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ASTRONOMY AND ASTROPHYSICS

A Note on Hydrodynamic Viscosity and Selfgravitation in Accretion Disks

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Abstract. We propose a generalized accretion disk viscosity prescription based on hydrodynamically driven turbulence at the critical effective Reynolds number. This approach is consistent with recent re-analysis by Richard & Zahn (1999) of experimental results on turbulent Couette-Taylor flows. This new β-viscosity formulation is applied to both selfgravitating and non-selfgravitating disks and is shown to yield the standard α-disk prescription in the case of shock dissipation limited, non-selfgravitating disks. A specific case of fully selfgravitating β-disks is analyzed. We suggest that such disks may explain the observed spectra of protoplanetary disks and yield a natural explanation for the radial motions inferred from the observed metallicity gradients in disk galaxies. The β-mechanism may also account for the rapid mass transport required to power ultra luminous infrared galaxies.

Key words: Accretion, accretion disks – Hydrodynamics – Turbulence – Stars: pre-main sequence – Galaxy: evolution – galaxies: evolution

1. Introduction

One of the major shortcomings of the current theoretical descriptions of accretion disks is lack of detailed knowledge about the underlying physics of viscosity in the disk. This problem is significant because almost all detailed modelling of the structure and evolution of accretion disks depends on the value of the viscosity and its dependence on the physical parameters. There is general agreement that molecular viscosity $\nu_{\rm mol}$ is totally inadequate and that some kind of turbulent viscosity is required. Moreover, the Reynolds number in the disk flow is extremely high in any astrophysical context and this in itself is likely to lead to strong turbulence regardless of the details of the actual instability involved.

However, there is far less certainty about how to prescribe such a turbulent viscosity in the absence of a proper physical theory of turbulence. Most investigators adopt the so-called α-*ansatz* introduced by Shakura (1972) and Shakura & Sunyaev (1973) that gives the viscosity (ν) as the product of the pressure scale height in the disk (h), the velocity of sound ($c_{\rm s}$), and a parameter α that contains all the unknown physics:

$$\nu = \alpha h c_{\rm s}. \quad (1)$$

One interprets this as some kind of isotropic turbulent viscosity $\nu = \nu_{\rm t} = l_{\rm t} v_{\rm t}$ where $l_{\rm t}$ is an (*a priori* unknown) length scale and $v_{\rm t}$ an (*a priori* unknown) characteristic velocity of the turbulence. One may then write $\alpha = (v_{\rm t}/c_{\rm s}) \cdot (l_{\rm t}/h)$. On general physical grounds neither term in parentheses can exceed unity so that $\alpha \leq 1$. If initially $v_{\rm t} > c_{\rm s}$, shock waves would result in strong damping and hence a return to a subsonic turbulent velocity. The condition $l_{\rm t} > h$ would require anisotropic turbulence since the vertical length scales are limited by the disk's thickness, which is comparable to h.

While it is always, in a trivial way, possible to calculate a value α, a parameterization of this sort for ν is only useful if the proportionality parameter, α, is (approximately) constant. One can expect this to happen only if the scaling quantities are chosen in a physically appropriate manner. Models for the structure and evolution of accretion disks in close binary systems (e.g., dwarf novae and symbiotic stars) show that Shakura & Sunyaev's parameterization with a constant α leads to results that reproduce the overall observed behaviour of the disks quite well. Time dependent model calculations of the outbursts of dwarf novae (e.g., Meyer & Meyer-Hofmeister 1984) and X-ray transients (e.g., Cannizzo 1996) demonstrate that, over a wide range of physical states of a disk in different phases of its evolution, the derived values of the viscosity parameter α do not vary by more than approximately an order of magnitude and are not too different from unity. As a result of this success, the α-ansatz is now used in essentially all accretion disk models.

It is noteworthy that, despite this success, the α-ansatz retains no information about the mechanism generating the turbu-

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lence but only about physical limits to its efficiency in a disk. In fact we would expect any high Reynolds number astrophysical shear flow to exhibit some kind of turbulent viscosity regardless of whether or not it happens to be in a disk. We therefore conclude that a more general prescription underlies the α-ansatz for accretion disks.

In recent years, Balbus & Hawley (1991) and their collaborators (e.g., Hawley, Gammie & Balbus 1995) have shown that for non-selfgravitating magnetic accretion disks, an instability exists that can give rise to turbulence with the required formal dependence and—if only marginally—-the required amount. We also note that in substantial regions of proto-stellar and proto-planetary disks, the charge density is unlikely to be high enough to sustain a significant magnetically mediated viscosity, although this phenomenon may be relevant elsewhere.

Whether a purely hydrodynamic turbulence can sustain the viscosity in the angular momentum profile of an accretion disk and can result in an angular momentum transport towards regions with larger specific angular momentum is still a matter of debate. Balbus, Hawley & Stone (1996), for instance, argue against it, based on numerical experiments, albeit for a rather low effective Reynolds number. Dubrulle (1992) and Kato & Yoshizawa (1997), among others, argue in favor of it, mainly based on analytical considerations. Experiments dating back to the 1930s on the Couette-Taylor flow between co-axial rotating fluids (Wendt 1933, Taylor 1936a, b) show clearly the existence of a purely hydrodynamic instability. While the flow is essentially incompressible, turbulence is generated above a critical Reynolds number, independent of the radial profile of angular momentum[1]. A modern review has been given by DiPrima & Swinney (1985). Most recently Richard & Zahn (1999) (hereafter RZ) have undertaken a reanalysis of Taylor's experimental results and, for high Reynolds number flow, have interpreted them in terms of a turbulent viscosity (see also Sect. 2).

In this contribution, we adopt the view that hydrodynamically driven turbulence can sustain the viscosity in accretion disks. We suggest, in Sect. 2, a viscosity prescription, the β-ansatz, that represents the maximum attributable to hydrodynamic turbulence. We show that in the limit of low mass, thin disks, hydrodynamic turbulence will result in the Shakura-Sunyaev prescription. We then discuss the implications of the proposed formulation for the structure and evolution of self-gravitating disks, noting that even for these disks, the viscosity prescription differs from the α-ansatz and hence removes a difficulty first noted by Paczyński (1978). Finally, we discuss protostellar, galactic and galactic center disks as examples where the β-ansatz may be relevant.

