

A Dynamical Solution to the Problem of a Small Cosmological Constant and Late-time Cosmic Acceleration

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Increasing evidence suggests that most of the energy density of the universe consists of a dark energy component with negative pressure, a “cosmological constant” that causes the cosmic expansion to accelerate. In this paper, we address the puzzle of why this component comes to dominate the universe only recently rather than at some much earlier epoch. We present a class of theories based on an evolving scalar field where the explanation is based entirely on internal dynamical properties of the solutions. In the theories we consider, the dynamics causes the scalar field to lock automatically into a negative pressure state at the onset of matter-domination such that the present epoch is the earliest possible time, consistent with nucleosynthesis restrictions, when it can start to dominate.

Introduction. Observations of large scale structure, searches for Type Ia supernovae, and measurements of the cosmic microwave background anisotropy all suggest that the universe is undergoing cosmic acceleration and is dominated by a dark energy component with negative pressure.¹ The dark energy may consist of a cosmological constant (vacuum density) or quintessence,² such as a scalar field with negative pressure. In either case, a key challenge is the “cosmic coincidence” problem: Why is it that the vacuum density or scalar field dominates the universe only recently? Until now, either cosmic initial conditions or model parameters (or both) had to be tuned to explain the low density of the dark energy component.

In this paper, we explore a new class of scalar field models with novel dynamical properties that avoid the fine-tuning problem altogether. A feature of these models is that the negative pressure results from the non-linear kinetic energy of the scalar field, which we call, for brevity, k -field or k -essence. (This consideration is inspired by earlier studies of k -inflation, kinetic energy driven inflation.³⁻⁵) As we will show, for a broad class of theories, there exist attractor solutions which determine the equation-of-state of k -essence during different epochs depending on the equation-of-state of the background. Effectively, the scalar field changes its speed of evolution in dynamic response to changes in the background equation-of-state. During the radiation-dominated epoch, k -essence is led to be subdominant and to mimic the equation-of-state of radiation. Hence, the ratio of k -essence to radiation density remains fixed. When the universe enters the dust-

dominated epoch, though, k -essence is unable to mimic the dust-like equation-of-state for dynamical reasons. Instead, the energy decreases rapidly by several orders of magnitude and freezes at a fixed value. After a period (typically corresponding roughly to the current age of the universe), the field overtakes the matter density and drives the universe into cosmic acceleration. Ultimately, the k -essence equation-of-state slowly relaxes to an asymptotic value between 0 and -1. (The reader may wish to sneak a peek at Fig. 3 which illustrates the behavior in a specific numerical example.)

The scenario bears some likeness to the quintessence “tracker models” discussed by Zlatev *et al.*,^{6,7} but is in fact very different from them. For a certain class of tracker potentials, the quintessence scalar field converges to an attractor solution in which the energy density in the quintessence field mimics the equation-of-state of the background (matter or radiation) energy density. However, for tracker potentials, it does not make a difference what the equation-of-state of the background is. Only after the scalar field passes a certain critical value does the quintessence develop a negative pressure, and, then, the energy density becomes fixed. The weakness in this model is that the energy density for which the pressure becomes negative is set by an adjustable parameter, which has to be fine tuned to explain why cosmic acceleration is happening at the present epoch in the history of the universe.

The distinctive feature of k -essence models is that tracking of the background energy density can only occur in the radiation epoch. At the matter-radiation equality, a sharp transition of k -essence from positive to negative pressure is automatically triggered by dynamics. The k -essence cannot dominate before matter-radiation equality because it is exactly tracking the radiation background. It also cannot dominate immediately after dust-domination because its energy density necessarily drops several orders of magnitude at the transition to dust-domination. However, since its energy density decreases more slowly than the matter density as the universe expands, k -essence must dominate not too long thereafter, at roughly the current epoch. The resolution of the cosmic coincidence problem boils down to the fact that we live at the “right time” after matter-radiation equality.

As noted above, the remarkable behavior comes at the cost of introducing a non-linear kinetic energy density functional of the scalar field and adjusting it to obtain the desired attractor behavior. This kind of action may

describe a fundamental scalar field or be a low-energy effective action. For example, in string and supergravity theories, non-linear kinetic terms appear generically in the effective action describing moduli and massless degrees of freedom (superpartners) due to higher order gravitational corrections to the Einstein action.^{8,9} The attractor behavior of our models relies on certain broad conditions on the form of these terms. Our initial examples are admittedly contrived for the purposes of numerical illustration. A systematic study of model-building will appear in a forthcoming paper,¹⁰ although, having seen here the relatively simple basic principles, the reader should be equipped to explore more attractive and better-motivated forms.

