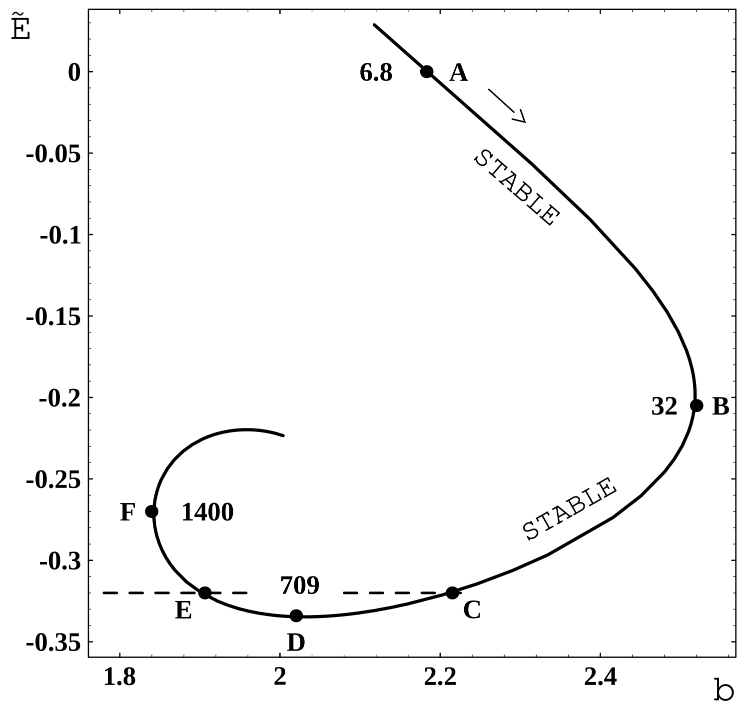
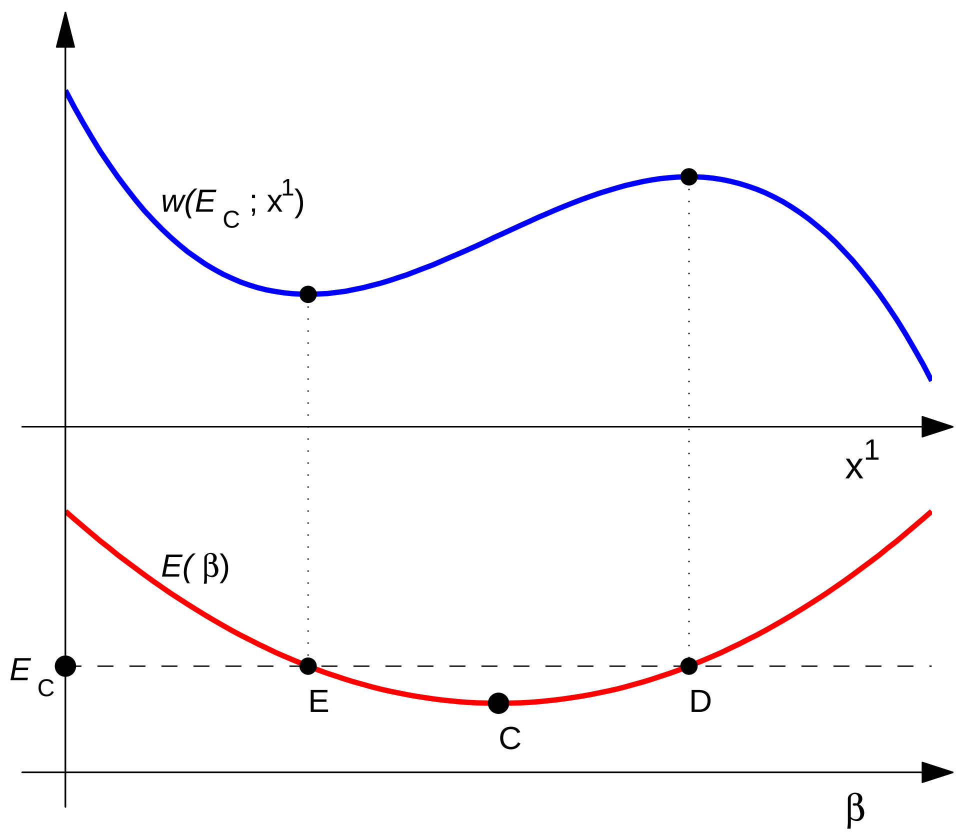


$\tilde{E}$
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-0.35
1.8
2
2.2
2.4
b
6.8
A
STABLE
32
B
STABLE
F
1400
709
E
C
D



w(E C ; x1)
x1
E(β)
E C
E
C
D
β

FLUCTUATIONS IN ISOTHERMAL SPHERES

Joseph Katz†

The Racah Institute of Physics, 91904 Jerusalem, Israel

and

Isao Okamoto‡

Division of Theoretical Astrophysics, National Astronomical Observatory,

Mitaka, Tokyo 181-8588, Japan

Sunday, December 19, 1999

ABSTRACT

Isolated isothermal spheres of N gravitationally interacting points with equal mass are believed to be stable when density contrasts do not exceed 709. That stability limit does, however, not take into consideration fluctuations of temperature near the onset of instability. These are important when N is finite.

Here we correlate *global mean quadratic temperature fluctuations* with onset of instability. We show that such fluctuations trigger instability when the density contrast reaches a value near $709 \cdot \exp(-3.3N^{-1/3})$. These lower values of limiting density contrasts are significantly smaller than 709 when N is not very big and this suggests (i) that numerical calculations with small N may not reflect correctly the onset of core collapse in clusters with big N and (ii) that a greater number of globular clusters than is normally believed may already be in an advanced stage of core collapse because most of observed globular clusters whose parameters fit quasi-isothermal configurations are close to marginal stability.