2. Prescription for Turbulent Viscosity

2.1. Reynolds viscosity as the general case

As noted in Sect. 1 the need for some kind of turbulent viscosity in accretion disks is generally recognized, as is the very high Reynolds number of the flow in the absence of such a viscosity. Here, we wish to investigate in particular the case of accretion disks where the magnetic fields do not play an important role. In these circumstances, it seems reasonable to assume that the turbulence is driven by the velocity field in the disk, which itself has characteristic length and velocity scales s (the radius of the orbit) and v_ϕ (the azimuthal velocity), respectively.

As has been pointed out, for example, by Lynden-Bell & Pringle (1974) and Thompson et al. (1977), the high corresponding Reynolds number $\Re = sv_\phi/\nu$ should lead to the generation of turbulence and hence to a steady enhancement in the effective viscosity. This will continue until the Reynolds number has been reduced to approximately its critical value $\Re_{\rm crit}$. Typical values for $\Re_{\rm crit}$ in laboratory flows are of the order of $\sim 10^2 - 10^3$. This limiting Reynolds viscosity can, in this case, be as high as

$$\nu = \nu_{\rm R} = \beta s v_\phi \tag{2}$$

where β is a constant satisfying

$$\beta \lesssim \frac{1}{\Re_{\rm crit}} \sim 10^{-3} - 10^{-2}. \tag{3}$$

In terms of previously introduced quantities we may write $v_{\rm t} \sim \beta_1 v_\phi$ and $l_{\rm t} \sim \beta_2 s$ so that $\beta \sim \beta_1\beta_2 \sim 10^{-2\ldots-3}$.

In support of this choice of s as the natural length scale we note that it is the only length scale which is relevant for angular momentum transport and which contains information about the driving agent for the turbulence—namely the rotation field; likewise, the orbital velocity v_ϕ is the only velocity scale containing such information.

This approach receives further support from the reanalysis by RZ of the Wendt (1933) and Taylor (1936a, b) experiments on turbulent viscosity generated in the flow between coaxial rotating cylinders. We note, however, that it is difficult to make precise comparisons between accretion disk and rotating cylinders in view of quite different constraints on the fluid flow.

Using a definition of $\Re \sim R\Delta\Omega\Delta R/\nu$ appropriate to the experimental situation (here R is the average cylinder radius and $\Delta\Omega$ and ΔR are the relative angular velocity and gap size between the cylinders), RZ derive expressions for $\Re_{\rm crit}$ as a function of relative gap size $\Delta R/R$. For small gap size they find $\Re_{\rm crit} \lesssim 2000$ independent of gap size. For large relative gap size they find that $\Re_{\rm crit} = \Re_{\rm grad}(\Delta R/R)^2$ where their *gradient Reynolds number* $\Re_{\rm grad} \sim r^3(d\Omega/dR) \sim 10^6$ is essentially constant. Thus for small gaps the experimental data yield essentially the same value of $\Re_{\rm crit}$ as we have adopted in Eq. (3). For large gap sizes, the constancy of $\Re_{\rm grad}$ leads to essentially the same *functional* form as in Eq. (2) but with a significantly smaller value of β. RZ arrive at similar conclusions for the two regimes of gap sizes from an analysis of the torques exerted on the cylinders in cases where the flow is turbulent. We believe that the small gap limit is the more relevant to the accretion disk case, for which the speed of rotation is constrained at each radius by the gravitational field. The flux of angular momentum at each radius is thus determined by the imposed

[1] One of the cases investigated by Wendt indeed has a rotation law which approximates closely the profile in a Keplerian disk.

orbital velocity field. By contrast, the experimental configuration constrains the flow velocity only at the inner and outer radius with the flow in the gap region able to take up a velocity profile determined by viscosity and in which the angular momentum flux is independent of radius. In our opinion the experimental results provide strong support for a turbulent viscosity generated by hydrodynamically driven turbulence. While we recognize that more work is required on the question, we also believe that the small gap results provide significant support for the viscosity prescription given in Eqs. (2) and (3).

In the following we will refer to Eq. 2 as the β-ansatz and to the disk structure arising from this viscosity prescription as β-disks. We suggest that the β-ansatz is the most appropriate initial formulation for accretion disks since it is directly connected to the driving mechanism. It establishes the maximum value of the viscosity that can arise from hydrodynamically driven turbulence.

The actual viscosity may, however, be limited to lower values by such phenomena as shock dissipation of turbulent energy if the implied turbulent velocities exceed the local sound speed. As we show in Sect. 2.2, this yields the α-ansatz in conditions relevant to these non-selfgravitating accretion disks. However, it leads to a different prescription in shock limited selfgravitating disks. In the Couette-Taylor case, all velocities were subsonic (the flow was essentially incompressible) so that no additional constraints applied. This would also be the case in astrophysical disks in which $\beta_1 < c_\mathrm{s}/v_\phi$.

2.2. *α-viscosity as the limiting case for shock dissipation limited low mass accretion disks.*

If the accretion disk is such that the local sound speed is less than the turbulent velocities implied by the β-ansatz, i.e., $\beta_1 > c_\mathrm{s}/v_\phi$, we may rewrite Eq. 2 as

$$\nu = v_\mathrm{t} l_\mathrm{t} \sim \Delta v_\phi \Delta s \tag{4}$$

where Δv_ϕ and Δs are the maximum representative velocity and length scales allowed by local conditions. Furthermore, we may write

$$\Delta v_\phi = \frac{\partial v_\phi}{\partial s}\Delta s \sim \frac{v_\phi}{s}\Delta s \tag{5}$$

so that a restriction on Δv_ϕ implies a constraint also on Δs and vice-versa.

