{\phi}^n > 0$, the relative full mass class is defined by

$$\mathcal{E}(X,\theta,\phi) := \{ u \in \mathrm{PSH}(X,\theta) : u \preceq \phi, \ \int_X \langle \theta_u^n \rangle = \int_X \langle \theta_\phi^n \rangle \}.$$

For the case in which ϕ is a model potential, Darvas, Di Nezza, and Lu [18] achieved a series of significant results on pluripotential theory and the (non-pluripolar product) complex Monge–Ampère equation in the relative full mass class $\mathcal{E}(X, \theta, \phi)$. Their results play a key role in our article, see Section 4.2 for details. That is,

Theorem 2.6. ([18, Theorem 5.17]) Let $\phi \in PSH(X, \theta)$ be a model potential and $\int_X \langle \theta_{\phi}^n \rangle > 0$. Assume that μ is a non-pluripolar positive measure such that $\mu(X) = \int_X \langle \theta_{\phi}^n \rangle$. Then, there exists a unique $u \in \mathcal{E}(X, \theta, \phi)$ such that $\langle \theta_u^n \rangle = \mu$ and $\sup_X u = 0$.

Theorem 2.7. ([18, Theorem 5.20]) Let $\phi \in PSH(X, \theta)$ be a model potential and $\int_X \langle \theta_{\phi}^n \rangle > 0$. Let $u \in \mathcal{E}(X, \theta, \phi)$ with $\sup_X u = 0$. If $\langle \theta_u^n \rangle = \mu$, where μ is a positive measure such that $\mu = f\omega^n$, $f \in L^p(\omega^n)$ for some p > 1, then u has the same singularity type as ϕ . More precisely:

$$\phi - C\left(f, p, \|f\|_{L^p}, \omega, \theta, \int_X \langle \theta_{\phi}^n \rangle\right) \le \varphi \le \phi.$$

Remark. Note that in the above theory, the condition of semi-positive is not necessary. But we only need the semi-positive case.

2.2. The complex Monge–Ampère operators on compact Kähler manifolds. In this section, we assume that θ is a smooth, closed semi-positive real (1, 1)-form on X whose cohomology class is big.

Definition 2.8. Fix $\varphi \in PSH(X, \theta)$. We say that the complex Monge–Ampère measure $(\theta + dd^c \varphi)^n$ is well defined and write $\varphi \in DMA(X, \theta)$ if there exists a Radon measure μ such that for any sequence $\{\varphi_j\}$ of bounded θ -psh functions decreasing to φ on X, the Monge–Ampère measures $(\theta + dd^c \varphi_j)^n$ converge weakly to μ . We set

$$MA_{\theta}(\varphi) = (\theta + dd^c \varphi)^n := \mu.$$

It follows from [22, 3.1-3.5] and [22, Theorem 3.18] that we have $PSH(X, \theta) \cap L^{\infty}(X) \subset DMA(X, \theta)$. Moreover, when $\varphi \in PSH(X, \theta) \cap L^{\infty}(X)$, we have that $MA_{\theta}(\varphi) = (\theta + dd^{c}\varphi)^{n}$.

In the local setting, assume that $\Omega \subset \mathbb{C}^n$ is a bounded domain. We denote $\mathcal{D}(\Omega) \subset \mathrm{PSH}^-(\Omega)$ as the largest subclass of the class of (negative-)psh functions in which the complex Monge-Ampère operator can be well defined, named the Blocki-Cegrell class; i.e., if $\varphi \in \mathcal{D}(\Omega)$, then there exists a measure μ such that for any sequence $\{\varphi_j\} \subset \mathrm{PSH} \cap L^\infty_{loc}(\Omega), \ \varphi_j \searrow \varphi$, the sequence $(dd^c \varphi_j)^n$ converges weakly to μ . We denote $\mathrm{MA}(\varphi) = (dd^c \varphi)^n := \mu$. It follows from [4, Section 2] and [5, Section 1] that we have

Theorem 2.9. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain and $u \in PSH^-(\Omega)$. Then, the following conditions are equivalent:

(i). $u \in \mathcal{D}(\Omega);$

(ii). There exists a measure μ in Ω such that if $U \subset \Omega$ is open and a sequence

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 $u_j \in PSH \cap C^{\infty}(U)$ is decreasing to u in U, then $MA(u_j)$ converges weakly to μ in U;

(iii). For every open $U \subset \Omega$ and any sequence $u_j \in PSH \cap C^{\infty}(U)$ decreasing to u in U, the sequence $MA(u_j)$ is locally weakly bounded in U;

(iv). For every open $U \subset \Omega$ and any sequence $u_j \in PSH \cap C^{\infty}(U)$ decreasing to u in U, the sequence

(2.1)
$$|u_j|^{n-p-2} du_j \wedge d^c u_j \wedge (dd^c u_j)^p \wedge (dd^c |z|^2)^{n-p-1}, \ p = 0, 1, ..., n-2,$$

is locally weakly bounded in U;

(v). For every $z \in \Omega$, there exists an open neighborhood U of z and a sequence $u_j \in PSH \cap C^{\infty}(U)$ decreasing to u in U such that the sequences (2.1) are locally weakly bounded in U.

The bounded hyperconvex domain $\Omega \subset \mathbb{C}^n$ is a bounded domain such that there exists a bounded psh function $h: \Omega \to (-\infty, 0)$ that satisfies $\{h < c\} \in \Omega$ for all c < 0. The unit ball in \mathbb{C}^n is clearly a bounded hyperconvex domain.

When Ω is a bounded hyperconvex domain, Blocki [5, Theorem 2.4] showed that $\mathcal{D}(\Omega)$ coincides with the class $\mathcal{E}(\Omega) \subset \mathrm{PSH}^{-}(\Omega)$ defined by Cegrell in [13]. That is, if $0 \geq \varphi \in \mathcal{E}(\Omega)$, then, for $\forall U \Subset \Omega$, there exists a decreasing sequence $\varphi_j \in \mathcal{E}_0(\Omega)$ such that $\varphi_j \searrow \varphi$ is in U and

$$\sup_{j} \int_{\Omega} (dd^c \varphi_j)^n < +\infty.$$

Where $\mathcal{E}_0(\Omega) := \{ \varphi \in \mathrm{PSH}^- \cap L^\infty(\Omega) : \lim_{z \to \zeta} \varphi(z) = 0, \forall \zeta \in \partial\Omega \text{ and } \int_\Omega (dd^c \varphi)^n < +\infty \}.$

Now, we can consider the Blocki–Cegrell class in the global setting.

Definition 2.10. Let $\mathcal{D}(X,\theta)$ be the set of functions $\varphi \in \text{PSH}(X,\theta)$ such that locally, on any small open holomorphic coordinate chart $U \subset X$, the psh function $\varphi|_U + \rho_U \in \mathcal{D}(U)$, where ρ_U is the local potential of θ on U such that $\varphi|_U + \rho_U \leq 0$.

It is easy to see that we have $\mathcal{D}(X,\theta) \subset DMA(X,\theta)$. Furthermore, when $\theta := \omega$ is a Kähler form, Coman, Guedj, and Zeriahi characterized $\mathcal{D}(X,\omega)$ in [15, Section 3]. They found that $\mathcal{D}(X,\omega)$ is much smaller than $DMA(X,\omega)$. Now we have

TABLE 1. Relations between several classes

Ω is a bounded domain	$L^{\infty}_{loc} \cap \mathrm{PSH}^{-}(\Omega) \subset \mathcal{D}(\Omega)$
Ω is a bounded hyperconvex domain	$\mathcal{E}_0(\Omega) \subset L^{\infty}_{loc} \cap \mathrm{PSH}^-(\Omega) \subset \mathcal{D}(\Omega) = \mathcal{E}(\Omega)$
X is a compact Kähler manifold	$L^{\infty}(X) \cap \mathrm{PSH}(X,\theta) \subset \mathcal{D}(X,\theta) \subset DMA(X,\theta)$

Let $\varphi \in \mathcal{D}(X, \theta)$ and $U \subset X$ be a holomorphic coordinate such that $\theta = dd^c g$ in U. Let $\chi \in C(X)$ such that $\operatorname{supp} \chi \Subset U$. Set $\varphi_k := \max\{\varphi, -k\}$; then, we have

$$\int_X \chi \mathrm{MA}_{\theta}(\varphi) = \lim_k \int_X \chi(\theta + dd^c \varphi_k)^n$$
$$= \lim_k \int_U \chi(dd^c (g + \varphi_k))^n = \int_U \chi \mathrm{MA}(\varphi + g)$$

by the definition of the complex Monge–Ampère operator. Hence,

$$\chi \mathrm{MA}_{\theta}(\varphi) = \chi \mathbf{1}_U \mathrm{MA}(\varphi + g),$$

where $\mathbf{1}_U MA(\cdot)$ represents the complex Monge–Ampère operator in U. We will use this concept frequently and will not explain it in detail again. Many properties of the complex Monge–Ampère operator that are local can be generalized to the global setting by the partition of unity.