Keyword: Globular clusters - Instabilities - Gravothermal catastrophe

† E-mail: jkatz@vms.huji.ac.il
‡ E-mail: okamoto@yso.mtk.nao.ac.jp

1. Introduction

Bounded isothermal spheres of pointlike stars of equal mass interacting gravitationally are unrealistic but instructive models to learn about the early stages of gravitational collapse in stellar clusters. Their great advantages are mathematical simplicity and the fact that other factors of evolution than the "gravothermal catastrophe" like escape of stars to infinity, equipartition of energy among stars of different masses or formation of hard binaries are switched off by the model. Isothermal spheres have been studied in statistical mechanics (Antonov 1962, Horwitz and Katz 1978), in thermodynamics (Lynden-Bell and Wood 1968, Katz 1978) and in numerical experiments (Hachisu et al. 1978, Inagaki 1980). The various studies came up with consistent results and confirmed the following picture of slow evolution of isothermal spheres through different stages of global quasi-static isothermal equilibrium with increasing entropy and density contrasts. It is nicely described In *Galactic Dynamics* by Binney and Tremaine (1987).

Consider a bounded isolated isothermal sphere of radius R with N point particles of mass m attracting each other gravitationally. The total energy E is a function of the inverse global temperature T shown in figure 1. The winding-in of pairs of thermodynamic equilibrium quantities like E and $\beta = 1/kT$ as shown in figure 1 is also a property of polytropic gas spheres (Chandrasekhar 1934), cold white dwarfs (Harrison et al. 1965), hot isentropic stars (Thorne 1966) and relativistic gases with P/ρ =constant (Chandrasekhar 1972). Each point on the $E(\beta)$ line, like A, B, C, etc. is associated with an equilibrium configuration. The branch of thermodynamically stable equilibria indicated on figure 1 ends at point D where the density contrast is $\simeq 709$ and the energy is also the smallest for equilibrium configurations (Antonov 1962; Lynden-Bell and Wood 1968). All points of density contrasts bigger than $\simeq 709$ represent unstable configurations (Katz 1978).

Evolution is down the slope in figure 1 say from point A towards point D, towards higher entropy and higher density contrasts. What happens to a slowly evolving isothermal sphere when it becomes unstable? Various interesting thought experiments have been performed and vividly described in Lynden-Bell and Wood and in Binney and Tremaine. The system becomes unstable when it reaches point D and a "gravothermal catastrophe" develops. Notice that from B to D the entropy is a local maximum, while the heat capacity $C_V < 0$ since the slope is positive. From D to F the entropy extremum is a minimum but $C_V > 0$.

Notice also that those thought experiments are based on the assumption that the system reaches a new local entropy minimum in a *finite* time. In fact the system spends a limited time at each point of the linear series in (quasi-)equilibrium. It is thus assumed that during the evolution, fluctuations, and in particular mean global temperature fluctuations (which we shall precisely define in a moment)are small, i.e. that the probability of a big fluctuation is negligibly small. This assumption is certainly not correct near point D where fluctuations of global temperature become infinite (Horwitz and Katz 1977 referred to below as HK77). One of the main objects of the present work is to try to assess how close to point D the evolution can proceed before the system becomes unstable as a result of big fluctuations. If the system spends an unlimited time in a stable equilibrium with energy less negative than E_D, it will eventually evolve to a non isothermal state because big fluctuations of temperature have a chance to develop and bring the system to a state of entropy which would increased indefinitely (see below) .

Now a few words about temperature fluctuations in isothermal spheres which are at the center of our considerations. The global temperature T is a global equilibrium parameter but it has also a local meaning (see Lynden-Bell and Wood again); T is related to the mean square velocity $\overline{v^2}$ by the same relation as in an ideal gas $\frac{3}{2}kT = \frac{1}{2}m\overline{v^2}$. The mean square velocity is that of the stars which happen to be near a point at a given moment. At later times stars left the vicinity and other stars came around. A local temperature can also be defined if the cluster is not in global thermal equilibrium but such a local temperature would differ from point to point and from time to time. A global equilibium temperature does then not exist. The proper method to tackle the problem is then in kinetic theory not statistical mechanics.

However, if an out of equilibrium entropy can be defined in statistical physics for a system with a given energy E [see HK 1977], say w, than a global out of equilibrium temperature $\tilde{T}$ can be defined as well by $\partial w/\partial E = 1/k\tilde{T} = \tilde{\beta}$. That $\tilde{T}$ will clearly be equal to T at the point where w is equal to its maximum S and fluctuations of T can be calculated like in classical statistical physics (see Landau and Lifshitz 1985).Thus we may state without proof (but we shall give the prove below) that the mean quadratic fluctuations of the global temperature $\overline{(\Delta T)^2}/T^2$ (from now on temperature will always mean global temperature) is, like in classical thermodynamics, of order $1/N$. Thus, if N is big, $\overline{(\Delta T)^2}/T^2$ is small and fluctuations may be expected to remain very small along the linear series up to or rather very near the stability limit at point D. In gravitational thermodynamics N is much smaller than in atomic physics; $N \sim 10^{10}$ in galactic nuclei, $10^{5\sim 6}$ in globular clusters and $10^{3\sim 4}$ in present day numerical simulations (Meylan and Heggie 1997). Fluctuations in these different cases are wildly different and some have measurable effects as we shall see.

In this connection, it is worthwhile recalling Monaghan's (1978) application of the theory of *hydrodynamic* fluctuations to a self-gravitating gas. He showed that density fluctuations become large before the point of ordinary stability is reached. Statistical mechanics confirms Monaghan's finding.

In the present work we apply our own version of fluctuation theory in N body systems in statistical physics with gravitational forces (Parentani et al. 1995). The theory, first sketched in Okamoto et al. (1995), has been applied to evolutionary sequences of quasi-static equilibrium configurations of self-gravitating radiation in the presence of a black hole. Such systems exhibit a first order phase transition and fluctuation theory plays a useful role in explaining when superheated black holes or superheated radiation become unstable.