Equations. In the theories we consider the Lagrangian density for φ is taken to be

$$\mathcal{L} = -\frac{1}{6}R + \frac{1}{\varphi^2}\tilde{p}_k(X) + \mathcal{L}_m \quad (1)$$

where R is the Ricci scalar, $X \equiv \frac{1}{2}(\nabla\varphi)^2$, \mathcal{L}_m is the Lagrangian density for dust and radiation and we use units where $8\pi G/3 = 1$. The energy density of the k -field φ is $\rho_k = (2X\tilde{p}_{,X} - \tilde{p})/\varphi^2$; the pressure is $p_k = \tilde{p}/\varphi^2$; and the speed of sound of k -essence is $c_s^2 = p_{k,X}/\rho_{k,X}$, where the subscript means derivative with respect to X .^{3,4}

The attractor behavior can be explained most easily by changing variables from X to $y = 1/\sqrt{X}$ and rewriting the k -field Lagrangian as:

$$\mathcal{L}_k = \tilde{p}_k(X)/\varphi^2 \equiv g(y)/\varphi^2 y. \quad (2)$$

In this case, the energy density and pressure are $\rho_k = -g'/\varphi^2$ and $p_k = g/\varphi^2 y$, where prime indicates derivative with respect to y . The equation-of-state is

$$w_k \equiv p_k/\rho_k = -g/yyg' \quad (3)$$

and the sound speed is

$$c_s^2 = \frac{p'_k}{\rho'_k} = \frac{g - g'y}{g''y^2}. \quad (4)$$

In order to have a sensible, stable theory, we require $\rho_k > 0$ and $c_s^2 > 0$. These conditions are satisfied if $g' < 0$ and $g'' > 0$ in the region where p'_k is positive. Therefore, a general, convex, decreasing function $g(y)$, such as shown in Fig. 1, satisfies these necessary conditions. Using the Friedmann equation: $H^2 = \rho_{tot} = \rho_k + \rho_m$, where ρ_m is the energy density of matter (radiation and dust), and the energy conservation equations, $\dot{\rho}_i = -3\rho_i(1+w_k)$ for the k -essence ($i \equiv k$) and matter ($i \equiv m$) components, we obtain the following equations of motion

$$\dot{y} = \frac{3}{2} \frac{(w_k(y) - 1)}{r'(y)} \left[r(y) - \sqrt{\frac{\rho_k}{\rho_{tot}}} \right] \quad (5)$$

$$\left(\frac{\rho_k}{\rho_{tot}} \right)' = 3 \frac{\rho_k}{\rho_{tot}} \left(1 - \frac{\rho_k}{\rho_{tot}} \right) (w_m - w_k(y)), \quad (6)$$

where

$$r(y) \equiv \left(-\frac{9}{8}g' \right)^{1/2} y(1+w_k) = \frac{3}{2\sqrt{2}} \frac{(g-g'y)}{\sqrt{-g'}}, \quad (7)$$

and dot denotes derivative with respect to $N \equiv \ln a$. These are the master equations describing the dynamics of k -essence models. Once some general properties of $g(y)$ are specified, the attractor behavior described in the introduction follows from these coupled equations.

Dynamics. We are seeking a tracker solution $y(N)$ in which the k -essence equation-of-state is constant and exactly equal to the background equation-of-state, $w_k(y(N)) = w_m$, and the ratio ρ_k/ρ_{tot} is fixed. Generically, this requires $y(N)$ be a constant y_{tr} and therefore $\rho_k/\rho_{tot} = r^2(y_{tr})$. The last condition can only be satisfied if $r(y_{tr})$ is less than unity. Hence, given a convex function such as shown in Fig. 1, we can first identify those ranges of y where $r(y)$ is greater than unity or less than unity. In ranges where $r(y)$ is less than unity, we can seek values of y where w_k in Eq. (3) is equal to w_m , the equation-of-state of the matter or radiation. The value of y_{tr} changes depending on the epoch and w_m . These are the attractor solutions. In ranges where $r(y)$ exceeds unity, there are no attractor solutions.