If we consider turbulent elements in a *smoothed out* background gas with sound speed c_s we may impose the limit that the turbulent velocity will approach but may not exceed c_s. Thus Eq. 5 gives

$$\frac{v_\phi}{s}\Delta s \sim \Delta v_\phi = \zeta c_\mathrm{s} \tag{6}$$

or

$$\Delta s \sim \frac{s}{v_\phi}\Delta v_\phi = \zeta \frac{s}{v_\phi} c_\mathrm{s} \tag{7}$$

with ζ a quantity smaller than but of order unity. This estimate of Δs may be interpreted as the distance a hydrodynamically driven turbulent element can travel before loosing its identity due to shock dissipation.

In a standard geometrically thin, non-selfgravitating accretion disk (i.e., a Shakura-Sunyaev, or α-disk) hydrostatic equilibrium in the vertical direction implies

$$\frac{h}{s} = \frac{c_\mathrm{s}}{v_\phi} \tag{8}$$

Using this in Eq. 7, we find for Shakura-Sunyaev disks

$$\Delta s \sim \zeta h \tag{9}$$

and hence that

$$\nu \sim \alpha h c_\mathrm{s} \tag{10}$$

with $\alpha = \zeta^2$ again not too much smaller than unity. This derivation of the Shakura-Sunyaev scaling, starting from the assumption of a Reynolds driven turbulence, depends on the disk mass being negligible, i.e., a vertical hydrostatic equilibrium of the form of Eq. 8 has to apply. For selfgravitating disks, Eq. 8 no longer applies, and thus the functional form of ν will differ from that of Eq. 10. Note that the upper bound to Δs *implies* approximately isotropic turbulence. This is the standard α-ansatz but derived from considerations of rotationally generated turbulence.

It is worthwhile noting that this derivation of the Shakura-Sunyaev prescription not only yields its functional form, $\nu \propto h c_\mathrm{s}$, but also the order of magnitude for the scaling parameter, $\alpha \sim \zeta^2$, where α is close to but less than unity. This value is consistent with values derived by comparing α-disk models with observations of disks, for instance in dwarf novae (Cannizzo, Shafter & Wheeler 1988).

From the above, it is clear that the viscosity in accretion disks depends not only on the generation of hydrodynamic turbulence but also on the limitation arising from the requirement that the turbulence be subsonic. It also depends on whether or not the disk is selfgravitating. In the following Sect., we use he same logic to investigate the viscosity prescription in selfgravitating accretion disks, in which turbulence is limited by shock dissipation.

3. Viscosity in Thin Selfgravitating Accretion Disks

3.1. *Conditions for Selfgravity in Accretion Disks*

In the following we assume that the accretion disks are geometrically thin in the vertical direction, symmetric in the azimuthal direction, and stationary. We approximate the vertical structure by a one zone model. Then a disk model is specified by the central mass M_*, the radial distributions of surface density $\Sigma(s)$, central plane temperature $T_\mathrm{c}(s)$, and effective temperature $T_\mathrm{eff}(s)$ or the radial mass flow rate $\dot{M}$. The relevant material functions are the equation of state, the opacity and the viscosity prescription.

One can estimate the importance of selfgravity by comparing the respective contributions to the local gravitational accelerations in the vertical and radial directions.

The vertical gravitational acceleration at the disk surface is $2\pi G\Sigma$ and GM_*h/s^3, for the selfgravitating and the purely Keplerian case, respectively. Selfgravitation is thus dominant in the vertical direction when

$$\frac{M_\mathrm{d}}{M_*} \sim \frac{\pi s^2 \Sigma}{M_*} > \frac{1}{2}\frac{h}{s}, \tag{11}$$

where $M_\mathrm{d}(s)$ is the mass enclosed in the disk within a radius s and is given approximately by $M_\mathrm{d} \sim \pi s^2 \Sigma$. Typical numbers for h/s are in the range $10^{-2 \dots -1}$.

Similar considerations lead to the condition

$$M_\mathrm{d} > M_* \tag{12}$$

for selfgravitation to dominate in the radial direction. Thus, we can define three regimes as follows:

- Non-selfgravitating (NSG) disks in which $M_\mathrm{d}(s) \leq (1/2)(h/s)M_*$ (i.e., the classical Shakura-Sunyaev disks)
- Keplerian selfgravitating (KSG) disks in which selfgravity is significant only in the vertical direction and which satisfy the constraint $(1/2)(h/s)\,M_* \leq M_\mathrm{d}(s) \leq M_*$
- Fully selfgravitating (FSG) disks which satisfy $M_* \leq M_\mathrm{d}(s)$

Because $M_\mathrm{d}(s)$ is a monotonically increasing function of s, all three regimes will arise in sufficiently massive, thin ($h/s \ll 1$) disks.

3.2. Selfgravitating Disks

In this Sect., we will review the structure of selfgravitating (SG) accretion disks within the framework of the assumptions introduced above. Compared to the standard NSG models, both the KSG and FSG disks require modification of the equation of hydrostatic support in the direction perpendicular to the disk. Thus, while in the standard model the local vertical pressure gradient is balanced by the z component of the gravitational force due to the central object, in the SG case we have balance between two local forces, namely the pressure force and the gravitational force due to the disk's local mass. In the KSG case, in the radial direction centrifugal forces are still balanced by gravity from a central mass (*Keplerian* approximation), while in the fully selfgravitating case we have to solve Poisson's equation for the rotation law in the disk.

For an SG disk, hydrostatic equilibrium in the vertical direction yields

$$P = \pi G \Sigma^2 \tag{13}$$

(Paczyński 1978), where P is the pressure in the central plane ($z = 0$), Σ is the surface mass density integrated in the z direction, and G is the gravitational constant.

Since details of the thermodynamics in the z direction are of no particular relevance to our argument, we shall assume the disk to be isothermal in the vertical direction.