Fluctuation theory plays a very different role in an evolutionary sequence of quasi-static isothermal spheres, for instance. These are in a state of *local* entropy maximum and the maximum is unique.The global entropy maximum is in fact infinite as shown by Antonov. The non-equilibrium entropy has thus a local maximum and a local minimum. It is qualitatively like the dashed curve in figure 2. Large enough fluctuations in equilibrium configurations near marginal stability at point D may bring the system along the dashed line at a point where the entropy can grow towards the global maximum, at infinity. Our aim is to find a limit where *global temperature fluctuations* become large enough to put the system out of equilibrium. Such a limit exists if the mean square amplitude of temperature fluctuations grows exponentially. This is the case in general and as we shall see it is certainly so in isothermal spheres.

Our theory of fluctuations applies not only to isothermal spheres but to any thermodynamic equilibrium with long range forces which has a *unique* but local minimum of entropy and satisfy some additional conditions which we describe in detail below.

In section 2 we review the theory of fluctuations that has been presented in Parentani et al. (1995) and in section 3 we apply the theory to isothermal spheres. As we shall see, the stability limit induced by fluctuations depends exponentially on a power of N and the density contrast at which instability develops may be significantly lower than 709 for "small" N. The significance of these calculations is analyzed in section 4.

2. Fluctuation theory with long range forces

(i) *Steepest descent expression for the entropy of a finite isolated system*

Consider a system of gravitationally interacting particles in a finite volume V, with total mass M and fixed energy E. The total Gibbs entropy is $S = k \ln \Omega$, where Ω is the volume of available phase space; it can be evaluated by a steepest descent method, the saddle point value giving the mean field entropy (HK77). Analysis of the quadratic fluctuations yields thermodynamical stability conditions and the next order term in calculating the entropy beyond the mean field approximation (see for instance Horwitz 1971). For a recent review on the subject of statistical mechanics of gravitating systems, see Padmanabhan (1990). In HK77's scheme Ω is given by a functional integral but one needs only consider continuous functions (Ginibre 1971) and, under general conditions (Courant and Hilbert 1953) which are usually met in physical problems, the continuous functions can be expanded in an eigenfunction series which converges absolutely and uniformly in the whole domain of existence. Under such circumstances, the functional integral can be replaced by discrete integrations in terms of an infinite but denumerable set of variables say x^i. Ω assumes thus a form like this

$$\Omega(V, M, E) = \frac{1}{A} \int_{L \to \infty} e^{w(V,M,E;x^i)} dx^1 dx^2 ... dx^L. \tag{2.1}$$

In the steepest descent evaluation, w is extremized,

$$\frac{\partial w}{\partial x^i} = 0. \tag{2.2}$$

w is then expanded in powers of $(x^i - X^i) = \Delta x^i$ where $X^i(V, M, E)$ is a solution of (2.2). We shall assume that the x^i's are chosen in such a way that at the point of extremum indicated by a sub-index e, the matrix of second derivatives of w is diagonal; thus

$$\left(\frac{\partial^2 w}{\partial x^i \partial x^j}\right)_e = -\delta_{ij}\lambda_j. \tag{2.3}$$

The minus sign is for convenience‡. We can always denumerate the λ_i's in such a way that $\lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots$ . . Expansion of w in powers of Δx^i may thus be written as follows (if A is properly chosen)

$$w = \frac{1}{k} S(V, M, E) - \frac{1}{2} \sum_{i=1}^{\infty} \lambda_i (\Delta x^i)^2 + O_3. \tag{2.4}$$

The λ_i's are known as Poincaré's coefficients of stability following Poincaré's (1885) seminal analysis of stability. In writing (2.4) we have nowhere assumed that we deal with pure or not pure gravitational interactions. (2.4) is therefore quite general and applies to non-extensive systems.

Equation (2.4) cries for qualifying restrictions:

(1) The summation like the integral is in general not convergent (see HK77 for details). What makes them converge is a short distance cutoff in the gravitational interaction. The effect of the cutoff is to modify the higher values in the spectrum λ_i which play no role in stability and fluctuation considerations (see below). The effect of short distance cutoffs on stability was analyzed in Aronson and Hansen (1972).

(2) w must admit second order derivatives with respect to x^i. We shall assume that w is twice differentiable also with respect to V, M and E in the vicinity of the maximum otherwise thermodynamic coefficients would not exist. In what follows we shall concentrate

‡ By assuming that variables may be chosen such that (2.3) holds, we bypass of course the main technical problem of stability theory which is *to find* the λ_i's...

on the E dependence of equilibrium configurations and take V and M fixed. Therefore we write $S(E)$, $\beta(E)$ etc...

(3) We shall now regard the smallest λ_i's to be different from each other. This assumption is more than we shall need and is made for convenience; it simplifies explanations. In the paper by Horwitz and Katz of (1978) it was shown that in stable isothermal spheres $\lambda_1 > 0$ and that at density contrasts near but greater than 709, $\lambda_1 < 0$ but $\lambda_2 > 0$. Another case in which a few of the smallest λ_i have been shown to be different is in models for stars of cold catalyzed matter; this includes cold neutron stars (see in Harrison et al. 1965). At each turning point of the linear series $-\mu(N)$ where μ is the chemical potential§ $\mu = dE/dN$ there is a change of stability at each of the three turning points calculated and each corresponds to *one* and only one square eigenfrequency (ω^2) of dynamical perturbations changing sign. Degenerate spectra of λ_i's are in fact rare in astrophysical applications though not unknown. Degeneracy corresponds to bifurcations of equilibria; see for instance the many bifurcations in liquid ellipsoids linear series (Chandrasekhar 1969). They usually correspond to an excess of symmetry which can be lifted with small arbitrary perturbations of the potential energy [see Thompson (1979), for Maclaurin ellipsoids in particular see Katz (1979)]. There are a few exceptions of physical interest in which bifurcations cannot be removed (see Arnold 1986). When all we are interested in is the passage of stability to instability it is enough for λ_1 to be different from λ_2. Some interesting results about the onset of instability may also be obtained without assuming that the spectrum of λ_i's is non degenerate (Sorkin 1981).