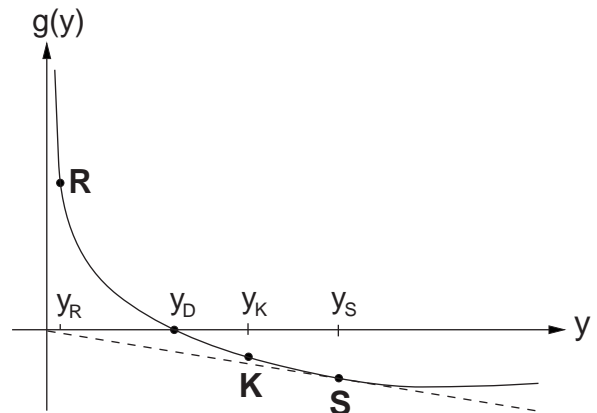


FIG. 1. A plot of $g(y)$ vs. y (see Eq. (3) for definition) indicating the points discussed in the text. **R** corresponds to the attractor solution during the radiation-dominated epoch; **S** is the de Sitter attractor at the onset of matter-domination; and **K** is the attractor as k -essence dominates. For our range of $g(y)$, there is no dust-like attractor solution at $y = y_D$.

Although there is no dust attractor, it is quite possible for there to be a radiation attractor. The radiation attractor corresponds to positive pressure, so it can be located only at $y < y_D$. Hence, we must have $g(y)$ such that $r(y)$ is less than unity for some range at $y < y_D$ which includes some point y_R where $w_k(y_R) = 1/3$. During the radiation-dominated epoch, the ratio of k -essence to the total density remains fixed on this attractor and equal to $(\rho_k/\rho_{tot}) = r^2(y_R)$.

In Fig. 1, the pressure $p_k = g/\varphi^2 y$ is positive above the

y -axis, and negative below the y -axis. The dust equation-of-state $p_k = 0$ can only be obtained at $y = y_D$ where $g(y)$ goes through zero. However, this point can be an attractor, only if, the second condition, $r(y_D) < 1$, is satisfied. If it so happens that $r(y_D) > 1$, then there is no dust attractor in the matter-dominated epoch. This is precisely what we want for our scenario, and this is possible for a broad class of functions g .

If g possesses a radiation attractor but no dust attractor, what happens at dust-radiation equality? To answer this question let us study the solutions of the master equations Eqs. (5-7) in two limiting cases, when the energy density of k -essence is either much smaller or much greater than the matter energy density. If $\rho_k/\rho_m \ll 1$, one can neglect the last term in the equation (5) and it is obvious that $y(N) \simeq y_S$, where y_S satisfies the equation $r(y_S) = 0$, is an approximate solution of the equations of motion. The point S satisfies $g(y_S) = g'(y_S)y_S$, so the tangent of g at y_S passes through the origin, as shown in Fig. 1. Since $r \propto (1 + w_k)$, the equation of state of k -essence at y_S corresponds to $w_k(y_S) \approx -1$; we call this solution the de Sitter attractor and denote it by **S** in Fig. 1. From Fig. 1, it is clear that y_S nearly always exists for convex decreasing functions g . We stress that $y(N) \approx y_S$ is an approximate solution to the equations-of-motion only when matter strongly dominates over k -essence. So, if ρ_k during the radiation dominated epoch is significantly less than the radiation density which it tracks, which is both typical and required to satisfy nucleosynthesis constraints, then k -essence proceeds to the de Sitter attractor immediately after dust-radiation equality.¹⁰

As the transition to dust-domination occurs, ρ_k first drops to a small, fixed value, as can be simply understood. Suppose that $(\rho_k/\rho_{tot})_R = r^2(y_R) = \alpha < 10^{-2}$ during the radiation dominated epoch, where the bound is set by nucleosynthesis constraints. From the equation of state, Eq. (3), we have the relation: $g(y_R) = -g'(y_R)y_R/3$. The condition $r(y_D) \geq 1$ is required in order to have no dust attractor solution. Combining these relations, we obtain:

$$\frac{g'_R y_R^2}{g'_D y_D^2} \leq \frac{9}{16} \alpha < 10^{-2} \quad (8)$$

On the other hand, it is apparent from Fig. 1 that $-g'_R > -g'_D$, so $y_R \ll y_D$ if $\alpha \ll 1$. In particular, the tangent at y_D falls below $g(y_R)$, so $g'_D(y_R - y_D) \approx -y_D g'_D \leq g(y_R) = -y_R g'_R/3$. Using this relation, we obtain

$$\frac{y_R}{y_D} \leq \frac{3}{16} \alpha < 2 \cdot 10^{-3} \quad \text{and} \quad \frac{g'_D}{g'_R} \leq \frac{\alpha}{16} < 7 \cdot 10^{-4}. \quad (9)$$

Since $\rho_k = -g'/\varphi^2$ and $|g'(y_S)| \leq |g'(y_D)|$ we conclude that after radiation domination, when the k -field reaches the vicinity of the **S**-attractor, the ratio of energy densities in k -essence and dust does not exceed

$(\rho_k/\rho_{tot})_R \times g'_D/g'_R$; that is, $\rho_k/\rho_{dust} < \alpha^2/16 < 7 \cdot 10^{-6}$. Hence, provided $(\rho_k/\rho_{tot})_R \leq 10^{-2}$ at dust-radiation equality, the k -essence field loses energy density on its way to the **S**-attractor down to a value below 7×10^{-6} .

