THE DENSITY PROBABILITY DISTRIBUTION FUNCTION IN TURBULENT, ISOTHERMAL, MAGNETIZED FLOWS IN A SLAB GEOMETRY

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We investigate the behavior of the magnetic pressure, b^2 , in fully turbulent MHD flows in " $1+2/3$ " dimensions by means of its effect on the probability density function (PDF) of the density field. We start by reviewing our previous results for general polytropic flows, according to which the value of the polytropic exponent determines the functional shape of the PDF. A lognormal density PDF appears in the isothermal ($\gamma = 1$) case, but a power-law tail at either large or small densities appears for large Mach numbers when $\gamma > 1$ and $\gamma < 1$, respectively. In the isothermal magnetic case, the relevant parameter is the field fluctuation amplitude, $\delta B/B$. A lognormal PDF still appears for small field fluctuations (generally the case for large mean fields), but a significant low-density excess appears at large fluctuation amplitudes (weak mean fields), similar to the behavior at $\gamma > 1$ of polytropic flows. We interpret these results in terms of simple nonlinear MHD waves, for which the magnetic pressure behaves linearly with the density in the case of the slow mode, and quadratically in the case of the fast wave. Finally, we discuss some implications of these results, in particular the fact that the effect of the magnetic field in modifying the PDF is strongest when the mean field is weak.

1 Introduction

A fundamental feature of compressible turbulence is the formation of density fluctuations, a property of central interest in astrophysics, as density excesses ("clouds") in the turbulent interstellar medium (ISM) exhibit numerous statistical properties over a range of sizes $3,1$ $3,1$ $3,1$ whose origin is not yet well understood. Although a full understanding of star formation requires knowledge of the full (multiple-point) statistics in order to determine mean densities as a function of region size, a first step towards this goal is the description and physical understanding of the one-point statistics, or *probability density function* ("PDF") of the mass density field.

Previous studies have shown that the density PDF in turbulent compressible flows has a lognormal form in the isothermal case, 14,9,7,8 14,9,7,8 14,9,7,8 14,9,7,8 14,9,7,8 14,9,7,8 14,9,7,8 14,9,7,8 14,9,7,8 but exhibits a

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power-law tail at densities larger (smaller) than the mean for flows with poly-tropic exponents smaller (larger) than unity.^{[12](#page-7-0),[11](#page-7-0),[6](#page-7-0)} It should be noted that some of those works referred to purely hydrodynamic flows, $14,11$ $14,11$ $14,11$ while the rest referred to magneto-hydrodynamic (MHD) flows.

In particular, ref. $\lceil 1 \rceil$ presented a heuristic model for the development of the lognormal and power-law PDFs, based essentially on the behavior of the speed of sound with density in those types of flows. In this sense, the density PDF is a diagnostic for the dependence of the pressure with density. However, such model applied only to non-magnetic flows, while the real ISM is most likely magnetized to a significant extent. In this paper we present preliminary results on the density PDF of turbulent magnetized flows in " $1+2/3$ " dimensions, as a first attempt to characterize the behavior of magnetic pressure with density in the fully turbulent regime. Previous works have focused on the pressure produced by weakly nonlinear Alfv \acute{e} n waves^{[5](#page-7-0),[13](#page-7-0)}, but here we consider fully turbulent regimes with arbitrarily large magnetic fluctuation amplitudes.