Lemma 2.11. Let $\varphi \in \mathcal{D}(X, \theta)$ and $\psi \in PSH(X, \theta)$ such that $\psi \succeq \varphi$. Then, $\psi \in \mathcal{D}(X, \theta)$.

Proof. Note that if $\psi \in \mathcal{D}(X, \theta)$, then $\psi + C \in \mathcal{D}(X, \theta)$, $\forall C \in \mathbb{R}$. Therefore, we can assume that $\psi \geq \varphi$. The proof follows from [5, Theorem 1.2].

And the following is derived from [5, 12]:

Lemma 2.12. Let $\varphi, \psi \in \mathcal{D}(X, \theta)$. Then, $(1 - t)\varphi + t\psi \in \mathcal{D}(X, \theta)$ for $0 \le t \le 1$, which means that $\mathcal{D}(X, \theta)$ is a convex set.

Proof. This lemma follows from the following: if Ω is a holomorphic coordinate chart of X, such that Ω is a bounded domain of \mathbb{C}^n , then $\mathcal{D}(\Omega)$ is a convex cone. Note that $\mathcal{D}(\Omega)$ is a cone, which is straightforward by Theorem 2.9. Hence, we only need to prove that it is convex.

1°. Let $\Omega' \subset \mathbb{C}^n$ be a bounded hyperconvex domain and $u, v \in \mathcal{E}(\Omega')$. Let $U \subseteq \Omega'$ and $u_j, v_j \subset \mathcal{E}_0(\Omega')$ such that $u_j \searrow u, v_j \searrow v$ in U, as in the definition of $\mathcal{E}(\Omega')$. Then, by the argument of [12, Page 188], we have

$$\int_{\Omega'} (dd^c (u_j + v_j))^n \le 3^n \left(\int_{\Omega'} (dd^c u_j)^n + \int_{\Omega'} (dd^c v_j)^n \right)$$

We thus obtain $u + v \in \mathcal{E}(\Omega')$.

2°. Suppose that $u, v \in \mathcal{D}(\Omega)$ and that $t \in [0, 1]$. It clearly follows from Theorem 2.9 that tu, $(1-t)v \in \mathcal{D}(\Omega)$. Then, for $\forall z \in \Omega$, there exists a bounded hyperconvex domain Ω' such that $z \in \Omega' \Subset \Omega$. By [5, Theorem 2.4], we have $tu, (1-t)v \in \mathcal{D}(\Omega') = \mathcal{E}(\Omega')$. Since the Cegrell class is convex, we have $tu + (1-t)v \in \mathcal{D}(\Omega')$. Hence, tu + (1-t)v satisfies the condition (v) in Theorem 2.9 near z for $\forall z \in \Omega$. By Theorem 2.9, we obtain $tu + (1-t)v \in \mathcal{D}(\Omega)$.

Set $MA_{\theta}(\varphi_1, ..., \varphi_n) := (\theta + dd^c \varphi_1) \wedge ... \wedge (\theta + dd^c \varphi_n), \ \varphi_i \in PSH(X, \theta) \cap L^{\infty}(X).$ We then have the following property:

Proposition 2.13. Let $\varphi_1, ..., \varphi_n \in \mathcal{D}(X, \theta)$ and $\varphi_i^j \in \text{PSH}(X, \theta) \cap L^{\infty}(X)$ such that $\varphi_i^j \searrow \varphi_i$ as $j \to +\infty$; then, $MA_{\theta}(\varphi_1^j, ..., \varphi_n^j)$ converges weakly to a Radon measure μ , and the limit measure does not depend on the particular sequence. We denote μ by $MA_{\theta}(\varphi_1, ..., \varphi_n)$.

Proof. 1°. We first prove that the similar conclusion holds in the bounded hyperconvex domain Ω . Suppose that $u_1, ..., u_n \in \mathcal{D}(\Omega)$ and $u_i^j \in \mathrm{PSH}^-(\Omega) \cap L^\infty_{loc}(\Omega)$ so that $u_i^j \searrow u_i$ as $j \to +\infty$. If $h \in \mathcal{E}_0(\Omega)$ and $m_j \searrow -\infty$, we can set $u_i^{j'} := \max\{u_i^j, m_j h\} \in \mathcal{E}_0(\Omega)$ such that $u_i^{j'} \searrow u_i$ as $j \to +\infty$. Then, $dd^c u_1^{j'} \land ... \land dd^c u_n^{j'}$ converges weakly to ν , which does not depend on the particular sequence by [13, Theorem 4.2].

Let $\{U_j\}_j$ be a normal exhaustion of Ω ; i.e., U_j open and $U_j \in U_{j+1} \in \Omega$ so that $\bigcup_j U_j = \Omega$. Set $M_j := \sup_{U_j} h < 0$, we can select a suitable m_j such that $m_j M_j \leq \min_i \{\inf_{U_{j+1}} u_i^j\}$. Then, we have $u_i^{j'} = u_i^j$ in U_j , which means that

$$dd^c u_1^{j'} \wedge \ldots \wedge dd^c u_n^{j'} = dd^c u_1^j \wedge \ldots \wedge dd^c u_n^j$$
 in U_i .

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Hence, $dd^c u_1^j \wedge \ldots \wedge dd^c u_n^j$ converges weakly to ν , which does not depend on the particular sequence.

2°. Let $\{U_i, z_i\}_{i=1}^N$ be the holomorphic atlas of X such that $z_i(U_i) \subset \mathbb{C}^n$ is the unit ball and $\theta = dd^c g_i$ on U_i . Let $\{\chi_i\}_{i=1}^N$ be the partition of unity of $\{U_i\}$. Then we have $\chi_i \operatorname{MA}_{\theta}(\varphi_1^j, ..., \varphi_n^j) = \chi_i \mathbf{1}_{U_i} \operatorname{MA}(\varphi_1^j + g_i, ..., \varphi_n^j + g_i)$ converges weakly to $\chi_i \nu_i$ on U_i as $j \to +\infty$. Set $\mu := \sum_i \chi_i \nu_i$. One can see that

$$\operatorname{MA}_{\theta}(\varphi_1^j, ..., \varphi_n^j) \to \mu$$

in the weak sense as $j \to +\infty$, which does not depend on the particular sequence.

Let $\varphi_1, ..., \varphi_n \in \mathcal{D}(X, \theta)$, and let μ be a measure defined in Proposition 2.13. We set $MA_{\theta}(\varphi_1, ..., \varphi_n) = \mu$, and

(2.2)
$$\operatorname{MA}_{\theta}\left(\varphi_{1}^{(k_{1})},...,\varphi_{n}^{(k_{n})}\right)$$

is defined by $MA_{\theta}(\psi_1, ..., \psi_n)$, where $\psi_1 = ... = \psi_{k_1} := \varphi_1$, $\psi_{k_1+1} = ... = \psi_{k_1+k_2} := \varphi_2, ..., \psi_n = \psi_{n-1} = ... = \psi_{n-k_n} = \varphi_n$.