By definition, the **S**-attractor is one in which $w \approx -1$ and the energy density is nearly constant. Hence, once ρ_k has reached its small but non-zero value, it freezes. In the further evolution of the universe, the matter density decreases, but the k -essence energy density remains constant, eventually overtaking the matter density of the universe. Note that, as ρ_k approaches ρ_m , the condition $\rho_k/\rho_m \ll 1$ is necessarily violated and a new attractor solution is found for the case where k -essence itself dominates the background energy density. This attractor is denoted **K** in Fig. 1.

To prove that the **K**-attractor exists, we consider the master equations, Eqs. (5-7), in the limit where $\rho_k/\rho_{tot} \rightarrow 1$. If y_K satisfies the equation $r(y_K) = 1$, then $y(N) \simeq y_K$ is an approximate solution of the equations of motion. When dust is not a tracker, there always exists a unique attractor y_K in the interval $y_D < y < y_S$.¹⁰ To prove this, note that, within this interval, the function $r(y)$ has a negative derivative. Recall that $r(y_S) = 0$ (definition of **S**-attractor) and $r(y_D) > 1$ (to avoid a dust attractor). Since $r(y)$ is a monotonically decreasing, continuous function, there exists a unique point y_K ($y_D < y_K < y_S$) where $r(y)$ becomes equal to unity. At $y > y_D$ the pressure of k -essence is negative. Hence, generically the **K** attractor, located near y_K , describes a universe dominated by a negative pressure component which induces power-law cosmic acceleration. As acceleration proceeds, ρ_k increasingly dominates and $y \rightarrow y_K$.

Following along using Fig. 1, the dynamics can be summarized as follows: k -Essence is attracted to $y = y_R$ during the radiation dominated epoch; at matter-domination, the energy density drops sharply as k -essence skips past $y = y_D$, because there is no dust attractor, and heads towards $y \approx y_S$. The energy density ρ_k freezes and, after a period, overtakes the matter density. As it does so, y relaxes towards y_K . In this scenario, our current universe would be making the transition from y_S to y_K . All this occurs for generic $g(y)$ satisfying broad conditions on its first and second derivatives. If the ratio of ρ_k to the radiation density is near the maximum allowed by nucleosynthesis (roughly equipartition initial conditions), the scenario predicts that the ρ_k dominates by the present epoch.

Numerical results. We have verified these analytic predictions numerically for a wide class of $g(y)$. As a strategy, we look for forms which are roughly linear,

$$g(y) \approx -\frac{1}{3}g'_R y_R + g'_R(y - y_R) + O\left((y - y_R)^2\right) \quad (10)$$

in the vicinity of radiation attractor **R** and parabolic

$$g(y) \approx \frac{g'_D y_D}{y_D^2 - y_S^2} (y - y_D) \left(y - \frac{y_S^2}{y_D} \right) + \dots \quad (11)$$

in the region $y_D \leq y \leq y_S$. One can easily check, that the points y_R , y_D and y_S here are, by construction, the places where the corresponding attractors are located and g'_R , g'_D are the derivatives of g at the appropriate points. The results are not sensitive to the precise form of g that interpolates between these regimes. The main constraints are that the attractor solution have a small ratio of ρ_k/ρ_{tot} during the radiation epoch and that there is no dust attractor.

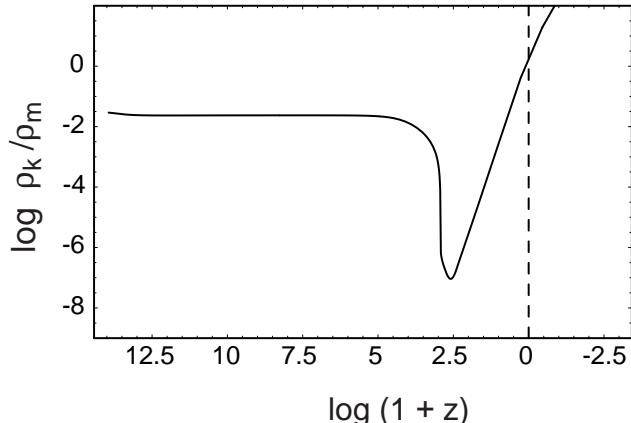


FIG. 2. The ratio of k -essence energy density, ρ_k , to the density in radiation and matter, ρ_m , vs. red shift. At the present epoch (dashed line), $\Omega_k \approx 0.7$.