Integrating the equation of conservation of angular momentum gives

$$\nu\Sigma = -\frac{\dot{M}}{2\pi s^3 \omega'}(s^2\omega - \xi) \tag{14}$$

with the radial mass flow rate[2] $\dot{M}$, the rotational frequency ω, its radial derivative ω', and a quantity ξ allowing for the integration constant or, equivalently, for the inner boundary condition. For a detailed discussion of ξ see, e.g., Duschl & Tscharnuter (1991), Popham & Narayan (1995) and Donea & Biermann (1996). For simplicity, we set the boundary condition $\xi = 0$ in the subsequent discussion. This does not alter the essence of our argument, and only changes details close to the disk's inner radial boundary, since the product $s^2\omega$ increases with s. In fact Eq. 14 applies in the general case (i.e., NSG and SG); for Keplerian disks (NSG and KSG), we may write $\omega' = -3\omega/2s$.

Finally, we have for the sound velocity

$$c_\mathrm{s}^2 = P \bigg/ \left(\frac{\Sigma}{2h}\right). \tag{15}$$

Eqs. 13 and 15 give

$$2\pi G \Sigma h = 4\pi G \overline{\rho} h^2 = c_\mathrm{s}^2 \tag{16}$$

where $\overline{\rho} = \Sigma/2h$ is a vertically averaged mass density.

On the other hand, the Jeans condition for fragmentation in the disk into condensations of radius R is

$$\frac{4\pi}{3} q G \overline{\rho} R^2 > c_\mathrm{s}^2 \tag{17}$$

(see Mestel 1965) where q is factor of order unity.

Thus, a selfgravitating disk is on the verge of fragmenting into condensations of radius $R \sim h$ unless these are destroyed by shear motion associated with the Keplerian velocity field. Thus Paczyński (1978) and later Kozłowski, Wiita & Paczyński (1979) and Lin & Pringle (1987) proposed that the viscosity prescription was directly coupled to the above gravitational stability criterion.

To solve for the dynamic and thermal structure of SG disks, a viscosity prescription has to be specified. As in the NSG case it is possible but not necessary that viscosity is limited by shock dissipation. In the absence of such dissipation, we would, as before, expect the β-viscosity to apply. It is instructive, however, to follow the logic of Sect. 2.2 in he case of SG disks in which turbulence is limited by shock dissipation.

[2] We choose the convention of radial mass flow rate $\dot{M}$ and radial velocity v_s *positive* for inward motion.

3.3. Viscosity in shock dissipation limited selfgravitating accretion disks

For a SG disk (whether Keplerian or not) Eq. 8 is no longer valid so the analysis of Eqs. 9 and 10 no longer applies. In physical terms, the scale height in the disk no longer reflects global properties of the disk (mass of and distance to the central star) but is set by local conditions.

For the selfgravitating case we have approximately

$$v_\phi^2 \sim \frac{G(M_* + M_\mathrm{d})}{s} \tag{18}$$

where conditions $M_* \gg M_\mathrm{d}$, and $M_* \ll M_\mathrm{d}$. distinguish between the Keplerian and the FSG cases, respectively, with the KSG disks as an intermediate case.

From Eq. 7

$$\Delta s \sim \zeta \frac{s c_\mathrm{s}}{v_\phi} \sim \zeta \left(\frac{2\pi h \Sigma s^3}{M_* + M_\mathrm{d}}\right)^{1/2} \sim \zeta \left(\frac{2h\Sigma s}{\Sigma_* + \Sigma}\right)^{1/2} \tag{19}$$

where Σ_* is defined by $M_* = \pi s^2 \Sigma_*$.

At the transition to the NSG regime, Eqs. 11 and 19 and the condition $M_* \gg M_\mathrm{d}$ give as before

$$\Delta s \sim \zeta h \tag{20}$$

and hence a smooth transition to the α-ansatz.

For the FSG regime, Eq. 19 and the condition $M_\mathrm{d} \gg M_*$ give the simple asymptotic form

$$\Delta s \sim \zeta (2hs)^{1/2} \tag{21}$$

and hence a viscosity of the form

$$\nu \sim \gamma (hs)^{1/2} c_\mathrm{s} \tag{22}$$

where γ is a factor of order unity.

The situation is more complex for intermediate values of M_d/M_*. The derived viscosity differs in all SG cases from the standard α-ansatz, but approaches that form as $M_\mathrm{d}/M_* \to 0$. Thus when hydrodynamically induced turbulence is limited by shock dissipation, the resultant viscosity reflects local conditions and takes the standard Shakura-Sunyaev form only when the disk mass is negligible. We show below that this new prescription removes a problem previously noted by Paczyński (1978) and others, with the structure of KSG disks with α-viscosity.

3.4. Structure of SG Disks with Shock Limited Viscosity

It follows from Eqs. 13, 14 and 15, with $\xi = 0$, that

$$c_\mathrm{s}^2 = -\frac{G h \dot{M}}{2\nu} \left(\frac{\mathrm{d}\ln s}{\mathrm{d}\ln \omega}\right). \tag{23}$$

If one adopts the standard α-prescription, this yields

$$c_\mathrm{s}^3 = -\frac{G \dot{M}}{2\alpha} \left(\frac{\mathrm{d}\ln s}{\mathrm{d}\ln \omega}\right). \tag{24}$$

For a KSG disk, this in turn yields

$$c_\mathrm{s}^2 = \left(\frac{2G\dot{M}}{3\alpha}\right)^{2/3} = \frac{kT_\mathrm{c}}{m_\mathrm{H}}, \tag{25}$$

or

$$T_\mathrm{c} = 2.42\,\mathrm{K} \left(\frac{1}{\alpha} \frac{\dot{M}}{10^{-6}\,\mathrm{M_\odot/yr}}\right)^{2/3} \tag{26}$$

A similar result arises for the FSG case, albeit with a different numerical factor resulting from the solution of Poisson's equation.

Thus, for a SG disk, the α-ansatz leads to the requirement of a constant temperature for all radii s (or, if $\xi \neq 0$ in Eq. 14, the temperature is prescribed as a function of s), independent of thermodynamics. While the exact constancy of the temperature may very well be an artefact of our simplified one-zone approximation for the vertical structure, there is no reason to expect that proper vertical integration of the structure will change this fundamentally.

