General Relativistic Modification of a Pulsar Electromagnetic Field

Kohkichi Konno*) and Yasufumi Kojima**)

Department of Physics, Hiroshima University, Higashi-Hiroshima 739-8526, Japan

(Received September 4, 2000)

We consider an exterior electromagnetic field surrounding a rotating star endowed with a dipole magnetic field in the context of general relativity. The analytic solution for a stationary configuration is obtained, and the general relativistic modifications and the implications for pulsar radiation are investigated in detail. We find that the general relativistic corrections of both the electric field strength and the curvature radii of magnetic field lines tend to enhance the curvature radiation photon energy.

§1. Introduction

In recent years, new aspects of rotating neutron stars have been revealed in about 1000 pulsars. Eleven X-ray pulsars [1)] and eight γ-ray pulsars [2)] have been detected in the past several years. Among these new objects, some exhibit quite different behavior in their pulse periods. [3)] The measurement of the period and its time derivative yields evidence of ultra-magnetized stars, possibly representing magnetars. [4)] Motivated by the recent observational situation, theoretical models have been studied. As for high-energy pulsars, two general classes of models have been proposed. One is the polar cap model [5)] and the other is the outer gap model. [6)] The main difference between these two models is in the assumed region of the acceleration of charged particles responsible for the radiation. Both models partially explain some observational features of the γ-rays. They will be discriminated after including more detailed radiation processes. Future observation may determine their validity.

An important element to be included in theoretical models is general relativistic effects, which are in particular crucial for polar cap models, since acceleration occurs under strong gravity near the surface of neutron stars. Gonthier and Harding [7)] considered the effects on the magnetic field configuration only. Their concern is the curvature radiation and the attenuation of pair production in a strong magnetic field. These processes result in a pair cascade and explain some aspects of pulsar radiation, including high-energy pulses in the γ-ray range. In addition to the magnetic fields, rotationally induced electric fields play an important role in the polar cap region (see, e.g., Ref. 8)). Charged particles are ripped off the surface and accelerated along the magnetic field lines by the electric fields. The magnetosphere is thereby eventually filled with charges. The accelerated particles may be seeds of subsequent curvature radiation. Muslimov and Tsygan [9)] discussed general relativistic effects not only in the case of magnetic fields but also electric fields. They derived general expressions

*) E-mail: konno@theo.phys.sci.hiroshima-u.ac.jp
**) E-mail: kojima@theo.phys.sci.hiroshima-u.ac.jp

including multipoles of arbitrary order using hypergeometric functions, assuming a vacuum outside the star. Their work, however, is limited to only analytic forms, and therefore not easy, e.g., to compare with the standard results in flat space-time. Order estimates of general relativistic effects are also lacking. In this paper, we derive analytic solutions again for both the electric and magnetic fields around a rotating star endowed with an aligned dipole magnetic field. The resultant expressions are rather cumbersome, and for this reason approximate expressions are also given. Such forms provide an estimate of the corrections to the results in flat space-time, as well as a concise, practical application. We also give detailed discussion concerning the difference between our results and those in Minkowski space-time. This discussion may become important in the future, with progress in observational technology.

As shown in Ref. 10), the deviation from spherical space-time is less than 10^{-3} if the rotation period is longer than 10 msec and the magnetic field at the surface is less than 10^{16} gauss. Therefore, the electric and magnetic fields are determined by solving the Maxwell equations in a fixed background space-time. The appropriate space-time metric is that for an external field surrounding a slowly rotating star. We can neglect the second-order rotational effects, except in the case of rapidly rotating stars. We also restrict ourselves to a stationary configuration, that is, the case in which the magnetic dipole moment $\boldsymbol{\mu}$ is aligned with the angular velocity $\boldsymbol{\Omega}$. This leads to the following form $A_\mu = (A_t, 0, 0, A_\phi)$ for the four-potential (see Ref. 11) and references therein), where A_t is related to the rotationally induced electric field, and therefore $A_t \sim O(\Omega) \times A_\phi$. Detailed calculations to solve the Maxwell equations are given in §2. Approximate expressions of these solutions are discussed in §3. Implications of the general relativistic effects with regard to the acceleration of charged particles and radiation in vacuum gaps are investigated in §4. Finally, we give discussion in §5. Throughout the paper, we use units in which $c = G = 1$.