(ii) *The standard limit of stability*

With (2.4), we now define a "temperature function" $\tilde{T}(E; x^i)$ or rather the more

§ The curve does not appear in Harrison et al.; it is represented schematically in Katz (1981) and is based on Harrison et al.'s tables.

convenient inverse temperature or $\tilde{\beta} = 1/k\tilde{T}$ which has the same units as E^{-1}:

$$\tilde{\beta} = \frac{\partial w}{\partial E} = \frac{1}{k}\frac{dS}{dE} + \sum_i \lambda_i \frac{dX^i}{dE}\Delta x^i + O_2 \equiv \beta + \sum_i \lambda_i \frac{dX^i}{dE}\Delta x^i + O_2. \tag{2.5}$$

The extremal value of $\tilde{\beta}$ is $\beta = 1/kT$, T is *the* global temperature of equilibrium. For small values of Δx^i's

$$\Delta\tilde{\beta} = \tilde{\beta} - \beta \simeq \sum_i \lambda_i \frac{dX^i}{dE}\Delta x^i. \tag{2.6}$$

These formal fluctuations are related to fluctuations of temperature in a small part of the system in the usual sense when the rest of the system can be treated as a reservoir [see Parentani et al. (1995) for a detailed discussion of this point]. That, however, is not normally the case in gravitating systems.

Since w is twice differentiable, derivatives of $\tilde{\beta}$ with respect to x^i and E exist; the derivative of $\tilde{\beta}$ defined in (2.5) with respect to a certain x^i, keeping E constant and all other $x^i = X^i$ is readily derived from (2.6)

$$\frac{\partial\tilde{\beta}}{\partial x^i} = \lambda_i \frac{dX^i}{dE} = \left(\frac{\partial\tilde{\beta}}{\partial x^i}\right)_e \qquad \text{[no summation on } i\text{]} \tag{2.7}$$

and the derivative of $\tilde{\beta}$ with respect to E, keeping *all* $x^i = X^i$ is obtained from (2.5)

$$\frac{\partial\tilde{\beta}}{\partial E} = \frac{d\beta}{dE} - \sum_i \lambda_i \left(\frac{dX^i}{dE}\right)^2 \tag{2.8}$$

Now replace dX^i/dE in (2.8) in terms of $(\partial\tilde{\beta}/\partial x^i)_e$ as given by (2.7); then $\partial\tilde{\beta}/\partial E$ can be written

$$\frac{\partial\tilde{\beta}}{\partial E} = \frac{d\beta}{dE} - \sum_i \frac{1}{\lambda_i}\left(\frac{\partial\tilde{\beta}}{\partial x^i}\right)_e^2. \tag{2.9}$$

We now turn our attention to the *stability limit.* The entropy is a maximum and the $X^i(E)$'s represent a stable thermodynamic mean field configuration when all the Poincaré coefficients are positive, i.e. when $\lambda_1 > 0$. Consider thus a linear series $\beta(E)$ of stable equilibrium configurations. Any change of stability or change in the degree of instability is characterized by a change of sign in one of the λ_i's. Thus when a series of *stable* equilibria becomes unstable, λ_1 becomes small and changes sign. Near instability, the sum in (2.9) is dominated by the $1/\lambda_1$ term *assuming* $\left(\partial\tilde{\beta}/\partial x^1\right)_e \neq 0$. If $\left(\partial\tilde{\beta}/\partial x^1\right)_e = 0$ we can get rid of the coincidence of zeros by adding a small perturbation $\delta w(x^i)$ to w, for instance $\epsilon x^1 E$ with ϵ small enough. Then $\left(\partial\tilde{\beta}/\partial x^1\right)_e = \epsilon$. This is the normal way bifurcations are removed in ordinary stability analysis (Hunt 1977)‡. Small perturbations will generally get rid of such degeneracies (but not always). We assume that $\left(\partial\tilde{\beta}/\partial x^1\right)_e \neq 0$ or, if necessary, is non zero with a small perturbation of w . If the sum in (2.9) is indeed dominated by the first term which diverges when $\lambda_1 = 0$, and if both $\partial\tilde{\beta}/\partial E$ and $(\partial\tilde{\beta}/\partial x^i)_e$ are finite, then for $\lambda_1 \to 0$, $d\beta/dE$ must diverge and it follows from (2.9) that

$$\frac{d\beta}{dE} \simeq \frac{1}{\lambda_1}\left(\frac{\partial\tilde{\beta}}{\partial x^1}\right)_e^2 \tag{2.10a}$$

or that

$$\lambda_1 \simeq \frac{dE}{d\beta}\left(\frac{\partial\tilde{\beta}}{\partial x^1}\right)_e^2 . \tag{2.10b}$$

This is Poincaré's turning point property of $E(\beta)$: the energy is maximum or minimum when a change of stability takes place. With (2.10) we can calculate the heat capacity C_V in stable configurations *near instability*:

$$C_V = \frac{dE}{dT} \simeq -k\lambda_1\beta^2\left(\frac{\partial\tilde{\beta}}{\partial x^1}\right)_e^{-2} < 0. \tag{2.11}$$

Thus $C_V < 0$ for a stable equilibrium ($\lambda_1 > 0$) and $C_V > 0$ for an unstable one. This property is true near a turning point. It is interesting to remind the reader that $C_V < 0$

‡ Notice however that we do not know what the x^i's are in general to begin with (see footnote p.7). For details on this point see Katz (1980).

in stable systems is a property of *finite* non-extensive microcanonical ensembles in thermal equilibrium and not of gravitating systems only. Notice that a microcanonical ensemble is here stable where a canonical ensembles is unstable ($C_V < 0$).