Corollary 2.14. Assume that $\varphi_1, ..., \varphi_s \in \mathcal{D}(X, \theta)$. Then, for $0 \leq t_1, ..., t_s \leq 1$ such that $\sum_i t_i = 1$, we have $\sum_i t_i \varphi_i \in \mathcal{D}(X, \theta)$ and

$$\operatorname{MA}_{\theta}\left(\sum_{i} t_{i}\varphi_{i}\right)$$
$$=\sum_{k_{i}\geq0,k_{1}+\ldots+k_{s}=n}C_{k_{1},\ldots,k_{s}}\cdot t_{1}^{k_{1}}\cdot\ldots\cdot t_{s}^{k_{s}}\cdot\operatorname{MA}_{\theta}\left(\varphi_{1}^{(k_{1})},\ldots,\varphi_{s}^{(k_{s})}\right),$$

where $C_{k_1,\ldots,k_s} := \frac{n!}{k_1!\cdots k_s!}$.

Proof. Let $\varphi_i^j \in \text{PSH}(X, \theta) \cap L^{\infty}(X)$, $i = 1, ..., s, j \in \mathbb{Z}^+$ such that $\varphi_i^j \searrow \varphi_i$ as $j \to +\infty$. Then, we have $\sum_i t_i \varphi_i^j \searrow \sum_i t_i \varphi_i$ and

$$\mathbf{MA}_{\theta}\left(\sum_{i} t_{i}\varphi_{i}^{j}\right)$$
$$=\sum_{k_{i}\geq0,k_{1}+\ldots+k_{s}=n}C_{k_{1},\ldots,k_{s}}\cdot t_{1}^{k_{1}}\cdot\ldots\cdot t_{s}^{k_{s}}\cdot\mathbf{MA}_{\theta}\left(\varphi_{1}^{j(k_{1})},\ldots,\varphi_{s}^{j(k_{s})}\right)$$

by the multinomial theorem and the symmetry of the complex Monge–Ampère operator. Moreover, it follows from Proposition 2.13 that we have

$$\mathrm{MA}_{\theta}\left(\sum_{i} t_{i} \varphi_{i}^{j}\right) \to \mathrm{MA}_{\theta}\left(\sum_{i} t_{i} \varphi_{i}\right) \text{ and}$$
$$\mathrm{MA}_{\theta}\left(\varphi_{1}^{j(k_{1})}, ..., \varphi_{s}^{j(k_{s})}\right) \to \mathrm{MA}_{\theta}\left(\varphi_{1}^{(k_{1})}, ..., \varphi_{s}^{(k_{s})}\right)$$

in the weak sense as $j \to +\infty$. The proof follows from the definition of $MA_{\theta}(\cdot, ..., \cdot)$.

Remark. Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain and $u_1, ..., u_n \in \mathcal{D}(\Omega)$. In the proof of Proposition 2.13, the measure $MA(u_1, ..., u_n)$ is well defined as a limiting measure.

Then, we can define MA $\left(u_1^{(k_1)}, ..., u_s^{(k_s)}\right)$ in the local setting, which is similar to the way it is defined in the global setting described in (2.2). By repeating the above arguments almost word for word, we can obtain the same conclusion in the local setting. That is, for $u_1, ..., u_s \in \mathcal{D}(\Omega)$ and $0 \leq t_1, ..., t_s$, we have $\sum_i t_i u_i \in \mathcal{D}(\Omega)$ and

$$\operatorname{MA}\left(\sum_{i} t_{i} u_{i}\right)$$
$$= \sum_{k_{i} \geq 0, k_{1} + \dots + k_{s} = n} C_{k_{1},\dots,k_{s}} \cdot t_{1}^{k_{1}} \cdot \dots \cdot t_{s}^{k_{s}} \cdot \operatorname{MA}\left(u_{1}^{(k_{1})},\dots,u_{s}^{(k_{s})}\right),$$

where $C_{k_1,\ldots,k_s} := \frac{n!}{k_1!\cdots k_s!}$.

2.3. Non-pluripolar energy. In this section, we consider mainly the Kähler form ω and $PSH(X, \omega)$. Let us now introduce the non-pluripolar energies E_k^{np} .

Definition 2.15. For $1 \leq k \leq n-1$, we define the non-pluripolar energy of $\varphi \in PSH(X, \omega)$ of order k as

$$E_k^{np}(\varphi) := \frac{1}{k+1} \sum_{j=0}^k \int_X \varphi \langle (dd^c \varphi + \omega)^j \rangle \wedge \omega^{n-j}.$$

Then, we can define the finite energy class $\mathcal{G}_k(X,\omega) := \{\varphi \in PSH(X,\omega) : E_k^{np}(\varphi) > -\infty\}.$

In particular, when k = n - 1, we set $E^{np} := E^{np}_{n-1}$, $\mathcal{G}(X, \omega) := \mathcal{G}_{n-1}(X, \omega)$.

Note that $\varphi \in \mathcal{G}(X, \omega)$ iff $\varphi \in L^1(\langle \omega_{\varphi}^k \rangle \wedge \omega^{n-k})$, k = 0, 1, ..., n-1 since φ is bounded from above. Hence, we can define the currents

$$[\omega + dd^c \varphi]^p := (\omega + dd^c \varphi) \land \langle \omega_{\varphi}^{p-1} \rangle = dd^c \left((h + \varphi) \langle (dd^c h + dd^c \varphi)^{p-1} \rangle \right)$$

and

$$S_p^{\omega}(\varphi) := [\omega + dd^c \varphi]^p - \langle \omega_{\varphi}^p \rangle,$$

for p = 1, 2, ..., n, where *h* is the local potential of ω . In [3, Proposition 3.3], they proved that $[\omega + dd^c \varphi]^p$ and $S_p^{\omega}(\varphi)$ are well-defined globally closed positive currents on *X* that only depend on the current $dd^c \varphi + \omega$ and not on the choice of ω as a Kähler representative in the class $[\omega]$.

Two useful results follow from [3]:

Theorem 2.16. ([3, Theorem 6.8]) Assume that $\varphi \in \mathcal{G}(X, \omega)$ and that η is a Kähler form in $[\omega]$ so that $\eta = \omega + dd^c g$, where g is a smooth function on X. Let $\varphi_l := \max\{\varphi, g - l\}$. Then, for $1 \le p \le n$,

$$(\omega + dd^c \varphi_l)^p \to [\omega + dd^c \varphi]^p + \sum_{j=1}^{p-1} S_j^{\omega}(\varphi) \wedge \eta^{p-j}, \ l \to \infty$$

in the weak sense.

Theorem 2.17. ([3, Theorem 1.11]) Let (X, ω) be a compact Kähler manifold of dimension n. Then,

 $\mathcal{D}(X,\omega) \subset \mathcal{G}(X,\omega).$

In particular, $\mathcal{D}(X,\omega) \subset DMA(X,\omega) \cap \mathcal{G}(X,\omega)$.

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3. The class of $DMA(X, \omega) \cap \mathcal{G}(X, \omega)$

In this section, we assume that (X, ω) is a compact Kähler manifold of complex dimension *n*. Note that if $\varphi \in DMA(X, \omega) \cap \mathcal{G}(X, \omega)$, by the definition of the complex Monge–Ampère operator and Theorem 2.16, we have

(3.1)
$$\mathrm{MA}_{\omega}(\varphi) = [\omega + dd^{c}\varphi]^{n} + \sum_{j=1}^{n-1} S_{j}^{\omega}(\varphi) \wedge \omega^{n-j} = \langle \omega_{\varphi}^{n} \rangle + \sum_{j=1}^{n} S_{j}^{\omega}(\varphi) \wedge \omega^{n-j}.$$

It may be helpful to consider the solution of the complex Monge–Ampère equation $MA_{\omega}(\varphi) = \mu$ when μ is allowed to be a normalized measure with a pluripolar part. A similar idea was proposed in [3, Remark 11.13], but they did not consider terms with $S_{j}^{\omega}(\varphi) \wedge \omega^{n-j}$, j = 1, ..., n - 1.