The α ansatz for a SG disk also requires that the disk structure satisfy

$$h\Sigma = \frac{c_\mathrm{s}^2}{2\pi G} = \frac{\dot{M}^{2/3}}{(3\alpha)^{2/3}\,(2\pi G)^{1/3}}. \tag{27}$$

In a standard NSG accretion disk the temperature is a free parameter which is determined by the energy released by the inward flow of the disk gas ($\dot{M}$), by the local viscosity, and by the respective relevant cooling mechanisms. The viscosity depends on T_c via Eq. (1) and on the equation of hydrostatic support in the direction normal to the disk (Eq. (8), which in the non-SG case replaces Eq. (13)).

In the SG case, it is the surface density Σ (and hence h) which must adjust in order to radiate the energy deposited by viscous dissipation and provided by the inward flowing material. While detailed solutions are beyond the scope of this paper, they clearly exist formally. On the other hand, the normal *thermostat* mechanism does not operate, at least in the steady state. Indeed in certain circumstances, the condition of constant mid-plane temperature appears to be inconsistent with the basic thermodynamic requirement that the average gas temperature in the disk exceed that of the black body temperature required to radiate away the energy dissipated by viscous stresses (see Appendix). It is therefore doubtful whether a physically plausible and stable quasi-steady state solution exists.

On the other hand, if one adopts the alternative prescription for shock limited viscosity proposed in Sect. 3.3, the above problem with constant or prescribed mid-plane temperature disappears, the temperature once again depends on h, and the normal thermostat can operate. While this does not prove the validity of the shock limited viscosity prescription given in Sect. 3.3, it is certainly an interesting consequence.

4. Selfgravitating β-Disks

4.1. General Observations

A general analysis of SG disks is complex and beyond the scope of this paper. In this Sect., we examine the structure of β-disks, in which the turbulence is subsonic at all radii. Before doing so, we make the following general observations.

First, with the β-viscosity prescription, Eqs. 14 and 23 give

$$\Sigma = -\frac{\dot{M}\omega}{2\pi\nu s\omega'} = -\frac{\dot{M}}{\beta s^3\omega'} \tag{28}$$

and

$$c_\mathrm{s}^2 = -\frac{Gh\dot{M}}{2\nu}\frac{\omega}{s\omega'} = -\frac{Gh\dot{M}}{2\beta s^3\omega'} \tag{29}$$

Thus the SG β-disks recover the thermostat property of the standard disk, namely that the temperature and scale height can adjust to accomodate (radiate away) the energy input to the system from viscous dissipation and inward motion.

Second, we note that if the disk matter distribution is clumpy (e.g., clouds within a low density smoothed out distribution) then there may be a formal connection between the α- and β-prescriptions. Since in the β-formulation the clump velocities are of order v_ϕ shock heating will tend to heat the low density inter-clump gas until its sound speed $c_\mathrm{s} \sim v_\phi$. The inter-clump gas then has a scale height $h \sim s$, the scale of the clumpy disk, and will hence be roughly a spherical structure. At this point the α- and β-prescriptions look formally identical but the scale height and sound speed now refer to a more or less spherical background distribution of hot gas in which a disk structure of cloudy clumps is imbedded.

4.2. Keplerian Selfgravitating β-Disks (KSG)

For the particular case of a KSG β-disk we have from Eq. 29

$$\frac{c_\mathrm{s}^2}{h} = \frac{3G\dot{M}}{3\beta}\frac{1}{(GM_*)^{1/2}s^{1/2}} \tag{30}$$

For the SG β-disk it follows immediately from Eq. (28) and from mass conservation in the disk that the radial inflow velocity v_s is given by

$$v_s = \frac{\dot{M}}{2\pi s\Sigma} = -\beta s^2\omega' \tag{31}$$

For the KSG β-disk, we then have

$$v_s = \frac{3}{2}\beta s\omega = \frac{3}{2}\beta v_\phi \tag{32}$$

Thus at each radius the inward velocity is the same fraction of the local orbital velocity. From Eq. (31) this, in fact, holds for any SG β-disk in which the angular velocity is a power law function of s with adjustment only to the numerical factor in Eq. (32). If β satisfies the constraint (3), then the approximation of centrifugal balance in the radial direction remains well justified.

Under these conditions the dissipation per unit area of a SG β-disk is given by

$$D = \frac{\dot{M}}{4\pi s}\left(\frac{v_\phi^2}{s}\right) = 2\sigma T_\mathrm{eff}^4 \tag{33}$$

where σ is the Stefan-Boltzmann constant. For an optically thick KSG disk this yields the same radial dependence of T_eff as for the standard disk, namely

$$T_\mathrm{eff} = \left(\frac{G\dot{M}M_*}{8\pi\sigma}\right)^{1/4} s^{-3/4} \tag{34}$$

This temperature dependence which is identical to that of the standard model then leads to the well known energy distribution for an optically thick standard disk of $F_\nu \propto \nu^{1/3}$. This also implies that—as long as the disks are not fully selfgravitating—it is hard to distinguish between an α- and a β-disk model observationally.