(iii) *Mean quadratic fluctuations of temperature and their probability*

Consider now a stable equilibrium configuration near a point of instability where $\lambda_1(>0) \to 0$ and $\lambda_i > 0$ for $i \geq 2$. All terms of the quadratic sum in (2.4) except the one with λ_1 may be integrated out in (2.1); they are by definition strongly stable if they have strongly negative exponentials which is generally the case when $N >> 1$. Thus, near instability, we are interested in the terms of w that are not (strongly) stable, i.e. in

$$w \simeq \frac{S(E)}{k} - \tfrac{1}{2}\lambda_1(\Delta x^1)^2 \tag{2.12}$$

with a slightly renormalized factor A in (2.1). We may replace λ_1 in (2.12) by its expression given in (2.10b). Near instability, *with all other x^i's integrated out*, the (new) temperature function $\tilde{\beta}$ calculated from $w(E;x^1)$ reduces to

$$\tilde{\beta} = \beta + \Delta\tilde{\beta} \simeq \beta + \left(\frac{\partial\tilde{\beta}}{\partial x^1}\right)_e \Delta x^1. \tag{2.13}$$

If we now replace Δx^1 in (2.12) in terms of $\Delta\tilde{\beta}$ given by (2.13) we obtain

$$w \simeq \frac{S(E)}{k} - \tfrac{1}{2}\frac{dE}{d\beta}(\Delta\tilde{\beta})^2. \tag{2.14}$$

Equation (2.14) is of the same form as equation (110.3) of Landau and Lifshitz's (1985) chapter on (non-quantum) fluctuation theory. We can now use the standard arguments of fluctuation theory and say that the probability dW for a fluctuation of $\tilde{\beta}$ in the range, $\beta + \Delta\tilde{\beta}$ and $\beta + \Delta\tilde{\beta} + d\tilde{\beta}$, is proportional to $\exp(w - S)$; thus, the properly normalized dW is given by

$$dW = \sqrt{\frac{1}{2\pi}\frac{dE}{d\beta}}\exp\left[-\tfrac{1}{2}\frac{dE}{d\beta}(\Delta\tilde{\beta})^2\right]d\tilde{\beta} \tag{2.15}$$

From (2.15) it follows that the mean quadratic fluctuations of the global temperature, as a function of the equilibrium parameters, is given by

$$\overline{(\Delta\tilde{\beta})^2} = \frac{d\beta}{dE} = -\frac{k\beta^2}{C_V}. \tag{2.16}$$

The analogy with classical fluctuation theory is flagrant except for the sign of C_V. Notice that our analysis does not use small subsystems treating the rest of the system as a heat bath. Small subsystems have no well defined energies independently of the big system when there are long range forces. The intriguing difference of signs of C_V and comparison with familiar results is discussed in detail in Parentani et al. (1995). Before getting any further it is good to remember that (2.15) and (2.16) are valid only close to the point of marginal stability, point D, where $dE/d\beta = 0$, $C_V = 0$ and $\overline{\Delta\tilde{\beta}^2} = \infty$. The derivative of $E(\beta)$ near point D may be approximated by the lowest order in a Taylor expansion:

$$\frac{dE}{d\beta} \simeq \left(\frac{d^2E}{d\beta^2}\right)_D (\beta - \beta_D). \tag{2.17}$$

and with (2.17) the exponential factor in dW can be written

$$\exp\left[-\tfrac{1}{2}\left(\beta^3\frac{d^2E}{d\beta^2}\right)_D \left(\frac{\beta-\beta_D}{\beta_D}\right)\left(\frac{\tilde{\beta}-\beta}{\beta_D}\right)^2\right]. \tag{2.18}$$

There appears here two dimensionless quantities which deserve a special symbol, say

$$\Delta U = \frac{\beta - \beta_D}{\beta_D} \quad \text{and} \quad \Delta\tilde{V} = \frac{\tilde{\beta}-\beta}{\beta_D} \tag{2.19}$$

ΔU depends on equilibrium parameters only, while $\Delta\tilde{V}$ is a fluctuation in units of β_D. The term in (2.18) that contains second order derivatives of E is non-dimensional. The virial theorem for particles in a container tells us that the total kinetic energy and the total potential energy are of the same order of magnitude. Thus the total energy E is itself of the same order of magnitude than the total kinetic energy. On the other hand $kT = 1/\beta$

is of the order of magnitude of the kinetic energy of one particle. Therefore the first factor in 2.18, which "goes like βE" must be of order N

$$\left(\beta^3 \frac{d^2 E}{d\beta^2}\right)_D = \Gamma_D N. \tag{2.20}$$

where the number Γ_{D} must be of order $1 << N$. In terms of ΔU, $\Delta\tilde{V}$ and Γ_D, the probability dW defined in (2.15) can be written

$$dW = \sqrt{\frac{\Gamma_D}{2\pi} N\Delta U} \exp\left[-\tfrac{1}{2}\Gamma_D N\Delta U(\Delta\tilde{V})^2\right] d\tilde{V}. \tag{2.21}$$

The expression for the mean quadratic fluctuations (2.16) can be rewritten in terms of U's and $\tilde{V}$'s in the following form

$$\overline{(\Delta\tilde{V})^2} = (\Gamma_D N\Delta U)^{-1}. \tag{2.22}$$

The rate at which a small fluctuation of temperature disappears is, by virtue of the fluctuation-dissipation theorem, the inverse of the time it takes to return to equilibrium, $1/t_{rel}$. This relaxation time is proportional to some power of N (see for instance Binney and Tremaine about stellar systems). But dW is a power of $e^{-N\Delta U}$ and if ΔU is not very small (say $\Delta U \sim 0.1$), the exponent becomes extremely small for large N(say $N > 100$). The probability per unit time dW/dt will thus also be dominated by the negative exponential of dW.

(v) *The destabilizing effect of fluctuations near a local maximum of entropy*

Let us now determine the role of global temperature fluctuations near marginal stability. Consider a sequence of quasi-equilibrium configurations evolving with a relative rate of change in volume $V(dt/dV)$ of a few t_{rel}. The quasi-static linear series has a local entropy maximum and a local entropy minimum (see figure 2). The evolutionary sequence approaches the turning point D through a succession of quasi-equilibria. However, as point

D is approached, the mean quadratic fluctuations tend to infinity as can been seen from (2.16) since $C_V \to 0$ and the system is unlikely to remain in thermal equilibrium. So how close to point D will the system survive in quasi-equilibrium? Let us try to define such a point, call it C, and find a posteriori if it has any physical sense.