2 The Non-Magnetic Case

The form of the density PDF in non-magnetic flows and its relation to the effective equation of state of the system has been understood in terms of a heuristic model by Passot and Vázquez-Semadeni,^{[11](#page-7-0)} herafter PVS98 (see also ref. $[6]$ $[6]$ $[6]$). As in ref. $[14]$ $[14]$ $[14]$, this model idealizes the generation of turbulent density fluctuations as a "multi-step" process, in which the local density at any given point in the flow is the result of a series of jumps due to the continuous passage of shock waves. The generation of densities is an iterative multiplicative process, in which every new density is obtained through a jump from the previous one, giving for the final density ρ_f in terms of the intermediate steps ρ_i : $\rho_f = \rho_0 \prod_i (\rho_{i+1}/\rho_i)$. In the isothermal case, the density jump is given by M^2 , where M is the Mach number of the shock and, because the speed of sound is constant, a given *velocity* jump is always characterized by the same Mach number, regardless of the local density, so that individual jumps can be regarded as independent, but extracted from the same distribution. In terms of the variable $s = \ln \rho$, the iterative process is *additive* so that the Central Limit Theorem can be applied in the limit of a large number of jumps, leading to a normal distribution for s and thus a lognormal distribution for ρ , as observed in the numerical experiments mentioned above.

Concerning the variance of the distribution, PVS98 have suggested (and confirmed numerically), through an analysis of the shock and expansion waves in the system, that for a large range of Mach numbers the typical size of the

logarithmic jump is expected to be $\sigma_s \sim M_{\rm rms}$, where $M_{\rm rms}$ is the rms Mach number. The mean s_0 of the distribution can be directly evaluated from the mass conservation condition $\langle \rho \rangle = \int_{-\infty}^{+\infty} e^{s} P(s) ds = 1$, where $P(s)$ is the PDF of s, yielding $s_0 = -\sigma_s^2/2$. The isothermal model PDF for s thus reads

$$
P(s)ds = \frac{1}{\sqrt{2\pi\sigma_s^2}} \exp\left[-\frac{(s-s_o)^2}{2\sigma_s^2}\right]ds,\tag{1}
$$

with $\sigma_s^2 = \beta M_{\text{rms}}^2$ and β a proportionality constant of order unity.

In the general polytropic case where the pressure P behaves as $P \propto \rho^{\gamma}$, with γ the polytropic exponent, the local Mach number $M(s)$ at a density $\rho = e^{s}$ is related to the one at the mean density (M) by $M(s) = M \exp^{(1-\gamma)s/2}$, suggesting an ansatz where the PDF keeps the same dependence on M , provided the above replacement is made. After relocating the term in s_0 from inside the exponential function to the normalization constant, the model PDF for the polytropic case reads

$$
P(s; \gamma)ds = C(\gamma) \exp\left[\frac{-s^2 e^{(\gamma - 1)s}}{2M^2} - \alpha(\gamma)s\right]ds.
$$
 (2)

This equation shows that when $(\gamma - 1)s < 0$, the PDF asymptotically approaches a power law, while in the opposite case it decays faster than a lognormal. Thus, for $0 < \gamma < 1$, the PDF approaches a power law at large densities $(s > 0)$, and at low densities $(s < 0)$ for $\gamma > 1$.

3 The Magnetic Case

In what follows we restrict ourselves to the isothermal case ($\gamma = 1$) and to a propagation along a uniform ambient magnetic field B_o and concentrate on the deviations from the corresponding lognormal PDF induced by the magnetic field.

3.1 Numerical Results

The numerical simulations solve the MHD equations using a pseudo-spectral method in a slab geometry $("1+2/3D")$ (variability is considered only with respect to the spatial variable x for the three components of the velocity and magnetic fields). A resolution of 2048 grid points is used allowing to handle large enough Mach and Reynolds numbers with only regular second-order viscosity. A random acceleration is applied on the y - and z-components of the velocity field, i.e. transverally to B_0 , thus inducing Alfvén waves into the flow. Since these waves have finite amplitudes, they in turn induce fast and slow

Figure 1. Density PDFs for two 1D simulations of polytropic turbulence, one with $\gamma = 0.3$ (left), and the other with $\gamma = 1.7$ (right), both with $M = 3$.

magnetosonic waves. The simulations are evolved over long times (several tens to a few hundred crossing times at the rms flow velocity) in order to obtain meaningful statistics for the density PDFs.