4.3. Fully Selfgravitating β-Disks (FSG)

We turn now to the case of the fully selfgravitating (FSG) β-disk, in which the disk mass is sufficiently great that it dominates the gravitational terms in the hydrostatic support equation in both the radial and vertical directions. While there are many potential solutions for the FSG disk structure, one is well known in both mathematical and observational terms, namely the constant velocity ($v_\phi = \mathrm{const.}$) disk. Within such a disk structure we have simultaneous solutions to the equation of radial hydrostatic equilibrium and Poisson's equation of the form

$$v_\phi = s\omega = v_0 \quad \text{and} \quad \Sigma \propto \Sigma_0\left(\frac{s}{s_0}\right)^{-1}. \tag{35}$$

(Toomre 1963, Mestel 1963). For the FSG disk, Eq. (28) then leads to

$$\Sigma = \frac{\dot{M}}{2\pi\beta s v_0} \tag{36}$$

which has the same radial dependence as the structural solution shown in Eq. (35). Thus Eq. (36) may be viewed as giving the rate of mass flow through the disk for a FSG β-disk with constant rotational velocity v_0. Finally, the equation of continuity provides a constraint if the structure is to maintain a basically steady state structure. For the $v_\phi = v_0 =$ const. disk, this yields

$$s\frac{\partial\Sigma}{\partial t} = \frac{\partial}{\partial s}(sv_s\Sigma) = \frac{\partial}{\partial s}(\beta v_\phi s\Sigma) = 0 \tag{37}$$

Thus the constant velocity disk represents a steady state solution in regions sufficiently far from the inner and outer boundaries of the β-disk.

It is then possible, in the spirit of the discussion of Eqs. (33) and (34), to calculate the energy dissipation rate per unit area D for the constant velocity β-disk. We then find

$$D = 2\sigma T_\mathrm{eff}^4 = \frac{\dot{M}v_0^2}{4\pi s^2} \tag{38}$$

so that

$$T_{\rm eff} = \left(\frac{\dot{M} v_0^2}{8\pi\sigma}\right)^{1/4} s^{-1/2} \tag{39}$$

The flux density, F_ν, emitted by an optically thick, constant velocity β-disk it then given by

$$F_\nu \propto \nu^{-1}. \tag{40}$$

In reality, a sufficiently massive disk may be expected to have an inner Keplerian (standard) zone, a Keplerian selfgravitating zone (KSG), and a fully selfgravitating zone (FSG). We should therefore expect a smooth transition in the spectral energy distribution from the $\nu^{1/3}$ spectrum of the inner two zones to the ν^{-1} spectrum arising at longer wavelengths from the FSG zone. The transition frequency $\nu_{\rm trans}$ may be derived by solving Eqs. (34) and (39) for s, determining a value of transition temperature $T_{\rm trans}$ and setting

$$\begin{aligned} \nu_{\rm trans} &= \frac{kT_{\rm trans}}{h} = \frac{k\dot{M}^{1/4} v_0^{3/2}}{h\,(8\pi\sigma)^{1/4}\,(GM)^{1/2}} = \\ &= \left(\frac{k}{h}\right)\left(\frac{\dot{M}}{8\pi\sigma}\right)^{1/4}\left(\frac{v_0^3}{GM}\right)^{1/2}. \end{aligned} \tag{41}$$

One could turn this argument around and argue that, if no other components contribute to the spectrum, the flatness of the νF_ν distribution is a measure for the importance of selfgravity and thus for the relative mass of the accretion disk as compared to the central accreting object. This, of course, applies only to the optically thick case which may not arise frequently in strongly clumped disks.

4.4. Time Scales

The evolution of accretion disks can be described by a set of time scales. For our purposes, the dynamical and the viscous time scale are of particular interest.

The dynamical time scale $\tau_{\rm dyn}$ is given by

$$\tau_{\rm dyn} = \frac{1}{\omega}. \tag{42}$$

While this formulation applies to all cases, selfgravitating or not, it is only in the non-SG and in the KSG cases that ω is given by the mass of the central accretor and by the radius. In the FSG case, ω is determined by solving Poisson's equation.

The time scale of viscous evolution $\tau_{\rm visc}$ is given by

$$\tau_{\rm visc} = \frac{s^2}{\nu} \tag{43}$$

In the standard non-SG and geometrically thin ($h \ll s$) case (α disks), this leads to

$$\tau_{\rm visc}^{\rm non-SG} = \left(\frac{s}{h}\right)^2 \frac{\tau_{\rm dyn}}{\alpha} \gg \frac{\tau_{\rm dyn}}{\alpha}. \tag{44}$$

In KSG and FSG disks (β-disks), $\tau_{\rm visc}$ is given by

$$\tau_{\rm visc}^{\rm KSG} = \tau_{\rm visc}^{\rm FSG} = \tau_{\rm visc}^{\rm SG} = \frac{\tau_{\rm dyn}}{\beta} \tag{45}$$

With $\alpha < 1$ and $\beta \ll 1$ (Eq. 3) under all circumstances $\tau_{\rm visc} \gg \tau_{\rm dyn}$. In the SG cases the ratio between the two time scales decouples from the disk structure. In all cases the models are self-consistent in assuming basic hydrostatic equilibrium in the vertical direction.

5. Possible Applications

5.1. Protoplanetary Accretion Disks

T Tauri stars have infrared spectral energy distributions νF_ν which can be approximated in many cases by power laws $\nu F_\nu \propto \nu^n$ with a spectral index n in the range $\sim 0 \ldots 1.3$. Assuming this spectral behaviour to be due to radiation from an optically thick disk, it translates into a radial temperature distribution $T_{\rm eff} \propto s^{-q}$ with $n = 4 - 2/q$.

An optically thick non-selfgravitating accretion disk which radiates energy that is liberated through viscous dissipation, i.e., an *active* accretion disk shows a spectral distribution with $q = 3/4$ or $n = 4/3$. This immediately excludes optically thick non-selfgravitating standard accretion disks as the major contributor to T Tau spectra.

Adams, Lada & Shu (1988) were the first to discuss the possibility of a non-standard radial temperature distribution with $q \neq 3/4$. Using q as a free parameter, they find that for flat spectrum sources, their best fits require disk masses that are no longer very small compared to the masses of the accreting stars. They already mention the possibility that the flatness of the spectrum and selfgravity of the disk may be related. On the other hand, at that time this indirect argument was the only evidence for large disk masses. Beckwith et al. (1990) in their survey of circumstellar disks around young stellar objects also find preferentially disk spectra that are considerably flatter than predicted by the standard optically thick disk models. For more than half of their objects they derive disk masses that correspond to the KSG and FSG cases. On the other hand, Natta (1993) proposed that flat disk spectra are the consequence of dusty envelopes engulfing a star with a standard disk around it. Recently, Chiang & Goldreich (1997) have investigated in detail non-selfgravitating *passive* accretion disks, i.e., disks that are heated by radiation from the star and re-radiate this energy. Depending on the details of the flaring of the disk, this can lead to considerably flatter spectra than expected from active disks.