Let T_C be the temperature at point C and $\overline{(\Delta \tilde{T})^2}_C$ its mean quadratic fluctuation. We define point C as one at which (see figure 2) the mean quadratic fluctuation is just equal to $(T_E - T_C)^2$. For smaller *mean* fluctuations big *real* fluctuations have only a negligible probability to put the system in a state of entropy at the left hand side of the minimum of entropy in figure 2 (in one relaxation period). If mean quadratic fluctuations are bigger than $(T_E - T_C)^2$ the probability for a real big fluctuation in a relaxation period increases also. Big fluctuations may bring the system in a state in which the entropy can increase indefinitely and if it can it will so that thermal equilibrium will be lost. Point C defines a limit between these different behaviors; at C

$$\sqrt{\overline{(\Delta \tilde{T})^2}_C} = T_E - T_C \simeq 2(T_D - T_C) \tag{2.23}$$

The last quasi-equality comes from a quadratic approximation near the horizontal tangent. In terms of U's and $\tilde{V}$'s defined in (2.19), (2.23) can be written

$$[\overline{(\Delta \tilde{V})^2}]_C = [2(\Delta U)_C]^2. \tag{2.24}$$

and if (2.24) holds then, following (2.22),

$$(\Delta U)_C = (4\Gamma_D N)^{-1/3} \tag{2.25}$$

This equality defines ΔU_C and thus β_C and also T_C. At point C the exponent in (2.18) is just equal to

$$-\tfrac{1}{8}\left(\frac{\Delta \tilde{V}_C}{\Delta U_C}\right)^2 = -\tfrac{1}{8}\left(\frac{\tilde{\beta} - \beta_C}{\beta_D - \beta_C}\right)^2 = -\tfrac{1}{8}\left(\frac{\tilde{T} - T_C}{T_D - T_C}\right)^2. \tag{2.26}$$

(2.25) gives the following value for β_C

$$\beta_{\rm C} = \beta_D[1 + (\Delta U)_{\rm C}] = \beta_D[1 + (4\Gamma_D N)^{-1/3}]. \tag{2.27a}$$

The corresponding value of the energy E_C can be estimated by expanding $E(\beta)$ near point D in powers of $\beta_C - \beta_D$, remembering that $(dE/d\beta)_D = 0$. Taking account of (2.20), we have, to second order in $\beta_C - \beta_D$, using also (2.27a):

$$E_C \simeq E_{\rm D}\left[1 + \frac{(\Gamma_D/2)^{1/3}}{4(\beta E/N)_{\rm D}} N^{-2/3}\right] \tag{2.27b}$$

(β_C, E_C) are the coordinates of point C.

3. Applications to Isothermal spheres

Isothermal spheres have been studied in great detail by many people. We refer to Binney and Tremaine for a modern presentation of the theory of isothermal spheres. With a Maxwellian distribution of energy per particle, the density $\rho(r)$ depends exponentially on the gravitational potential $\rho \sim e^{-\beta\Phi}$. This explains why Newton's equation can be written in the form [see equation (371) in Chandrasekhar (1934) or (4.15b) in Binney and Tremaine]

$$\frac{d}{dr}\left(r^2 \frac{d\ln\rho}{dr}\right) = -4\pi G m \beta r^2 \rho. \tag{3.1}$$

The equation can be replaced by a set of two first-order differential equations (Emden 1907). Here we use the (u, v) variables introduced by Chandrasekhar; first define the mass within a radius r as $\mu(r)$

$$\mu = \int_0^r 4\pi\rho r^2 dr. \tag{3.2}$$

Then introduce the variables

$$u = \frac{4\pi\rho r^3}{\mu}, \quad v = 4\pi G m \beta \rho r^2 \quad \text{and} \quad \zeta = \ln r. \tag{3.3}$$

Equation (3.1) is equivalent to the following pair of equations for u and v (Chandrasekhar's equations (404) and (405))

$$\frac{du}{d\zeta} = u(3-u) - v, \tag{3.4}$$

$$\frac{dv}{d\zeta} = 2v - \frac{v^2}{u} \quad . \tag{3.5}$$

Tables of solutions of these equations have been published by Emden (1907) and Chandrasekhar and Wares (1949). The total energy E of isothermal spheres is calculable in terms of u_B and v_B, the boundary values at $r = R$ of u and v. The non-dimensional total energy and inverse temperature in figure 1 are the following functions of u_B and v_B;

$$E^* = \left(\frac{R}{GNm^2}\right)\frac{E}{N} = \frac{u_B}{v_B}(u_B - \frac{3}{2}) \tag{3.6}$$

and from (3.3) alone, since $\mu(R) = \mu_B = M = Nm$,

$$b = \frac{GNm^2}{R}\beta = \frac{v_B}{u_B}. \tag{3.7}$$

Reciprocally, in terms of E^* and b

$$u_B = bE^* + \frac{3}{2}, \quad v_B = b\left(bE^* + \frac{3}{2}\right). \tag{3.8}$$

From (3.8), (3.7) and from (3.4), (3.5) at $r = R$ we obtain a differential equation for the linear series $E^*(b)$ represented in figure 1:

$$\frac{dE^*}{db} = -\frac{1}{b^2}\left(2bE^* + b - \tfrac{1}{2} + \frac{b-2}{bE^* + \frac{1}{2}}\right). \tag{3.9}$$

Equation (3.9) can be integrated directly; however the parametric form (3.4)-(3.5) deduced from (3.6)-(3.7) is more convenient for numerical integration; with $\zeta_B = \ln R$ we have

$$\frac{dE^*}{d\zeta_B} = -\frac{1}{b}\left[2(bE^* + \tfrac{1}{2})^2 + (b - \tfrac{3}{2})(bE^* + \tfrac{1}{2}) + b - 2\right], \tag{3.10}$$

$$\frac{db}{d\zeta_B} = b\left(bE^* + \tfrac{1}{2}\right). \tag{3.11}$$

The density contrast $\mathcal{R}$ is obtained by integrating twice equation (3.1). In terms of u, v and ζ, a first integration gives

$$\frac{d\ln\rho}{d\zeta} = -\frac{v}{u}. \tag{3.12}$$

With (3.7), (3.11) and (3.12) it then follows that

$$\mathcal{R} = \frac{\rho(0)}{\rho(R)} = \exp\int_0^b \frac{dx}{xE^*(x) + \frac{1}{2}} \tag{3.13}$$

Figure 1 has been obtained by integrating (3.10) and (3.11) setting $\zeta = 0$ at a point with $b_0 = 1.549$ and $E_0 \simeq 0.327$ calculated from Emden's tables.