In particular, when (X, ω) is a compact Kähler surface, the above argument becomes more interesting. Let $f \in C^{\infty}(X)$ and $C \in \mathbb{R}^+$ so that $\eta := C^{-1}dd^c f + \omega > 0$. Set $g := C^{-1}f$ and $\varphi'_l := \max\{\varphi, g - l\}, \varphi_l := \max\{\varphi, -l\}$. Since $\varphi'_l, \varphi_l \in PSH(X, \omega) \cap L^{\infty}(X)$ and φ'_l, φ_l decrease to φ as $l \to \infty$, it follows from Theorem 2.16 and the definition of the complex Monge–Ampère operator that

(3.2)
$$\operatorname{MA}_{\omega}(\varphi) = [\omega + dd^{c}\varphi]^{2} + S_{1}^{\omega}(\varphi) \wedge \omega$$
$$= [\omega + dd^{c}\varphi]^{2} + S_{1}^{\omega}(\varphi) \wedge (\omega + dd^{c}g).$$

Hence, for $w \in C^{\infty}(X)$, it follows from (3.2) that

$$\int_X w dd^c g \wedge S_1^{\omega}(\varphi) = \frac{1}{C} \int_X w dd^c f \wedge S_1^{\omega} = 0$$

When w = f, we obtain $\int_X \left(\sqrt{-1}\partial f \wedge \bar{\partial}f\right) \wedge S_1^{\omega}(\varphi) = 0$, $\forall f \in C^{\infty}(X)$. Then, we can construct a positive (1,1)-form α such that $\alpha > \omega$ and $\int_X \alpha \wedge S_1^{\omega}(\varphi) = 0$. Hence $\int_X S_1^{\omega}(\varphi) \wedge \omega = 0$, then we have $S_1^{\omega}(\varphi) \wedge \omega = 0$, which means

(3.3)
$$\langle \omega + dd^c \varphi \rangle \wedge \omega = (\omega + dd^c \varphi) \wedge \omega.$$

We thus obtain the following result:

Proposition 3.1. Let (X, ω) be a compact Kähler surface and $\varphi \in DMA(X, \omega) \cap \mathcal{G}(X, \omega)$. Then,

$$\begin{aligned} \mathrm{MA}_{\omega}(\varphi) &= [\omega + dd^{c}\varphi]^{2} = \langle \omega + dd^{c}\varphi \rangle \wedge (\omega + dd^{c}\varphi) \\ &= dd^{c}\varphi \wedge \langle \omega + dd^{c}\varphi \rangle + \omega \wedge (\omega + dd^{c}\varphi). \end{aligned}$$

In particular, when $\varphi \in \mathcal{D}(X, \omega)$, the above formula holds.

Furthermore, for the class $\mathcal{D}(X, \omega)$, we have

Theorem 3.2. Let $\varphi_1, ..., \varphi_n \in \mathcal{D}(X, \omega)$. Then, $\varphi_1 \in L^1(\langle \omega_{\varphi_2} \land ... \land \omega_{\varphi_n} \rangle \land \omega)$. In particular, if X is a compact Kähler surface and $\varphi, \psi \in \mathcal{D}(X, \omega)$, then

$$\mathrm{MA}_{\omega}(\varphi,\psi) = \frac{1}{2} \left[(\omega + dd^{c}\varphi) \wedge \langle \omega + dd^{c}\psi \rangle + (\omega + dd^{c}\psi) \wedge \langle \omega + dd^{c}\varphi \rangle \right]$$

The proof needs the following Lemma:

Lemma 3.3. Let φ be a quasi-psh function on X, and let T be a positive (n-1, n-1)-current such that $\varphi \in L^1(\omega \wedge T)$. Then, for $\varphi_j \in C^{\infty}(X)$ such that $\varphi_j \searrow \varphi$, we have

$$dd^c \varphi_i \wedge T \to dd^c \varphi \wedge T$$

in the weak sense.

Proof. Without loss of generality, we can assume that $\varphi_1 \leq 0$. Set $f \in C^{\infty}(X)$ and $C \in \mathbb{R}^+$ such that $-C\omega \leq dd^c f \leq C\omega$. We have

$$\begin{split} \int_X f dd^c \varphi \wedge T &= \int_X \varphi dd^c f \wedge T \\ &= \int_X \varphi (C\omega + dd^c f) \wedge T - \int_X \varphi \ C\omega \wedge T. \end{split}$$

By the monotone convergence theorem, we have

$$\int_{X} \varphi(C\omega + dd^{c}f) \wedge T - \int_{X} \varphi C\omega \wedge T$$
$$= \lim_{j} \left(\int_{X} \varphi_{j}(C\omega + dd^{c}f) \wedge T - \int_{X} \varphi_{j} C\omega \wedge T \right)$$
$$= \lim_{j} \int_{X} \varphi_{j} dd^{c}f \wedge T = \lim_{j} \int_{X} f dd^{c}\varphi_{j} \wedge T,$$

where in the last line we used Stokes theorem.

Proof of Theorem 3.2. 1°. We first prove that $\varphi_1 \in L^1(\langle \omega_{\varphi_2} \wedge ... \wedge \omega_{\varphi_n} \rangle \wedge \omega)$. Without loss of generality, we can assume that $\varphi_i \leq 0, i = 1, ..., n$.

It follows from Corollary 2.14 and Theorem 2.17 that

$$\frac{1}{n}\left(\sum_{i}\varphi_{i}\right)\in L^{1}\left(\langle(\omega+dd^{c}\frac{1}{n}(\sum_{i}\varphi_{i}))^{n-1}\rangle\wedge\omega\right).$$

This means that

$$-\int_X \varphi_1 \omega \wedge \langle (n\omega + dd^c (\sum_i \varphi_i))^{n-1} \rangle < +\infty.$$

By [22, Proposition 8.16], there exists $\varphi_1^j \in \text{PSH}(X, \omega) \cap C^{\infty}(X)$ such that $\varphi_1^j \searrow \varphi_1$ as $j \to +\infty$. Then, we have

$$-\int_{X} \varphi_{1}^{j} \omega \wedge \langle (n\omega + dd^{c}(\sum_{i} \varphi_{i}))^{n-1} \rangle$$
$$\leq -\int_{X} \varphi_{1} \omega \wedge \langle (n\omega + dd^{c}(\sum_{i} \varphi_{i}))^{n-1} \rangle < +\infty$$

It thus follows from Proposition 2.1 and the multinomial theorem that

$$-\int_{X} \varphi_{1}^{j} \omega \wedge \langle (n\omega + dd^{c}(\sum_{i} \varphi_{i}))^{n-1} \rangle$$

$$= -\sum_{k_{1}+\ldots+k_{n}=n-1} C_{k_{1},\ldots,k_{n}} \cdot \int_{X} \varphi_{1}^{j} \omega \wedge \langle (\omega + dd^{c}\varphi_{1})^{k_{1}} \wedge \ldots \wedge (\omega + dd^{c}\varphi_{n})^{k_{n}} \rangle$$

$$\leq -\int_{X} \varphi_{1} \omega \wedge \langle (n\omega + dd^{c}(\sum_{i} \varphi_{i}))^{n-1} \rangle = M < +\infty,$$

where $k_i \ge 0$ and $C_{k_1,...,k_n} := \frac{(n-1)!}{k_1! \cdots k_n!}$. Hence, $-\int_X \varphi_1^j \omega \wedge \langle (\omega + dd^c \varphi_2) \wedge \dots \wedge (\omega + dd^c \varphi_n) \rangle \le M < +\infty.$

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Now, again by the monotone convergence theorem, we obtain

$$0 \leq -\int_{X} \varphi_1 \omega \wedge \langle (\omega + dd^c \varphi_2) \wedge \dots \wedge (\omega + dd^c \varphi_n) \rangle < +\infty.$$

 2° . Next, we prove that

$$\mathrm{MA}_{\omega}(\varphi,\psi) = \frac{1}{2} \left[(\omega + dd^{c}\varphi) \wedge \langle \omega + dd^{c}\psi \rangle + (\omega + dd^{c}\psi) \wedge \langle \omega + dd^{c}\varphi \rangle \right].$$