The simulations are essentially characterized by the sonic and Alfvénic Mach numbers, respectively defined as $M_s \equiv u/c_s$ and $M_a \equiv u/v_A$, where u is the rms velocity, c_s the sound speed and $v_A \equiv B_o^2/\rho$ the Alfvén speed. In order to investigate the effect of the magnetic field exclusively, we consider two runs (denoted I and II) with approximatively the same value of M_s (4.06 and 3.74 respectively) but very different values of M_a (0.36 and 1.66 resp.). These quantities are integrated over the duration of the run, excluding the first few time units in order to avoid including the uniform-density initial conditions. The rms relative field fluctuations $\delta B/B$ are for runs I and II respectively 0.75 and 6.65.

In fig. [2](#page-4-0) we show the PDFs for the two magnetic runs, together with lognormal fits (dashed lines). It is clearly seen that, contrary to what one

Figure 2. Density PDFs of two magnetic simulations with similar values of the sonic Mach number M_s but different values of the Alfvénic Mach number M_a . The simulation with small Ma (large mean field) is seen to have a nearly lognormal PDF, while the one with large Ma (weak mean field) is seen to have a large excess at low densities, indicative of an effective "magnetic" polytropic exponent larger than unity.

might expect, the high- M_a run, which should be closest to the pure-hydro case, exhibits a strong deviation from lognormality, while the low- M_a PDF is quite close to a lognormal. This implies that the case with the weakest mean field is the one with the strongest effect of magnetic pressure on the density PDF.

3.2 Interpretation in Terms of Simple MHD waves

A preliminary interpretation of the behavior reported in the previous section can be given in terms of the so-called "simple MHD waves" (see, e.g., ref. [[4](#page-7-0)]). These are the finite-amplitude equivalent of linear MHD waves, and have the same three well-known modes: Alvén, slow and fast. A simple wave refers to a solution that depends only on a single variable, any combinations of the spatial (x) and temporal (t) independent variables. However, as mentioned above, simple waves have finite amplitudes, and in general they develop shocks in a finite time.

A number of properties of simple waves has been reported by Mann[4](#page-7-0) (see also Jeffrey and Taniuti[2](#page-7-0)). Among them, most relevant for the present

discussion are the propagation velocities of the three modes, and the relations between the density and magnetic field fluctuations. The Alfvén mode with speed V such that $V^2 = v_{A_x}^2$ is not associated with density fluctuations. The fast and slow speed are given by

$$
V_{\pm}^{2} = \frac{v_{A}^{2} + c_{s}^{2}}{2} \left\{ 1 \pm \left[1 - \frac{4c_{s}^{2}v_{A}^{2}}{(v_{A}^{2} + c_{s}^{2})^{2}} \right]^{1/2} \right\}
$$
(3)

and the relation between the magnitude of the magnetic field b and the density is given by

$$
\frac{db}{d\rho} = \frac{V^2 - c_s^2}{b},\tag{4}
$$

where $v_{A_x}^2 = b_x^2/\rho$, $v_A^2 = b^2/\rho$, and c_s is the sound speed. An important point to note is that eqs. (3) and (4) imply a positive correlation between b and ρ for the fast mode (V_+) and an anticorrelation for the slow mode $(V_-),^4$ $(V_-),^4$.

Inserting eq. (3) in eq. (4), one can find the density dependence of the magnetic pressure. Excluding the case where the β of the plasma is of order unity with at the same time small to moderate field distorsions, this dependence simplifies and reads (general numerical solutions have been presented by $Mann⁴$ $Mann⁴$ $Mann⁴$:

$$
\frac{b^2}{b_x^2} \approx a_1 - a_2 \frac{c_s^2 \rho}{b_x^2}
$$
 (slow mode) (5)

$$
b^2 \approx 2a_3\rho^2 \quad \text{(fast mode)},\tag{6}
$$

where $b^2 = b_x^2 + b_y^2$, with $b_y^2 = b_y^2 + b_z^2$ being the magnitude of the magnetic field fluctuation. Note that in our $1+2/3$ geometry, $b_x(=B_0)$ is constant. The quantities a_i denote integration constants.