Let us now evaluate the quantities related to the stability limit triggered by fluctuations. Point D, has coordinates b_D, E^*_D and a density contrast $\mathcal{R}_D$ all equal to

$$b_D = 2.03, \quad E^*_D = -0.335, \quad \mathcal{R}_D = 709. \tag{3.14}$$

The number Γ_D associated with point D, and defined in (2.20) can be derived from (3.9); it is non-dimensional and can be written using (3.6) and (3.7) like this

$$\Gamma_D = \left(b^3\frac{d^2E^*}{db^2}\right)_D = -b_D\left(2E^*_D + 1 + \frac{2E^*_D + \frac{1}{2}}{(E^*_D b_D + \frac{1}{2})^2}\right) \simeq 9.95 \approx 10 \tag{3.15}$$

Thus, following (2.27), the coordinates of the stability limit due to fluctuations are

$$b_C \simeq b_D(1 + \Delta b_C/b_D) = 2.03(1 + 0.29N^{-1/3}) \tag{3.16a}$$

$$E^*_C \simeq E^*_D(1 + \Delta E^*_C/E^*_D) = -0.335(1 - 0.63N^{-2/3}) \tag{3.16b}$$

at which point the density contrast, according to (3.13), is

$$\mathcal{R}_C = \mathcal{R}_D \exp\int_{b_D}^{b_C} \frac{dx}{xE^*(x) + \frac{1}{2}} \simeq \mathcal{R}_D \exp\left(\frac{\Delta b_C}{b_C E^*_D + \frac{1}{2}}\right) \tag{3.17}$$

Thus, taking account of (3.16) we obtain

$$\mathcal{R}_C \simeq 709 \cdot \exp\left(-3.30N^{-1/3}\right) \tag{3.18}$$

4. Remarks on these results and some observational implications

The Table gives the stability limit induced by fluctuations for values of N in the range $10 \leq N \leq \infty$. In addition to the coordinates (b_C, E_C^*) and the corresponding values of density contrasts $\mathcal{R}_C$, the table also provides relative changes in temperature $\Delta b_C/b_D$ and energy $\Delta E_C^*/E_D^*$ which give indications about the validity of the linear approximation near point D. Notice that relative corrections of b_C and E_C^* do not exceed 15% for N as small as 10. The relative reduction of the density contrasts $\Delta\mathcal{R}_C/\mathcal{R}_D$, on the other hand changes significantly.

The numbers have a simple interpretation: in isolated sphere in slow quasi-static evolution towards higher and higher density contrast as described in the Introduction the gravitational catastrophe or core collapse appears at lower density contrasts than 709. The change from $\mathcal{R}_D$ to $\mathcal{R}_C$ is very small for $N = 10^{6\sim 5}$ but quite significant for $N \simeq 10^3$ and becomes drastic at lower values of N.

A gravothermal catastrophy, as is well known, is not like an avalanche; the central parts of the stellar system gets hotter while the outer parts are left behind [see Lynden-Bell (1999) for a recent revue on gravothermal catastrophy]. The instability induces a change in the evolution which accelerates progressively. Our point here is to note that the change does not happen at point D but rather earlier, at point C.

How sharply is point C defined? To appreciate the sharpness of this new stability limit, consider the probability distribution dW at an inverse (non-dimensional) temperature $b > b_C$. Consider also a fluctuation $(\tilde{b} - b)$ (refere to figure 1 again) as big as $2(b - b_D)$, big enough to put the system in a state of ever growing entropy. The probability distribution for such a fluctuation defined by (2.21) can here be written in terms of b's rather than U's;

we obtain for dW:

$$dW = .465 N^{1/3} r^{1/2} \exp\left(-\tfrac{1}{2} r^3\right) d\tilde{V} \tag{4.1}$$

where

$$r = \frac{b - b_D}{b_C - b_D} \tag{4.2}$$

Thus for a given r

$$b = b_C + (r-1)(b_C - b_D). \tag{4.3}$$

For $N = 10^6$ (or $N = 10^2$), at point $b \simeq 1.01$ (1.12) higher than b_C with a factor $r = 3$, the exponential factor $\exp\left(-\frac{1}{2} r^3\right) \simeq 10^{-6}$ already. At $b \simeq 1.15$ (1.18) higher than b_C with a factor $r = 4$, $\exp\left(-\frac{1}{2} r^3\right) \simeq 10^{-14}$. Thus, the probability (per unit time) for fluctuations to be big enough to render the system unstable changes cery steeply for small departures from b_C. *In this respect, point C is rather sharply defined* .

The gravothermal catastrophe must thus take place at lower density contrasts because of the fluctuations and this should show up in N-body calculations with isothermal spheres evolving through quasi-static configurations. Possible implications for real globular clusters are as follows. Galactic globular clusters are of course not closed isothermal spheres. They are correctly modeled by Michie (1963)-King (1966) models which have truncated Gaussian distributions and are not in thermodynamic equilibrium. However to the extent that our results for isothermal spheres have any bearing on the behaviour of observed globular clusters, we speculate that a greater number of globular clusters appearing in the tables of Trager et al. (1993) than indicated there may actually be in an advanced stage of core collapse because most observed globular clusters whose parameter allows fitting them to energy truncated gaussian distributions are close to marginal stability, that is to point D (Katz 1980).