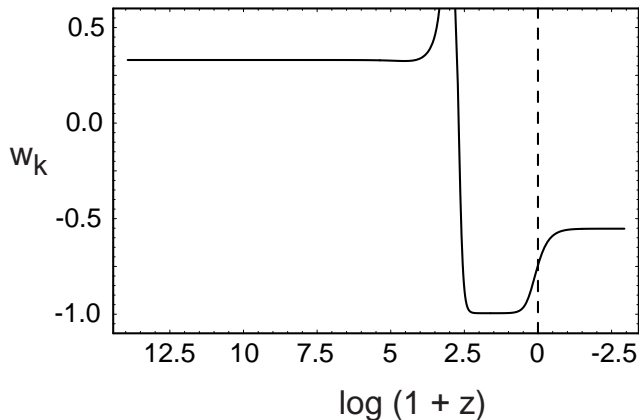


FIG. 3. The k -essence equation-of-state vs. red shift. The three attractors in the radiation-, matter-, and k -essence-dominated epochs are evident. At the present epoch, $w_k \approx -0.77$.

For illustrative purposes, we have used these principles to obtain a sample $g(y)$ and have transformed it to a form for $\tilde{p}(X)$ in the action, Eq. (1):

$$\tilde{p}_k(X) = -2.01 + 2\sqrt{1+X} + 0.03(aX)^3 - (bX)^4, \quad (12)$$

where $a = 10^{-5}$ and $b = 10^{-6}$. For small values of X , after a field redefinition, this Lagrangian density reduces to one equivalent to a canonical scalar field with exponential potential (curiously, a tracker model⁶). The distinctive dynamical attractor in our models relies deviations from

linearity at large X .

The results of a numerical integration are presented in Figs. 2 and 3. We see that k -essence tracks the radiation ($w_k \approx 1/3$) during the radiation-dominated epoch. Then, at the onset of matter-domination, w_k starts to change and the energy density of k -essence suddenly drops by several orders of magnitude becoming of the order of 10^{-7} times the critical density at red shifts about $z \simeq 1000$ as the **S**-attractor is approached and $w \rightarrow -1$. At about red shift $z \sim 3 - 5$, ρ_k becomes non-negligible and w_k starts to increase, ultimately reaching $w_k \simeq -0.77$ at $z = 0$. The ratio of the k -essence energy density to the critical density today is $\Omega_k \approx 0.74$. In the future, w_k in this model has to approach the value -0.55 , corresponding to the **K**-attractor solution, and the universe will enter the period of power law k -inflation.

Summary. In this paper, we have presented a scenario in which cosmic acceleration occurs late in the history of the universe due to an inevitable sequence of events caused by attractor dynamics. We view the present work as a demonstration of principle; hence, we have emphasized general conditions and an analytic understanding of the scenario. The specific example illustrated in this paper is admittedly complex, composed to illustrate the concept, but we know of no fine-tuning or other requirement that poses a barrier to finding simpler and better-motivated forms. By changes of variables and other simple techniques, one can quickly enlarge the class of actions considered here, which may suggest other types of attractive models.

A prediction of k -essence models that distinguishes them from models based on tracker potentials⁷ is that w_k is in the process of increasing today from -1 towards its asymptotic value at the **K** attractor, whereas, for trackers, w_k is undergoing a transition from $w \approx 0$ towards $w = -1$. A consequence is that the effective value of w_k for k -essence – that is, the Ω_k weighted average of w_k between the present and $z = 1$ – can be significantly lower than for the tracker potential case, which is bounded below by $w_{eff} \approx -0.75$.⁷ In the numerical example above, the effective $w_{eff} = -0.84$, for example. The current supernovae data suggest a lower value of w_k more consistent with k -essence.¹¹ Of course, the k -essence range for w_k is more difficult to distinguish from a cosmological constant ($w = -1$).

In future work,¹⁰ we discuss model-building using further examples and generalizations. We also explore interesting variations of the dynamical scenario with different kinds of attractors, including some which can lead to different long-term future outcomes, such as a return to a pressureless, unaccelerated expansion in the long-term future.

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