However, in the meantime, high resolution direct observations of protostellar disks yield independent strong evidence for comparatively large disk masses. Lay et al. (1994), for instance, find a lower limit for the disk masses in HL Tau—one of the sources in Adams, Lada & Shu's sample of flat spectrum T Tauri stars—of $\sim 0.02\,{\rm M}_\odot$.

We suggest that the flatness of the spectrum actually reflects the mass of the disk, i.e., the importance of selfgravity. For disk masses considerably smaller than $\sim 1/30 M_*$, the standard accretion disk models apply. For disks whose masses are larger but still small compared to M_* the spectral behaviour is not altered significantly, but the disk structure and the time scale of disk evolution ($\tau_{\rm visc}$, see Eqs. 44 and 45) change. For even more massive disks, we expect a clear trend towards flatter spectra that approach an almost constant νF_ν distribution if selfgravity in the disk becomes important in the radial as well as in the vertical direction.

5.2. Galactic Disks

The relevance of viscosity in the evolution of galactic disks has been the subject of discussion since von Weizsäcker (1943, 1951) and Lüst (1952) first raised the issue nearly fifty years ago. They noted then that, with an eddy viscosity formulation (a β-disk), the time scale for evolution of typical galactic disks was comparable to the age of the universe and suggested that this might account for the difference between spiral and elliptical galaxies.

With the subsequent realization that galactic disks moved primarily under the influence of extended massive halos, interest in FSG disks waned. However, as noted above, it is possible for a massive disk to exist and evolve under the influence of viscosity, while embedded in such a halo gravitational field. Indeed, in the event that such a structure forms, it must evolve under viscous dissipation and can achieve a quasi-steady state with essentially the same mass and energy dissipation distribution as for the FSG constant velocity disk. We refer to this case as an Embedded Self-Gravitating (ESG) disk.

The time scale for viscous evolution $\tau_{\rm visc}$ as given in Sect. 4.4 suggests a means of differentiating between the α- and β-formulations for this case. For a normal spiral galaxy with a suggested mean temperature in the gaseous disk of around 10^4 K and a scale height of around 300 pc, we obtain

$$\begin{aligned} \tau_{\rm visc}^{(\alpha)} &\sim 10^4 \tau_{\rm dyn} \sim 3\,10^{11}\,{\rm yr} \\ \tau_{\rm visc}^{(\beta)} &\sim 10^2 - 10^3 \tau_{\rm dyn} \sim 3\,10^9 - 3\,10^{10}\,{\rm yr} \end{aligned} \quad (46)$$

Thus, with these parameters, little evolution would take place in a Hubble time on the α-hypothesis but significant evolution is predicted on the β-hypothesis. This problem of the viscous time scale in a selfgravitating α accretion disk was also noted by Shlosman & Begelman (1987, 1989). Shlosman, Frank & Begelman (1989) proposed non-axisymmetric disturbances ("bars within bars") as an alternative way of transporting angular momentum in the radial direction within a sufficiently short time scale.

In terms of inflow velocities the β-ansatz suggests values in the range $0.3 - 3\,{\rm km\,s^{-1}}$ which would be exceedingly hard to measure directly. The α-ansatz suggests still lower values. On the other hand, it may be possible to provide limits on the viscosity through other observational constraints. For example, the build up of the 3 kpc molecular ring in our own galaxy can be interpreted as due to viscosity driven inflow in the constant velocity part of the galactic disk which ceases (or at least slows down) in the constant angular velocity inner regions (Icke 1979, Däther & Biermann 1990). Similarly, several authors have suggested that the radial abundance gradients observed in our own and other disk galaxies may be due to radial motion and diffusive mixing associated with the turbulence generating the eddy viscosity (Lacey & Fall 1985, Sommer-Larson & Yoskii 1990, Köppen 1994, Edmunds & Greenhow 1995, Tsujimoto et al. 1995). According to these authors, radial inflows of around $1\,{\rm km\,s^{-1}}$ at the galactic location of the Sun are required for optimum fits to the abundance gradient data within the context of the viscous disk hypothesis. Such inflow velocities are consistent with the β-ansatz but could, of course, be generated also by other means (e.g effects of bars, magnetic fields).

5.3. Ultraluminous galaxies

Recent high resolution imaging of ultraluminous galaxies in the near infrared and mm wavelengths bands shows dense gas and dust accretion disks in their galactic nuclei. The two nuclei in the merger galaxy Arp 220, for instance, have masses of the order of several $10^9\,{\rm M_\odot}$ within radii of $\lesssim 100\,{\rm pc}$ (Scoville 2000). Similar properties, albeit less well resolved than in Arp 220, seem to be typical for this class of galaxies (Solomon et al. 1997, Downes & Solomon 1998). Most, if not all, ultraluminous galaxies seem to be merging galaxies (Sanders & Mirabel 1996). In Arp 220, these gas masses are the major contributor to the dynamical mass in the two nuclei (Scoville 2000), i.e., these nuclear disks are selfgravitating. Most likely this is true for the nuclear disks in other ultraluminous galaxies as well. The merger process is presumably responsible for transporting large amounts of material into the central few hundred parsecs, thus filling a mass reservoir which is then available for subsequent disk accretion to the very center.