By Corollary 2.14 with $t_1 = t_2 = \frac{1}{2}$ and Proposition 3.1, we obtain

$$\begin{split} \mathrm{MA}_{\omega}(\varphi,\psi) &= 2\mathrm{MA}_{\omega}\left(\frac{1}{2}(\varphi+\psi)\right) - \frac{1}{2}\left(\mathrm{MA}_{\omega}(\varphi) + \mathrm{MA}_{\omega}(\psi)\right) \\ &= \frac{1}{2}\left(\omega \wedge \left(\omega + dd^{c}\varphi\right) + \omega \wedge \left(\omega + dd^{c}\psi\right)\right) \\ &- \frac{1}{2}\left(dd^{c}\varphi \wedge \left\langle\omega + dd^{c}\varphi\right\rangle + dd^{c}\psi \wedge \left\langle\omega + dd^{c}\psi\right\rangle\right) \\ &+ \left(\left(dd^{c}(\varphi+\psi)\right) \wedge \left\langle\omega + 2^{-1}dd^{c}(\varphi+\psi)\right\rangle\right). \end{split}$$

Therefore, we can complete the proof of the theorem if we can show that

$$\begin{aligned} dd^{c}(\varphi + \psi) \wedge \langle \omega + 2^{-1}dd^{c}(\varphi + \psi) \rangle \\ = &\frac{1}{2} \Big(dd^{c}\varphi \wedge \langle \omega + dd^{c}\varphi \rangle + dd^{c}\psi \wedge \langle \omega + dd^{c}\psi \rangle \\ &+ \frac{1}{2} dd^{c}\varphi \wedge \langle \omega + dd^{c}\psi \rangle + dd^{c}\psi \wedge \langle \omega + dd^{c}\varphi \rangle \Big). \end{aligned}$$

Indeed, analogous to 1°, we can find smooth functions ψ_j and φ_j that decrease to ψ and φ , respectively. Applying Proposition 2.1 and Lemma 3.3, makes it straightforward to obtain the desired conclusion. Therefore, the proof is complete.

4. Solving the complex Monge-Ampère equation

4.1. Cegrell–Lebesgue decomposition. First, let us recall the Cegrell–Lebesgue decomposition theorem in the local setting. Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain.

Theorem 4.1. ([13, Theorem 5.11]) Let μ be a positive measure on Ω . Then, μ can be decomposed into a regular positive (non-pluripolar) measure μ_r , and the singular positive (pluripolar) measure μ_s has support on some pluripolar set such that

$$\mu = \mu_r + \mu_s.$$

We claim that the Cegrell–Lebesgue decomposition of μ is unique. Indeed, suppose that μ'_r and μ'_s are other Cegrell–Lebesgue decompositions, and that

$$\mu_s = \mathbf{1}_{\{u = -\infty\}} \mu_s = \mathbf{1}_{\{u = -\infty\}} \mu, \ \mu'_s = \mathbf{1}_{\{u' = -\infty\}} \mu'_s = \mathbf{1}_{\{u' = -\infty\}} \mu,$$

where $u, u' \in \text{PSH}(\Omega)$. Set $A := \{u = -\infty\} \cup \{u' = -\infty\} = \{u + u' = -\infty\}$. For $\forall f \in C_0(\Omega)$, since $\int_A \mu_r = \int_A \mu'_r = 0$, we have

$$\int f d\mu_s = \int \mathbf{1}_A f d\mu_s = \int_A f d\mu = \int \mathbf{1}_A f d\mu'_s = \int f d\mu'_s.$$

This means that $\mu_s = \mu'_s$; hence, $\mu_r = \mu - \mu_s = \mu - \mu'_s = \mu'_r$.

When $\mu := MA(u)$ for some $u \in \mathcal{D}(\Omega)$, from [8, Theorem 2.1] and [2, Page 7-8], we have the following property:

Proposition 4.2. Let $\mu := MA(u)$ for some $u \in \mathcal{D}(\Omega)$. Then, we have

$$\mu_r = \mathbf{1}_{\{u > -\infty\}} \mu = \lim_{k} \mathbf{1}_{\{u > -k\}} (dd^c \max\{u, -k\})^n \text{ and } \mu_s = \mathbf{1}_{\{u = -\infty\}} \mu_s$$

Moreover, we can make the following generalization:

Lemma 4.3. Let $u_1, ..., u_n \in \mathcal{D}(\Omega)$. Then, we have

$$\begin{aligned} \operatorname{MA}(u_1, \dots, u_n)_r \\ &= \lim_k \mathbf{1}_{\cap_i \{u_i > h_i - k\}} \left(dd^c \max\{u_1, h_1 - k\} \land \dots \land dd^c \max\{u_n, h_n - k\} \right), \\ for \ h_1, \dots, h_n \in L^{\infty} \cap \operatorname{PSH}(\Omega). \ In \ particular, \ we \ have \\ \operatorname{MA}(u_1, \dots, u_n)_r &= \mathbf{1}_{\cap_i \{u_i > -\infty\}} \operatorname{MA}(u_1, \dots, u_n) \\ &= \langle dd^c u_1 \land \dots \land dd^c u_n \rangle \end{aligned}$$

$$:= \lim_k \mathbf{1}_{\cap_i \{u_i > -k\}} \left(dd^c \max\{u_1, -k\} \land ... \land dd^c \max\{u_n, -k\} \right)$$

and

$$\mathrm{MA}(u_1, ..., u_n)_s = \mathbf{1}_{\bigcup_i \{u_i = -\infty\}} \mathrm{MA}(u_1, ..., u_n)$$

Proof. 1°. We first claim that $MA(u_1, ..., u_n)_s = \mathbf{1}_{\cup_i \{u_i = -\infty\}} MA(u_1, ..., u_n)$ and $MA(u_1, ..., u_n)_r = \mathbf{1}_{\cap_i \{u_i > -\infty\}} MA(u_1, ..., u_n).$

It follows from the Remark of Corollary 2.14 and Proposition 4.2 that we have

$$\operatorname{MA}\left(\sum_{i} u_{i}\right)_{r} = \mathbf{1}_{\{\sum_{i} u_{i} > -\infty\}} \operatorname{MA}\left(\sum_{i} u_{i}\right) = \mathbf{1}_{\bigcap_{i}\{u_{i} > -\infty\}} \operatorname{MA}\left(\sum_{i} u_{i}\right)$$
$$= \mathbf{1}_{\bigcap_{i}\{u_{i} > -\infty\}} \mu + n! \cdot \mathbf{1}_{\bigcap_{i}\{u_{i} > -\infty\}} \operatorname{MA}(u_{1}, ..., u_{n}),$$

where μ is the sum of the complex Monge–Ampère measures. Moreover, note that

$$\Omega = (\cap_i \{u_i > -\infty\}) \bigcup (\cup_i \{u_i = -\infty\}).$$

We thus obtain $\mathbf{1}_{\cap_i \{u_i > -\infty\}} MA(u_1, ..., u_n)$ is a non-pluripolar measure and

$$MA(u_1, ..., u_n) = \mathbf{1}_{\bigcap_i \{u_i > -\infty\}} MA(u_1, ..., u_n) + \mathbf{1}_{\bigcup_i \{u_i = -\infty\}} MA(u_1, ..., u_n)$$

By the uniqueness of the Cegrell–Lebesgue decomposition, we have proven the claim.

2°. Set $u_i^k := \max\{u_i, h_i - k\}$. When l > k, we have $\{u_i > h_i - k\} = \{u_i^l > h_i - k\}$. Hence,

$$\mathbf{1}_{\cap_i \{u_i > h_i - k\}} \mathrm{MA}(u_1^k, ..., u_n^k) = \mathbf{1}_{\cap_i \{u_i^k > h_i - k\}} \mathrm{MA}(u_1^k, ..., u_n^k).$$

Applying the maximum principle, [22, Theorem 3.27] for each $u_i^k = \max\{u_i^l, h_i - k\}$, i = 1, ..., n, we have

$$\begin{split} \mathbf{1}_{\cap_i \{u_i^l > h_i - k\}} \mathrm{MA}(u_1^k, ..., u_n^k) &= \mathbf{1}_{\cap_i \{u_i^l > h_i - k\}} \mathrm{MA}(u_1^l, u_2^k, ..., u_n^k) \\ &= ... = \mathbf{1}_{\cap_i \{u_i^l > h_i - k\}} \mathrm{MA}(u_1^l, ..., u_n^l) \\ &= \mathbf{1}_{\cap_i \{u_i > h_i - k\}} \mathrm{MA}(u_1^l, ..., u_n^l). \end{split}$$