From these relations, it can be seen that the magnetic pressure, $\propto b^2$, scales roughly linearly with the density (albeit inversely) in the case of the slow mode, but quadratically in the case of the fast mode. Note also that, in the case of small field fluctuations (most often the case if $\beta \equiv c_s^2/(b_x^2/\rho) \ll 1$) the coefficient of ρ in eq. (5) is small, implying that, for the slow mode, large density fluctuations can occur even for small variations of b. Thus, *the slow mode is expected to dominate density fluctuation production in the case of small field fluctuations*. Instead, at large enough field fluctuations, the quadratic dependence of b^2 on ρ of the fast mode eventually overwhelms the linear dependence of the slow mode (and also of the thermal pressure), so *the fast mode is expected to dominate the density fluctuation production*.

The dependence of the magnetic pressure on the density seems to describe the observations of sec. [3.1](#page-2-0) adequately because, according to the results of the

non-magnetic case, a pressure that depends linearly on the density produces a lognormal density PDF, while one with a higher-than-linear dependence produces a near power law at low densities.

4 Conclusions

4.1 Summary

In this paper we have reviewed our previous results^{[11](#page-7-0)} on the development of the density PDF as a consequence of the effective (polytropic) equation of state, and applied them to an understanding of the PDF in the magnetic case.

In the context of the model presented in ref. $[11]$ $[11]$ $[11]$, isothermal non-magnetic flows have lognormal mass density PDFs while in the non-isothermal cases, a power-law tail develops at high Mach number, at either $\rho > \langle \rho \rangle$ or $\rho < \langle \rho \rangle$ depending on whether $\gamma < 1$ or $\gamma > 1$, respectively.

In the magnetic case, we considered only isothermal cases and a propagation parallel to the ambient magnetic field, but reported that a deviation from the lognormal PDF occurs *when the magnetic fluctuations are large*, a case expected when the mean field is small. We interpreted this effect in terms of so-called "simple", finite-amplitude nonlinear MHD waves. Indeed, for the slow mode of these waves, the magnetic pressure $b²$ behaves linearly with the density ρ , while for the fast mode, the magnetic pressure behaves quadratically with ρ . Together with the fact that for the slow wave the density is weighted by a small factor, this suggests that the production of density fluctuations is dominated by the slow waves at small field deviations (i.e., B_0 large) giving a lognormal PDF again, while for weak mean fields (large field deviations) the quadratic density dependence of $b²$ produces an excess at small densities in the PDF, corresponding to the $\gamma > 1$ case of the polytopic description. A more detailed study, including in particular the case of non-parallel propagation is in progress.

4.2 Astrophysical Implications

The results of this paper have a number of implications in the astrophysical context. First, we have seen that the main parameter determining the effect of the magnetic field appears to be the magnetic fluctuation amplitude, $\delta B/B$, rather than β alone, which is the parameter most frequently used to characterize MHD flows. In other words, it is important to have a knowledge of the relative importance of the turbulent fluctuations (as given by the Alfvénic Mach number) in addition to simply the ratio of thermal to magnetic pressures (as given by β).

Second, as far as the density PDF is concerned, the effect of the magnetic field is most notorious when the field fluctuations are large (generally, when the mean field is weak). This suggests that the limit of vanishing field does not approach the non-magnetic case, and in this sense the latter is singular. This may imply that it is inadequate to model the ISM as a non-magnetic flow even if the degree of magnetization is very low, as has started to be recently contended at the level of molecular clouds (see, e.g., ref. $[10]$).

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cray00^{\cdot 3: submitted to **World Scientific** *on December 19, 2018* 8}