Acknowledgements

J.K. is grateful for the kind hospitality of the Division of Earth Rotation and Mizusawa Astrogeodynamics Observatory in Mizusawa, where much of this work was done in a peaceful and inspiring setting. He also thanks very much Donald Lynden-Bell for clarifying discussions when we were at the Department of Theoretical Physics of Charles University in Prague. Gerald Horwitz of the Racah Institute helped to understand the valuable clarifications demanded by the second referee; many thanks to both of them.

Bibliography

Antonov V.A. 1962, Vestnik Lenigrad Univ. **7**, 135 [translated in english in *Dynamics of Star Clusters* 1985, eds. Goodman J. and Hut P. , IAU Symposium No 113 (Reidel, Dordrecht)

Arnold V.I. 1986, *Catastrophe Theory*, 2nd ed, (Springer Verlag, Berlin, Tokyo)

Aronson E.B. and Hansen C.J. 1972, Astrophys.J. **177**, 145

Binney J. and Tremaine S. 1987, *Galactic Dynamics* (Princeton University Press, New Jersey)

Chandrasekhar S. 1934, *An Introduction to the Study of Stellar Structure* (Dover 1961, New York)

Chandrasekhar S. 1969, *Ellipsoidal Figures in Equilibrium* (Yale Univ. Press, New Haven)

Chandrasekhar S. 1972, in *General Relativity - papers in honor of J.L. Synge* Ed. O'Raifearthaigh OUP, Oxford)

Chandrasekhar S. and Wares G.W. 1949, Astrophys.J. **109** 551

Courant R. and Hilbert D. 1953, *Methods in Mathematical Physics* Vol. 1, (Interscience, New York)

Emden R. 1907, *Gaskügeln* (Leipzig)

Ginibre J. 1971 in *Statistical Mechanics and Quantum Field Theory* eds. DeWitt C. and Stora R. (Gordon and Breach, New York)

Harrison B.K., Thorne K.S., Wakano M. and Wheeler J.A. 1965, *Gravitation Theory and Gravitational Collapse* (University Press, Chicago)

Hachisu I., Nakada Y., Nomoto K. and Sugimoto D. 1978, Prog. Theor. Phys. **60** 393

Inagaki S. 1980, Publ. Astr. Soc. Japan **32** 213

Horwitz G. 1971, J. Math. Phys. **14** 658

Horwitz G. and Katz J. 1977, Astrophys.J. **211** 226 (referred to as HK77)

Horwitz G. and Katz J. 1978, Astrophys.J. **222** 941

Hunt G.W. 1977 Proc. R. Soc. London **A 357** 193

Katz J. 1978, MNRAS **183** 765

Katz J. 1979, MNRAS **189** 817

Katz J. 1980, MNRAS **190** 497

Katz J. 1981, *Linear Series and Catastrophies in Astronomy* in Proceedings of the 5th Götingen-Jerusalem Symposium (Vandenhock, Ruprecht)

King I.R. 1966, Astrophys.J. **71** 64

Landau L.D. and Lifshitz E.M. 1985, *Statistical Physics* 3rd ed. (Pergamon Press, Oxford)

Lynden-Bell D. 1999 Physica **A263** 293

Lynden-Bell D. and Wood R. 1968, MNRAS **138** 495

Meylan G. and Heggie D.C. 1997, Astron. Astrophys. Rev. **8** 1

Michie P.W. 1963, MNRAS **138** 495

Monaghan J.J. 1978, MNRAS **184** 25

Okamoto I., Katz J. and Parentani R. 1995, Class. Quantum Grav. **12** 443

Parentani R., Katz J., and Okamoto I. 1995, Class. Quantum Grav. **12** 1663

Padmanabhan T. 1990, Physics Reports **188** No. 5, 285

Poincaré H. 1885, Acta Math. **7** 259

Sorkin R.D. 1981 Astrophys.J. **249** 254

Thorne, K.S. 1966, in *High Energy Astrophysics* eds. DeWitt C., Schatzman E. and Véron P. (Gordon and Breach, New York)

Trager S.C., Djorgovski S. and King I.R. 1993, in *Structure and Dynamics of Globular Cluster* ASP Conference Series 50, eds. Djorgovski S. and Meylan G.

Thompson J.M.T. 1979, Phil. Trans. R. Soc. Lond. **292** 1386

TABLE

N	b_C	$-E^*_C$	$\mathcal{R}_C$	$\frac{\Delta b_C}{b_D}$	$-\frac{\Delta E^*_C}{E^*_D}$	$\frac{\Delta \mathcal{R}_C}{\mathcal{R}_D}$
∞	2.03	0.335	709	0	0	1
10^6	2.04	0.335	686	.003	$6.0 \cdot 10^{-5}$	0.97
10^5	2.04	0.335	660	0.006	$3.0 \cdot 10^{-4}$	0.93
10^4	2.06	0.334	608	0.014	$1.4 \cdot 10^{-3}$	0.86
10^3	2.09	0.333	510	0.029	$6.3 \cdot 10^{-3}$	0.72
10^2	2.16	0.325	348	0.063	$3.0 \cdot 10^{-3}$	0.49
10^1	2.30	0.289	154	0.140	$1.4 \cdot 10^{-1}$	0.22

Figure Captions

Figure 1

The figure represents the dimensionless expression of the total energy $E^* = RE/GN^2m^2$ of isothermal spheres as a function of a dimensionless inverse temperature $b = GNm^2/RkT$, where k is Boltzmann's constant, and G the gravitational constant. Numbers along the line represent density contrasts $\mathcal{R} = \rho(0)/\rho(R)$, $\rho(0)$ is the density at the centre, $\rho(R)$ is that at the boundary. $\mathcal{R}$ grows continuously in the direction of the arrow.

Figure 2

The full line represents the schematic behaviour of the linear series $E(\beta)$ *near* point D with corresponding points C and E of equal energy E_C. The dashed line is a schematic curve representing $w(E_\mathrm{C}; x^1)$ as a function of x^1, $x^i(i > 1)$ =constant.