Then, by letting $l \to +\infty$, we arrive at

$$\mathbf{1}_{\bigcap_{i}\{u_{i}>h_{i}-k\}}\mathrm{MA}(u_{1}^{k},...,u_{n}^{k}) = \mathbf{1}_{\bigcap_{i}\{u_{i}>h_{i}-k\}}\lim_{l}\mathrm{MA}(u_{1}^{l},...,u_{n}^{l})$$
$$= \mathbf{1}_{\bigcap_{i}\{u_{i}>h_{i}-k\}}\mathrm{MA}(u_{1},...,u_{n}).$$

Where in the second line we used the definition of $MA(u_1, ..., u_n)$. By the claim of 1° , we obtain

$$\lim_{k} \mathbf{1}_{\cap_{i}\{u_{i}>h_{i}-k\}} \mathrm{MA}(u_{1}^{k},...,u_{n}^{k}) = \lim_{k} \mathbf{1}_{\cap_{i}\{u_{i}>h_{i}-k\}} \mathrm{MA}(u_{1},...,u_{n})$$
$$= \mathbf{1}_{\cap_{i}\{u_{i}>-\infty\}} \mathrm{MA}(u_{1},...,u_{n}) = \mathrm{MA}(u_{1},...,u_{n})_{r}.$$

Which completes the proof.

Let (X, ω_X) be a compact complex manifold of complex dimension n equipped with a Hermitian form, and let μ be a positive measure on X. Let $\{U_i, z_i\}_{i=1}^N$ be the holomorphic atlas of X such that $z_i(U_i)$ is the unit ball of \mathbb{C}^n and $\{\chi_i\}_{i=1}^N$ is the partition of unity of $\{U_i\}_i$. According to Theorem 4.1, we have $\chi_i \mu = \mu_r^i + \mu_s^i$ for each i, where μ_r^i is a positive non-pluripolar measure on U_i and μ_s^i is supported on the set $\{u_i = -\infty\}$, $u_i \in \text{PSH}(U_i)$. Set $\mu_r := \sum_i \mu_r^i$ and $\mu_s := \sum_i \mu_s^i$, we have the Cegrell–Lebesgue decomposition $\mu = \mu_r + \mu_s$ in the global setting.

By the same method as above, the decomposition is unique in the global setting. Furthermore, thanks to [23, Theorem 1.1](see also [22, Theorem 12.5] for the Kähler case), there exists $u \in \text{PSH}(X, \omega_X)$ such that μ_s is supported on the set $\{u = -\infty\}$.

From the above discussion, we have the following:

Corollary 4.4. Let (X, ω_X) be a compact complex manifold of complex dimension n equipped with a Hermitian form. Let μ be a positive measure on X. Then, μ can be uniquely decomposed into a regular positive (non-pluripolar) measure μ_r and a singular positive (pluripolar) measure μ_s support on $\{u = -\infty\}$, $u \in PSH(X, \omega_X)$ such that

$$\mu = \mu_r + \mu_s.$$

Now, we return to compact Kähler manifolds. Let θ be a smooth, closed semipositive real (1,1)-form on X whose cohomology class is big. As in the special case of [19, Question 12], the main problem of classes $DMA(X,\theta)$ and $\mathcal{D}(X,\theta)$ is characterizing the range of the complex Monge–Ampère operator on $DMA(X,\theta)$ and $\mathcal{D}(X,\theta)$.

From Corollary 4.4, for a normalized measure μ , a natural idea is to decompose μ into regular and singular parts μ_r and μ_s and consider

(4.1)
$$\operatorname{MA}_{\theta}(\varphi)_r = \mu_r, \ \operatorname{MA}_{\theta}(\varphi)_s = \mu_s,$$

respectively.

For this reason, we provide a key theorem:

Theorem 4.5. Let $\varphi_1, ..., \varphi_n \in \mathcal{D}(X, \theta)$. Then, we have

$$\mathrm{MA}_{\theta}(\varphi_1, \dots, \varphi_n)_r = \langle \theta_{\varphi_1} \wedge \dots \wedge \theta_{\varphi_n} \rangle$$

and

$$\mathrm{MA}_{\theta}(\varphi_1,...,\varphi_n)_s = \mathbf{1}_{\cup_j \{\varphi_j = -\infty\}} \mathrm{MA}_{\theta}(\varphi_1,...,\varphi_n).$$

In particular, if $\varphi \in \mathcal{D}(X, \theta)$, then we have

$$\mathrm{MA}_{\theta}(\varphi)_r = \langle \theta_{\varphi}^n \rangle$$
 and $\mathrm{MA}_{\theta}(\varphi)_s = \mathbf{1}_{\{\varphi = -\infty\}} \mathrm{MA}_{\theta}(\varphi).$

Proof. Let $\{U_i, z_i\}_{i=1}^N$ be the holomorphic atlas of X such that $z_i(U_i) \subset \mathbb{C}^n$ is the unit ball and $\theta = dd^c g_i$ on U_i , where $g_i \in C^{\infty} \cap L^{\infty} \cap \text{PSH}(U_i)$. Let $\{\chi_i\}_{i=1}^N$ be the

partition of unity of $\{U_i\}$ and $\varphi_j^k := \max\{\varphi_j, -k\}$. Then, we have

$$\mathbf{1}_{\bigcap_{j}\{\varphi_{j}>-k\}}\mathrm{MA}_{\theta}(\varphi_{1}^{k},...,\varphi_{n}^{k}) = \sum_{i}\chi_{i}\mathbf{1}_{\bigcap_{j}\{\varphi_{j}>-k\}}\mathrm{MA}_{\theta}(\varphi_{1}^{k},...,\varphi_{n}^{k})$$
$$= \sum_{i}\left(\chi_{i}\mathbf{1}_{U_{i}}\mathbf{1}_{\bigcap_{j}\{\varphi_{j}+g_{i}>g_{i}-k\}}\cdot\right)$$
$$\mathrm{MA}(\max\{\varphi_{1}+g_{i},g_{i}-k\},...,\max\{\varphi_{n}+g_{i},g_{i}-k\}).$$

Note that the currents

$$\chi_i \langle dd^c(\varphi_1 + g_i) \wedge \dots \wedge dd^c(\varphi_n + g_i) \rangle, \ i = 1, \dots, N$$

are the well-defined positive currents on X. Taking $k \to +\infty$ and applying Lemma 4.3 with $\Omega := U_i$ and $u_j := \varphi_j + g_i$, $h_1 = \ldots = h_n := g_i$ for each U_i , we have (4.2)

$$\chi_{i} \lim_{k} \mathbf{1}_{U_{i}} \mathbf{1}_{\bigcap_{j} \{\varphi_{j}+g_{i}>g_{i}-k\}} \mathrm{MA}(\max\{\varphi_{1}+g_{i},g_{i}-k\},...,\max\{\varphi_{n}+g_{i},g_{i}-k\})$$
$$= \chi_{i} \mathbf{1}_{U_{i}} \mathbf{1}_{\bigcap_{j} \{\varphi_{j}+g_{i}>-\infty\}} \mathrm{MA}(\varphi_{1}+g_{i},...,\varphi_{n}+g_{i})$$
$$= \chi_{i} \langle dd^{c}(\varphi_{1}+g_{i}) \wedge ... \wedge dd^{c}(\varphi_{n}+g_{i}) \rangle.$$

Therefore, combining the above arguments, we have

where in the second line we used the definition of $\langle \theta_{\varphi_1} \wedge ... \wedge \theta_{\varphi_n} \rangle$, in the last line we used (4.2). Now, for each U_i , we have

$$\chi_{i} \mathrm{MA}_{\theta}(\varphi_{1}, ..., \varphi_{n}) = \chi_{i} \mathbf{1}_{U_{i}} \mathrm{MA}_{\theta}(\varphi_{1}, ..., \varphi_{n})$$
$$= \chi_{i} \mathbf{1}_{U_{i}} \mathrm{MA}(\varphi_{1} + g_{i}, ..., \varphi_{n} + g_{i}).$$

By the uniqueness of Cegrell–Lebesgue decomposition and Lemma 4.3, we arrive at

$$(\chi_i \mathrm{MA}_{\theta}(\varphi_1, ..., \varphi_n))_r = \chi_i \mathbf{1}_{U_i} \mathrm{MA}_{\theta}(\varphi_1, ..., \varphi_n)_r$$

