The "Swiss cheese" cosmological model has no extrinsic curvature discontinuity: A comment on the paper by G.A. Baker, Jr. (astro-ph/0003152)

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October 28, 2018

Abstract

Contrary to a claim, the Schwarzschild solution insertion in an expanding universe model, the so called "Swiss cheese" model, does not possess an extrinsic curvature discontinuity. We show that both the intrinsic metric and the extrinsic curvature are continuous, and point out the error that led to the claim.

1   Introduction

The "Swiss cheese" cosmological model is a general relativistic description of space-time. The name refers to the fact that in this model static spherical voids are created within a larger, time-dependent space-time. A void is constructed by removing the background material inside a spherical boundary and replacing the mass by a concentration of that mass at the centre of the sphere.

Mathematically, the model is realized by the matching of a Friedmann-Lemaître-Robertson-Walker (FLRW) metric as the exterior solution, to an exterior Schwarzschild metric as the interior solution, across a spherical boundary. The spherical boundary stays at a fixed coordinate radius in the FLRW frame, but changes with time in the Schwarzschild frame.

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The smooth matching of two space-times across a three-surface of discontinuity Σ is guaranteed if the Darmois junction conditions are satisfied: the first fundamental forms (intrinsic metrics) and the second fundamental forms (extrinsic curvatures) calculated in terms of the coordinates on Σ, are identical on both sides of the hypersurface [1]. The Darmois junction conditions allow us to use different coordinate systems on both sides of the hypersurface.

The continuity of the first and second fundamental forms on a matching hypersurface Σ implies the continuity of the fluid pressure on Σ (see e.g. [2]). In the case of the "Swiss cheese" model, it implies a dust filled (i.e. zero pressure) FLRW space-time.

Recently, Baker [3] claimed that the extrinsic curvatures for the Schwarzschild and the FLRW metrics used in the "Swiss cheese" model cannot be matched at a spherical boundary. In the following section we show that this claim is erroneous. We prove this by explicitly constructing a smooth matching between the Schwarzschild and FLRW space-times across a spherical hypersurface. In particular, we show that both the intrinsic metrics and the extrinsic curvatures (henceforth often refered to as the first and second fundamental forms respectively) are continuous on the hypersurface. We also verify that the pressure is continuous as required. We conclude by indicating the error in ref. [3] that led to the claim.

2 The Matching

The general FLRW metric can be written in spherical coordinates as

$$ds^2 = dt^2 - K^2(t) \left[r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) + \frac{dr^2}{1 - kr^2} \right] , \tag{1}$$

where $K(t)$ is the scale factor and $k = 0, \pm 1$ the curvature constant of space. We will show that the metric (1) can be joined smoothly on a spherical hypersurface Σ to the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2M}{\rho} \right) dT^2 - \rho^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) - \left(1 - \frac{2M}{\rho} \right)^{-1} d\rho^2 . \tag{2}$$

The first fundamental form is the metric which Σ inherits from the space-time in which it is imbedded, and may be written as

$$\Upsilon_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} , \tag{3}$$

where $u^\alpha = (u^1 \equiv u,\, u^2 \equiv v,\, u^3 \equiv w)$ is the coordinate system on the hypersurface. Greek indices run over $1, \ldots, 3$, while Latin indices over $1, \ldots, 4$.

The second fundamental form [4] is defined by

$$\Omega_{\alpha\beta} = \left(\Gamma^p_{\ ij} n_p - n_{i,j} \right) \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} , \tag{4}$$

where n_a is a unit normal to Σ, and $\Gamma^p{}_{ij}$ are the Christoffel symbols. If Σ is given by the function $f[x^a(u^\alpha)] = 0$, then n_i can be calculated from

$$n_i = -\frac{f_{,i}}{|g^{ab}f_{,a}f_{,b}|^{1/2}}\,, \tag{5}$$

where $,i$ denotes $\frac{\partial}{\partial x^i}$. To avoid confusion we will denote indexed quantities associated with the FLRW and Schwarzschild metrics by the letters F and S respectively.