Within the framework of β-disks, one finds that the viscous accretion time scale $\tau_{\rm visc}$ increases towards larger radii as long as the surface density Σ in the disk increases with radius s not steeper than $\Sigma \propto s$, which is most likely fulfilled. Then the viscous time scale at the disk's outer edge is an upper limit to its evolution time scale. For Arp 220 (disk mass $\sim 2\,10^9\,{\rm M_\odot}$; outer radius $\sim 100\,{\rm pc}$) one finds a time scale of $\sim 10^{5.5}\,{\rm yr}/\beta \sim 3\ldots30\,10^7$ years for $\beta = 10^{-2\ldots-3}$, which, in turn yields accretion rates $\dot{M} \sim 10^{1\ldots2}\,{\rm M_\odot\,yr^{-1}}$. Such rates lead to accretion luminosities $L_{\rm accr} = \eta \dot{M} c^2$ up to $\sim \eta 10^{47.5\ldots48.5}\,{\rm erg\,s^{-1}}$, where η (~ 0.1) is the conversion efficiency of gravitational energy into radiation and c is the speed of light. Such luminosities are large enough to power even the strongest AGN and the time scales are very much shorter than the Hubble time.

Assuming that these rates can be maintained during a sizeable fraction of $\tau_{\rm visc}$, a significant fraction of the disk's original gas mass could be accreted to much smaller radii, presumably to a black hole in the very center (some will be lost to star formation or winds). In this process the black hole gains a considerable amount of mass within a relatively short time scale. One may speculate that this is actually the process that produces the most massive black holes in the young universe. By contrast, galaxies that do not undergo mergers presumably have no way of rapidly collecting such large masses of gas within 10^2 pc. As a consequence, these disks are less likely to be selfgravitating and thus are likely to have longer $\tau_{\rm visc}$. The nuclei of such galaxies will accrete much smaller amounts of material over longer time scale, resulting in lower mass central black holes (Duschl 1988a, b). An example may be our own Galactic Center.

6. Summary

We propose a viscosity prescription based on the assumption that the effective Reynolds number of the turbulence does not fall below the critical Reynolds number. In this parametrization the viscosity is proportional to the azimuthal velocity and the radius (β-disks). This prescription yields physically consistent models of both Keplerian and fully selfgravitating accretion disks. Moreover, for the case of thin disks with sufficiently small mass, we recover the α-disk solution as a limiting case.

Such β-disk models may be relevant to protoplanetary accretion disks as well as to galactic and galactic center disks. In the case of protoplanetary disks they yield spectra that are considerably flatter than those due to non-selfgravitating disks, in better agreement with observed spectra of these objects. In galactic disks, they result in viscous evolution on time scales shorter than the Hubble time and thus offer a natural explanation for an inward flow that could account for the observed chemical abundance gradients. In galactic centers, β-disks may be the supply for powering AGN and for forming supermassive black holes within time scales short compared to the Hubble time.

Finally, β-disks yield a natural solution to an inconsistency in the α-disk models if the disk's mass is large enough for selfgravity to play a role. This problem arises even in Keplerian selfgravitating disks in which only the vertical structure is dominated by selfgravity while the azimuthal motion remains Keplerian.

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Appendix A: Thermodynamic Considerations for KSG α-disks

For a KSG α-disk, we have from Eq. 26 that

$$T_{\rm c} = 2.42\,{\rm K}\left(\frac{1}{\alpha}\frac{\dot M}{10^{-6}\,{\rm M}_\odot/{\rm yr}}\right)^{2/3}. \qquad (A.1)$$

If the disk is optically thick and advection is negligible, viscous dissipation leads to local effective temperature of

$$\begin{aligned} T_{\rm eff} &= \left(\frac{3}{8\pi\sigma}\right)(GM)^{1/4}\,\dot M^{1/4} s^{-3/4} \\ &= 8.53\,10^3\,{\rm K}\left(\frac{m\dot m}{s_{\rm A}^3}\right)^{1/4} \end{aligned} \qquad (A.2)$$

with m the mass of the central star in solar units and $s_{\rm A}$ the radius in astronomical units.

An essential thermodynamics requirement is that $T_{\rm c} > T_{\rm eff}$ or that

$$\frac{T_{\rm eff}}{T_{\rm c}} = 3.53\,10^{-2}\frac{m^{1/4}\dot m^{-5/12}}{s_{\rm A}^{3/4}}\alpha_{-1}^{2/3} < 1 \qquad (A.3)$$

This condition is satisfied provided that

$$\dot m > \dot m_{\rm T} = 3.27\,10^{-4}\frac{m^{3/5}\alpha_{-1}^{8/5}}{s_{\rm A}^{9/5}} \qquad (A.4)$$

and that the disk is selfgravitating in the vertical direction at $s_{\rm A}$. The latter condition leads to a second requirement on $\dot m$.

For a standard Keplerian disk, the mass flow rate is given (Eqs. 8, 10, 43) by

$$\dot M \approx \frac{M_{\rm d}}{\tau_{\rm visc}} \approx \frac{M_{\rm d}\nu}{s^2} = \alpha M_{\rm d}\left(\frac{h}{s}\right)^2\omega \qquad (A.5)$$

with $M_{\rm d}$ the disk's mass. From Eq. 11, the condition that the disk is non-selfgravitating is $M_{\rm d} < (h/2s)M_*$ and hence, from Eq. A.5, that

$$\dot M < \frac{\alpha}{2}\left(\frac{h}{s}\right)^3\left(\frac{GM_*^3}{s^3}\right)^{1/2} \qquad (A.6)$$

or

$$\dot m < \dot m_{\rm G} = 3.14\,10^{-1}\alpha_{-1}\left(\frac{h}{s}\right)^3\frac{m^{3/2}}{s_{\rm A}^{3/2}} \qquad (A.7)$$

A selfconsistent and physically acceptable solution can be obtained only if $\dot m_{\rm G} > \dot m_{\rm T}$, that is the disk becomes selfgravitating at values of $\dot m$ which are sufficiently high that thermodynamic requirements are not violated. This condition may then be written as

$$\frac{h}{s} > 1.01\,10^{-1} m^{-3/10} s_{\rm A}^{-1/10}\alpha_{-1}^{1/5} \qquad (A.8)$$

Thus *thin* KSG α-disks appear to be inconsistent with basic thermodynamic requirements if $m \lesssim 1$, $s_{\rm A} \lesssim 1$. There is no inconsistency if either or both of these quantities are sufficiently large.

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