= $\chi_i \mathbf{1}_{U_i} \mathbf{1}_{\cap_j \{\varphi_j > -\infty\}} \mathrm{MA}_{\theta}(\varphi_1, ..., \varphi_n)$
= $\chi_i \langle dd^c(\varphi_1 + g_i) \wedge ... \wedge dd^c(\varphi_n + g_i) \rangle.$

Hence,

$$MA_{\theta}(\varphi_{1},...,\varphi_{n})_{r} = \sum_{i} \chi_{i} MA_{\theta}(\varphi_{1},...,\varphi_{n})_{r} = \mathbf{1}_{\bigcap_{j} \{\varphi_{j} > -\infty\}} MA_{\theta}(\varphi_{1},...,\varphi_{n})$$
$$= \sum_{i} \chi_{i} \langle dd^{c}(\varphi_{1} + g_{i}) \wedge ... \wedge dd^{c}(\varphi_{n} + g_{i}) \rangle = \langle \theta_{\varphi_{1}} \wedge ... \wedge \theta_{\varphi_{n}} \rangle$$

Furthermore, we have

$$\mathrm{MA}_{\theta}(\varphi_1,...,\varphi_n) - \mathrm{MA}_{\theta}(\varphi_1,...,\varphi_n)_r = \mathbf{1}_{\cup_j \{\varphi_j = -\infty\}} \mathrm{MA}_{\theta}(\varphi_1,...,\varphi_n).$$

It follows from Corollary 4.4 that we completed the proof.

Remark. Theorem 4.5 provides a method for solving the complex Monge–Ampère equation on $\mathcal{D}(X, \theta)$. That is, we can consider the equation

$$\langle \theta_{\varphi}^n \rangle = \mu_r \text{ and } \mathbf{1}_{\{\varphi = -\infty\}} \mathrm{MA}_{\theta}(\varphi) = \mu_s, \ \varphi \in \mathcal{D}(X, \theta),$$

for some normalized measure μ . This makes the idea of (4.1) feasible.

Corollary 4.6. Let $\varphi_1, ..., \varphi_n, \psi_1, ..., \psi_n \in \mathcal{D}(X, \theta)$ such that $\varphi_j \preceq \psi_j, j = 1, ..., n$. Then,

$$\mathrm{MA}_{\theta}(\varphi_1, ..., \varphi_n)_s \geq \mathrm{MA}_{\theta}(\psi_1, ..., \psi_n)_s.$$

In particular, if $\varphi, \psi \in \mathcal{D}(X, \theta)$ so that $\varphi \leq \psi$, then

$$\mathrm{MA}_{\theta}(\varphi)_s \geq \mathrm{MA}_{\theta}(\psi)_s.$$

Proof. Since $\varphi_j \leq \psi_j$, without loss of generality, we can assume that $\varphi_j \leq \psi_j$, j = 1, ..., n. Let $\{U_i, z_i\}_{i=1}^N$ be the holomorphic atlas of X such that $z_i(U_i) \subset \mathbb{C}^n$ is the unit ball and $\theta = dd^c g_i$ on U_i , where $g_i \in C^{\infty} \cap L^{\infty} \cap \text{PSH}(U_i)$. Let $\{\chi_i\}_{i=1}^N$ be the partition of unity of $\{U_i\}$.

In each U_i , we have $\varphi_1 + g_i \leq \psi_1 + g_i$. Then, from [1, Lemma 4.1], we have

$$\mathbf{1}_{U_i} \mathbf{1}_{\{\varphi_1 + g_i = -\infty\}} \operatorname{MA}(\varphi_1 + g_i, \varphi_2 + g_i, ..., \varphi_n + g_i)$$

$$\geq \mathbf{1}_{U_i} \mathbf{1}_{\{\psi_1 + g_i = -\infty\}} \operatorname{MA}(\psi_1 + g_i, \varphi_2 + g_i, ..., \varphi_n + g_i).$$

By repeating the above method for each $\varphi_k + g_i \leq \psi_k + g_i$, where k = 2, ..., n, we obtain

(4.3)
$$\mathbf{1}_{U_i} \mathbf{1}_{\cup_j \{\varphi_j + g_i = -\infty\}} \mathrm{MA}(\varphi_1 + g_i, ..., \varphi_n + g_i) \\ \geq \mathbf{1}_{U_i} \mathbf{1}_{\cup_j \{\psi_j + g_i = -\infty\}} \mathrm{MA}(\psi_1 + g_i, ..., \psi_n + g_i).$$

Now, we have

$$\begin{split} \mathrm{MA}_{\theta}(\varphi_{1},...,\varphi_{n})_{s} &= \mathbf{1}_{\cup_{j}\{\varphi_{j}=-\infty\}}\mathrm{MA}_{\theta}(\varphi_{1},...,\varphi_{n}) \\ &= \sum_{i}\chi_{i}\mathbf{1}_{U_{i}}\mathbf{1}_{\cup_{j}\{\varphi_{j}+g_{i}=-\infty\}}\mathrm{MA}(\varphi_{1}+g_{i},...,\varphi_{n}+g_{i}) \\ &\geq \sum_{i}\chi_{i}\mathbf{1}_{U_{i}}\mathbf{1}_{\cup_{j}\{\psi_{j}+g_{i}=-\infty\}}\mathrm{MA}(\psi_{1}+g_{i},...,\psi_{n}+g_{i}) \\ &= \mathbf{1}_{\cup_{j}\{\psi_{j}=-\infty\}}\mathrm{MA}_{\theta}(\psi_{1},...,\psi_{n}) = \mathrm{MA}_{\theta}(\psi_{1},...,\psi_{n})_{s}, \end{split}$$

where in the third line we used (4.3) for each U_i , in the first and last line we used Theorem 4.5. This completes the proof.

When $\varphi \in \mathcal{D}(X, \omega)$, due to the uniqueness of the Cegrell–Lebesgue decomposition and (3.1), Theorem 4.5, we have the following

Corollary 4.7. Let $\varphi \in \mathcal{D}(X, \omega)$. Then,

$$\operatorname{MA}_{\omega}(\varphi)_{r} = \langle \omega_{\varphi}^{n} \rangle$$
 and $\operatorname{MA}_{\omega}(\varphi)_{s} = \sum_{j=1}^{n} S_{j}^{\omega}(\varphi) \wedge \omega^{n-j}.$

4.2. Solving the complex Monge–Ampère equation in $\mathcal{D}(X,\theta)$. In this section, we consider mainly $\varphi \in \mathcal{D}(X,\theta)$, where θ is a smooth, closed semi-positive real (1,1)-form on X whose cohomology class is big such that $\int_X \theta^n = 1$.

We set a normalized measure $\mu = \mu_r + \mu_s$ on X such that $\int_X \mu = 1$. If there exists $\varphi \in \mathcal{D}(X, \theta)$ such that $MA_{\theta}(\varphi) = \mu$, it follows from Corollary 4.4 and Theorem 4.5 that

$$\int_X \langle \theta_{\varphi}^n \rangle = \int_X \mu_r \text{ and } \mathbf{1}_{\{\varphi = -\infty\}} \mathrm{MA}_{\theta}(\varphi) = \mu_s.$$

Then we have

Proposition 4.8. Assume that $\varphi \in \mathcal{D}(X, \theta)$ and $\mu_s = \mathrm{MA}_{\theta}(\varphi)_s$. Then, for $\psi \in \mathrm{PSH}(X, \theta)$ so that $\varphi \preceq \psi \preceq P_{\theta}[\varphi]$, we have $\mathrm{MA}(\psi)_s = \mu_s$. Furthermore, if $\psi \succeq \varphi$ so that $\mathrm{MA}(\psi)_s = \mu_s$, then we have $\psi \preceq P_{\theta}[\varphi]$.