We consider a spherical hypersurface Σ given by the function $f_F(x_F^i) = r - r_0 = 0$, where r_0 is a constant, and parametrized by $x_F^1 = t = u$, $x_F^2 = \theta = v$, $x_F^3 = \phi = w$, and $x_F^4 = r = r_0$, in the FLRW frame. In the Schwarzschild frame we choose the parametrization $x_S^1 = T = T(u)$, $x_S^2 = \theta = v$, $x_S^3 = \phi = w$, and $x_S^4 = \rho = \rho(u)$. The condition $\Upsilon_{F\alpha\beta} = \Upsilon_{S\alpha\beta}$ then implies

$$1 = \left(1 - \frac{2M}{\rho}\right)\left(\frac{dT}{du}\right)^2 - \left(1 - \frac{2M}{\rho}\right)^{-1}\left(\frac{d\rho}{du}\right)^2\,, \tag{6}$$

$$K^2 r_0^2 = \rho^2\,. \tag{7}$$

We now turn to the two second fundamental forms. The (outward pointing) unit normal in the FLRW frame can be calculated from eq. (5) and $f_F(x_F^i) = r - r_0 = 0$. The result is $n_{Fi} = \delta_i^4 n_{F4}$, where $n_{F4} = -|g_{F44}|^{1/2}$. The unit normal is spacelike, i.e. $n_F^i n_{Fi} = -1$. The unit normal in the Schwarzschild frame cannot be obtained directly from eq. (5) since we do not know the form of f_S. However, n_{Si} must satisfy the two conditions

$$n_S^i n_{Si} \equiv n_F^i n_{Fi} = -1 \quad \text{and} \quad n_{Si}\frac{\partial x_S^i}{\partial u^\alpha} = 0\,, \tag{8}$$

where the second condition results from the partial differentiation of $f_S[x_S^i(u^\alpha)] = 0$ with respect to u^α. From (8) one obtains

$$\left.\begin{array}{c} n_{S2} = n_{S3} = 0, \\ n_{S1}\frac{dT}{du} + n_{S4}\frac{d\rho}{du} = 0, \\ \left(1 - \frac{2M}{\rho}\right)^{-1}n_{S1}{}^2 - \left(1 - \frac{2M}{\rho}\right)n_{S4}{}^2 = -1. \end{array}\right\} \tag{9}$$

With the help of eq. (6), equations (9) enable us to derive n_{Si} as a function of u^α:

$$n_{Si} = \left(\epsilon\frac{d\rho}{du},\ 0,\ 0,\ -\epsilon\frac{dT}{du}\right)\,, \quad \epsilon = \pm 1\,. \tag{10}$$

Because of the simple form of n_{Fi}, eq. (4) for the second fundamental form can be much simplified in the FLRW frame. In this case one obtains from (4)

$$\Omega_{F\alpha\beta} = \Gamma^4_{Fij}n_{F4}\frac{\partial x_F^i}{\partial u^\alpha}\frac{\partial x_F^j}{\partial u^\beta} - n_{F4,j}\frac{\partial x_F^4}{\partial u^\alpha}\frac{\partial x_F^j}{\partial u^\beta}$$

$$
\begin{aligned}
&= \Gamma^4_{F\mu\nu} n_{F4} \frac{\partial x^\mu_F}{\partial u^\alpha} \frac{\partial x^\nu_F}{\partial u^\beta}, \quad \text{since } \frac{\partial x^4_F}{\partial u^\alpha} = \frac{\partial r_0}{\partial u^\alpha} = 0, \\
&= \Gamma^4_{F\mu\nu} n_{F4} \delta^\mu_\alpha \delta^\nu_\beta \\
&= n_{F4} \Gamma^4_{F\alpha\beta} \\
&= -\frac{1}{2} |g_{F44}|^{1/2} g^{4i}_F (g_{F\alpha i,\beta} + g_{F\beta i,\alpha} - g_{F\alpha\beta,i}) \\
&= \frac{1}{2} |g_{F44}|^{1/2} g^{44}_F g_{F\alpha\beta,4},
\end{aligned}
\tag{11}
$$

so that

$$
\Omega_{F\alpha\beta} = -\frac{1}{2} |g_{F44}|^{-1/2} g_{F\alpha\beta,4}. \tag{12}
$$

Equation (12) with F's dropped, i.e.

$$
\Omega_{\alpha\beta} = -\frac{1}{2} |g_{44}|^{-1/2} g_{\alpha\beta,4}, \tag{13}
$$

is valid for any coordinate hypersurface $x^4 = $ constant, in an orthogonal coordinate system and parametrized by $x^\alpha = u^\alpha$. No similar simplification of the second fundamental form of Σ is possible in the Schwarzschild frame. Moreover, for us to calculate the second term in eq. (4) we need n_{Si} as a function of x^i_S. However, if we once more differentiate the second condition in eq. (8) with respect to u^α it follows that

$$
n_{Si,j} \frac{\partial x^i_S}{\partial u^\alpha} \frac{\partial x^j_S}{\partial u^\beta} = -n_{Si} \frac{\partial^2 x^i_S}{\partial u^\alpha \partial u^\beta},
$$

giving

$$
\Omega_{S\alpha\beta} = \Gamma^p_{Sij} n_{Sp} \frac{\partial x^i_S}{\partial u^\alpha} \frac{\partial x^j_S}{\partial u^\beta} + n_{Si} \frac{\partial^2 x^i_S}{\partial u^\alpha \partial u^\beta}. \tag{14}
$$

Using equations (12) and (14), we find that $\Omega_{S\alpha\beta} = 0 = \Omega_{F\alpha\beta}$, $\forall \alpha \neq \beta$. The remaining, diagonal components of $\Omega_{\alpha\beta}$ and the continuity condition $\Omega_{S\alpha\beta} = \Omega_{F\alpha\beta}$ on Σ, result in the following three differential equations

$$
\begin{aligned}
\Omega_{S11} &\equiv \Gamma^4_{S11} n_{S4} \left(\frac{dT}{du} \right)^2 + \Gamma^4_{S44} n_{S4} \left(\frac{d\rho}{du} \right)^2 + 2\Gamma^1_{S14} n_{S1} \frac{dT}{du} \frac{d\rho}{du} \\
&\quad + n_{S1} \frac{d^2 T}{du^2} + n_{S4} \frac{d^2 \rho}{du^2} = 0 \equiv \Omega_{F11},
\end{aligned}
\tag{15}
$$

$$
\Omega_{S22} \equiv \Gamma^4_{S22} n_{S4} = \alpha |K| r_0 \equiv \Omega_{F22}, \tag{16}
$$

$$
\Omega_{S33} \equiv \Gamma^4_{S33} n_{S4} = \alpha |K| r_0 \sin^2 \theta \equiv \Omega_{F33}, \tag{17}
$$

where the Christoffel symbols are given by

$$
\Gamma^1_{S14} = -\Gamma^4_{S44} = \frac{M}{\rho(\rho - 2M)},
$$

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$$\Gamma^4_{S11} = \frac{(\rho - 2M)M}{\rho^3},$$

$$\Gamma^4_{S22} = -(\rho - 2M),$$

$$\Gamma^4_{S33} = \sin^2\theta\, \Gamma^4_{S22},$$

and $\alpha \equiv (1 - kr_0^2)^{1/2}$.

Equations (16) and (17) being equivalent, we can use either of them and eq. (7) to obtain

$$\frac{dT}{du} = \frac{\epsilon\alpha\rho}{\rho - 2M}. \tag{18}$$

It then follows from eq. (6) that

$$\left(\frac{d\rho}{du}\right)^2 = \alpha^2 - \left(\frac{\rho - 2M}{\rho}\right). \tag{19}$$

Differentiating eqs. (18) and (19) w.r.t. u, one obtains

$$\frac{d^2T}{du^2} = -\epsilon\alpha \frac{2M}{(\rho - 2M)^2} \frac{d\rho}{du}. \tag{20}$$

and

$$\frac{d^2\rho}{du^2} = -\frac{M}{\rho^2}. \tag{21}$$

With equations (18)-(21), eq. (15) is now identically satisfied. Thus, both the first and second fundamental forms are continuous on Σ.