Proof. Assume that $\psi \in \text{PSH}(X, \theta)$ such that $\varphi \preceq \psi \preceq P_{\theta}[\varphi]$. By Proposition 2.2 and Lemma 2.11, we have $\psi \in \mathcal{E}(X, \theta, P_{\theta}[\varphi]) \cap \mathcal{D}(X, \theta)$. Applying Corollary 4.6, we obtain

$$\mathrm{MA}_{\theta}(\varphi)_s \geq \mathrm{MA}_{\theta}(\psi)_s.$$

Note that we have

$$\int_{X} \mathrm{MA}_{\theta}(\psi)_{s} = 1 - \int_{X} \mathrm{MA}_{\theta}(\psi)_{r} = 1 - \int_{X} \langle \theta_{\psi}^{n} \rangle$$
$$= 1 - \int_{X} \langle \theta_{\varphi}^{n} \rangle = 1 - \int_{X} \mathrm{MA}_{\theta}(\varphi)_{r} = \int_{X} \mathrm{MA}_{\theta}(\varphi)_{s},$$

where we used the definition of $\mathcal{E}(X, \theta, P_{\theta}[\varphi])$ and Theorem 4.5. By comparing the total mass, we obtain $\mu_s = \mathrm{MA}_{\theta}(\varphi)_s = \mathrm{MA}_{\theta}(P_{\theta}[\varphi])_s$.

If $\psi \succeq \varphi$ so that $MA(\psi)_s = MA(\varphi)_s$, then we have $\int_X \langle \theta_{\psi}^n \rangle = \int_X \langle \theta_{\varphi}^n \rangle$. It follows from Proposition 2.4 (ii). that $\psi \preceq P_{\theta}[\varphi]$.

Set $\nu = \nu_r + \mu_s$. If ν_r satisfies $\int_X \nu_r = 1 - \int_X \mu_s$ and $\nu = f\omega^n$ for some $f \in L^p(\omega), p > 1$, it follows from Theorem 2.6 and Theorem 2.7 that there exists $\psi \in PSH(X, \theta)$ such that

$$\psi \simeq P_{\theta}[\varphi]$$
 and $\operatorname{MA}_{\theta}(\psi)_r = \langle \theta_{\psi}^n \rangle = \nu_r.$

By Lemma 2.11 and Proposition 4.8, we have $\psi \in \mathcal{D}(X,\theta)$ and $MA_{\theta}(\psi)_s = MA_{\theta}(P_{\theta}[\varphi])_s = \mu_s$. We thus obtain

$$\begin{split} \mathrm{MA}_{\theta}(\psi) &= \mathrm{MA}_{\theta}(\psi)_r + \mathrm{MA}_{\theta}(\psi)_s \\ &= \langle \theta_{\psi}^n \rangle + \mu_s = \nu_r + \mu_s. \end{split}$$

In summary, we obtain the following Theorem.

Theorem 4.9. Let (X, ω) be a compact Kähler manifold of complex dimension n, and let θ be a normalized, smooth closed semi-positive real (1, 1)-form on X whose cohomology class is big such that $\int_X \theta^n = 1$. Let μ_s be a pluripolar measure on X supported on some pluripolar set. If there exists $\varphi \in \mathcal{D}(X, \theta)$ such that $MA_{\theta}(\varphi)_s = \mu_s$, then for all non-pluripolar measures ν_r such that $\int_X \nu_r = 1 - \int_X \mu_s$ and $\nu_r = f\omega^n$ for some $f \in L^p(\omega), p > 1$, there exists $\psi \in PSH(X, \theta)$ that is the solution of

(4.4)
$$\operatorname{MA}_{\theta}(\psi) = \nu_r + \mu_s, \ \psi \in \mathcal{D}(X, \theta).$$

4.2.1. Application. Let (X, ω) be a compact Kähler surface such that $\int_X \omega^2 = 1$. Let μ be a measure so that $\mu = \mu_s$. If there exists $\varphi \in \mathcal{D}(X, \omega)$ such that $\operatorname{MA}_{\omega}(\varphi) = \mu(\text{e.g. the quasi-psh Green functions, see [14]})$, then, for $\forall s \in [0, 1]$, it follows from Corollary 2.14, (3.3), and Theorem 3.2 that

$$\begin{aligned} \mathbf{MA}_{\omega}(s\varphi) &= (1-s)^{2}\mathbf{MA}_{\omega}(0) + 2(1-s)s\mathbf{MA}_{\omega}(0,\varphi) + s^{2}\mathbf{MA}_{\omega}(\varphi) \\ &= (1-s)^{2}\omega^{2} + 2(1-s)s\omega \wedge \langle \omega + dd^{c}\varphi \rangle + s^{2}\mu_{s}. \end{aligned}$$

This means that

$$\begin{aligned} \mathrm{MA}_{\omega}(s\varphi)_r &= (1-s)^2 \omega^2 + 2(1-s)s\omega \wedge \langle \omega + dd^c \varphi \rangle \text{ and} \\ \mathrm{MA}_{\omega}(s\varphi)_s &= s^2 \mu. \end{aligned}$$

Therefore, for any $t \in [0,1]$ and any positive non-pluripolar measure ν such that $\int_X \nu = 1$ and $\nu = f\omega^2$ for some $f \in L^p(\omega^2)$, p > 1, by Theorem 4.9, there exists $\psi_{\nu,t} \in \mathcal{D}(X, \omega)$ that satisfies

$$\mathrm{MA}_{\omega}(\psi_{\nu,t}) = (1-t)\nu + t\mu.$$

4.2.2. For more general case. Let μ be a positive measure such that there exists $\varphi \in \mathcal{D}(X,\theta)$, $\mathrm{MA}_{\theta}(\varphi)_s = \mu_s$. When $\psi \in \mathcal{E}(X,\theta,P_{\theta}[\varphi]) \cap \mathcal{D}(X,\theta)$, we have $\psi \preceq P_{\theta}[\varphi]$; hence, $\mathrm{MA}_{\theta}(\psi)_s \geq \mathrm{MA}_{\theta}(P_{\theta}[\varphi])_s = \mu_s$ by Corollary 4.6. Moreover, we have

$$\int_{X} \mathrm{MA}_{\theta}(\psi)_{s} = 1 - \int_{X} \langle \theta_{\varphi}^{n} \rangle = 1 - \int_{X} \langle \theta_{P_{\theta}[\varphi]}^{n} \rangle = \int_{X} \mathrm{MA}_{\theta}(P_{\theta}[\varphi])_{s}$$

by the definition of $\mathcal{E}(X, \theta, P_{\theta}[\varphi])$ and Theorem 4.5. Now, by comparing the total mass, we obtain $MA_{\theta}(\psi)_s = MA_{\theta}(P_{\theta}[\varphi])_s = \mu_s$. So we want to know

Problem 4.10. In the setting of Theorem 4.9, without the condition that $\nu_r = f\omega^n$, $f \in L^p(\omega^n)$, p > 1, given an appropriate condition of ν_r so that there exsits $\psi \in \mathcal{D}(X, \theta)$ satisfies $\mathrm{MA}_{\theta}(\psi) = \nu_r + \mu_s$.

If ν_r is arbitrary, by Theorem 2.6, stating the uniqueness and existence of a solution of

(4.5)
$$\langle \theta_{\psi}^n \rangle = \nu_r, \ \psi \in \mathcal{E}(X, \theta, P_{\theta}[\varphi]),$$

answering this question follows from proving that $\mathcal{E}(X, \theta, P_{\theta}[\varphi]) \subset \mathcal{D}(X, \theta)$.

Unfortunately, this is incorrect even when $P_{\theta}[\varphi] = 0$, see [15, Example 3.4]. So for arbitrary ν_r , the answer to this question is incorrect, because the case of $\mu_s = 0$ is a trivial counterexample.

Suppose $\int_X \langle \theta_{\varphi}^n \rangle > 0$ and ψ' is another solution of (4.4). If $\psi' \leq P_{\theta}[\varphi]$, then we have $\psi' \in \mathcal{E}(X, \theta, P_{\theta}[\varphi])$ since $\int_X \nu = \int_X \langle \theta_{P_{\theta}[\varphi]}^n \rangle$. By the uniqueness of a solution of (4.5), we have $\psi' - \psi \equiv \text{constant}$. So a natural question is

Problem 4.11. In the setting of Theorem 4.9, given an appropriate condition so that ψ is unique up to a constant.

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