It remains to verify that the pressure is also continuous across the spherical boundary. Since in the Schwarzschild space-time the pressure is zero, it must vanish in the FLRW space-time. The FLRW space-time is a perfect fluid space-time, and as such its Einstein field equations are given by

$$G^{ij} \equiv R^{ij} - \frac{1}{2}Rg^{ij} = 8\pi \left[(\mu + p)u^i u^j - pg^{ij}\right], \tag{22}$$

where G^{ij} is the Einstein tensor, R^{ij} the Ricci tensor, R the Ricci curvature scalar, μ the matter-energy density, p the pressure, and u^i the unit four-velocity. Specifically, eq. (22) implies

$$G^1_{\ 1} = \frac{3(\dot{K}^2 + k)}{K^2} = 8\pi\mu \tag{23}$$

and

$$G^2_{\ 2} = G^3_{\ 3} = G^4_{\ 4} = \frac{2\ddot{K}}{K} + \frac{(\dot{K}^2 + k)}{K^2} = -8\pi p, \tag{24}$$

where $\dot{K} \equiv \frac{dK}{du} \equiv \frac{dK}{dt}$, and $\ddot{K} \equiv \frac{d^2K}{du^2} \equiv \frac{d^2K}{dt^2}$.

To obtain $\dot{K}$ and $\ddot{K}$, we differentiate eq. (7) w.r.t. u. The results are

$$\dot{K} = \eta \frac{1}{r_0} \frac{d\rho}{du} \, , \tag{25}$$

and

$$\ddot{K} = \eta \frac{1}{r_0} \frac{d^2\rho}{du^2} \, , \tag{26}$$

where $\eta \equiv \frac{K}{|K|}$. Substituting eqs (19) and (21) into (25) and (26), and using $K = \eta \frac{\rho}{r_0}$ (from eq. (7)), it follows from (24) that $p = 0$. Finally, we note that, from eq. (23), $8\pi\mu = \frac{6M}{r_0^3 |R|^3}$, a positive quantity, which shows that the matching is physically admissible.

3 Discussion and Conclusions

We have shown, in a mathematically rigorous way, that an FLRW space-time can be joined smoothly to a Schwarzschild space-time across a spherical boundary, a construction used in the "Swiss cheese" cosmological model of the Universe. The boundary has a fixed coordinate radius in the FLRW frame, but changes with time in the Schwarzschild frame. In particular, we have shown that the intrinsic metric and extrinsic curvature are both continuous on the boundary as is required for a smooth, permanent matching. These matching requirements also imply the continuity of the pressure on the boundary, which has been verified here.

Our results differ from those of ref. [3]. Specifically, we have not found a discontinuity in the extrinsic curvature on the spherical boundary as claimed in [3]. The claim in [3] can be explained out as follows. The author, without justification, uses equation (13) (his eq. (2.10) with $N_\alpha = 0$) for calculating the components of $\Omega_{\alpha\beta}$ in both the FLRW and Schwarzschild frames. This results in the mismatch of $\Omega_{S\alpha\beta}$ and $\Omega_{F\alpha\beta}$ on the spherical hypersurface, leading to the claim that "the 'Swiss cheese' model is at best only an approximation, with a singular interface".

The use of eq. (13) in the Schwarzschild frame is incorrect. Assuming, *a priori,* (as the author of ref. [3] and we have done) that the boundary surface changes with time in the Schwarzschild frame, it follows that the boundary is not a coordinate boundary in that frame, and, therefore, eq. (4) for the extrinsic curvature, cannot be reduced to eq. (13). Equation (13) can be used for calculating $\Omega_{S\alpha\beta}$ only under the assumption that the matching hypersurface in the Schwarzschild frame is the coordinate hypersurface $\rho = $ constant. But then eq. (7) cannot be satisfied unless $R(t) = $ constant; obviously not an interesting case, since then the FLRW space-time reduces to the Minkowski space-time. Thus the use of eq. (13) in the Schwarzschild frame is an error.

References

[1] G. Darmois, *Mémorial des Sciences Mathématiques*, Fascicule XXV (Gauthier-Villars, Paris, 1927), Chap. V.

[2] H. Stephani, *General Relativity* (Cambridge University Press, Cambridge, 1990).

[3] G.A. Baker, Jr. *Bound systems in an expanding universe*, astro-ph/0003152, 10 March 2000.

[4] L.P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, 1949).