§2. The general relativistic solution for an exterior stellar electromagnetic field

We now derive expressions for an electromagnetic field surrounding a rotating, magnetized star using a general relativistic treatment. We solve the Maxwell equations in a fixed metric, assuming that the field is in a vacuum. The background metric outside the star with total mass M and angular momentum J is specified up to first order in the slow rotation approximation as

$$ds^2 = -e^{-\lambda(r)}dt^2 - 2\omega(r)r^2\sin^2\theta dt d\phi + e^{\lambda(r)}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2, \quad (2{\cdot}1)$$

where

$$e^\lambda = \left(1 - \frac{2M}{r}\right)^{-1}, \quad (2{\cdot}2)$$

$$\omega = \frac{2J}{r^3}. \quad (2{\cdot}3)$$

In the non-rotating limit, a poloidal magnetic field can be described by the A_ϕ

component only. In the slowly rotating case, the four-potential is given by $A_\mu = (A_t, 0, 0, A_\phi)$. The A_t component is rotationally induced as $A_t \sim O(\Omega) \times A_\phi$. The Maxwell equations for A_t and A_ϕ are given as

$$e^{-\lambda}\frac{\partial^2 A_\phi}{\partial r^2} - \lambda' e^{-\lambda}\frac{\partial A_\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A_\phi}{\partial \theta^2} - \frac{1}{r^2}\cot\theta\frac{\partial A_\phi}{\partial \theta} = 0, \tag{2·4a}$$

$$\begin{aligned} &e^{-\lambda}\frac{\partial^2 A_t}{\partial r^2} + \frac{2e^{-\lambda}}{r}\frac{\partial A_t}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A_t}{\partial \theta^2} + \frac{1}{r^2}\cot\theta\frac{\partial A_t}{\partial \theta} \\ &+ \left[\left(\lambda' + \frac{2}{r}\right)\omega + \omega'\right] e^{-\lambda}\frac{\partial A_\phi}{\partial r} + \frac{2}{r^2}\omega\cot\theta\frac{\partial A_\phi}{\partial \theta} = 0, \end{aligned} \tag{2·4b}$$

where the prime here denotes differentiation with respect to r. Note that the last two terms on the left-hand side of Eq. (2·4b) represent the coupling between the frame-dragging and the stellar magnetic field, and that these terms originate from a purely general relativistic effect.

From this point, we restrict our discussion to the case of a dipole magnetic field, so that Eq. (2·4a) can be solved in the form

$$A_\phi(r, \theta) = -a_\phi(r)\sin^2\theta. \tag{2·5}$$

In a similar way, the potential A_t can be written as

$$A_t(r, \theta) = a_{t0}(r) + a_{t2}(r)P_2(\cos\theta), \tag{2·6}$$

where P_2 is the Legendre polynomial of degree 2.

The solution for a_ϕ can easily be derived in the form [12)]

$$a_\phi = \frac{3\mu}{8M^3}r^2\left[\ln\left(1 - \frac{2M}{r}\right) + \frac{2M}{r} + \frac{2M^2}{r^2}\right], \tag{2·7}$$

where μ is the magnetic dipole moment with respect to an observer at infinity. The resulting dipole magnetic field in the local frame is given by

$$B_{(r)} = -\frac{3\mu}{4M^3}\left[\ln\left(1 - \frac{2M}{r}\right) + \frac{2M}{r} + \frac{2M^2}{r^2}\right]\cos\theta, \tag{2·8a}$$

$$B_{(\theta)} = \frac{3\mu}{4M^3}\left[\sqrt{1 - \frac{2M}{r}}\ln\left(1 - \frac{2M}{r}\right) + \frac{2M(r-M)}{r\sqrt{r(r-2M)}}\right]\sin\theta. \tag{2·8b}$$

Next, we discuss the electric field induced by the rigid rotation of the star. The solution for a_{t0} and a_{t2} can be obtained analytically as

$$a_{t0} = \frac{c_0}{r} + \frac{J\mu}{2M^3r^2}(3r - M) + \frac{J\mu}{4M^4r}(3r - 4M)\ln\left(1 - \frac{2M}{r}\right), \tag{2·9a}$$

$$\begin{aligned} a_{t2} = &\frac{c_1}{M^2}(r-M)(r-2M) \\ &+ c_2\left[\frac{2}{Mr}\left(3r^2 - 6Mr + M^2\right) + \frac{3}{M^2}\left(r^2 - 3Mr + 2M^2\right)\ln\left(1 - \frac{2M}{r}\right)\right] \end{aligned}$$

$$-\frac{J\mu}{2M^6r^2}\left(9r^4-3Mr^3-30M^2r^2+8M^3r+2M^4\right)$$
$$-\frac{J\mu}{2M^6r}\left(12r^3-36Mr^2+24M^2r+M^3\right)\ln\left(1-\frac{2M}{r}\right), \qquad (2\cdot 9\text{b})$$

where c_0, c_1 and c_2 are constants of integration. Since c_0 is understood as the net charge of the star, we set $c_0=0$. Furthermore, we derive

$$c_1=\frac{9J\mu}{2M^4}, \qquad (2\cdot 10)$$

from the regularity condition at infinity. The constant c_2 is fixed by the junction condition at the surface of the star. If we impose the assumption of a perfectly conducting interior, the magnetic field is frozen into the the fluid motion, i.e. $u^\mu F_{\mu\nu}=0$, where $u^\mu=(u^t,0,0,\Omega u^t)$ is the four-velocity of the fluid. From this condition at the surface, we have

$$c_2=\left\{\frac{\mu J}{M^5R^2}\left(12R^3-24MR^2+4M^2R+M^3\right)\right.$$
$$+\frac{\mu J}{2M^6R}\left(12R^3-36MR^2+24M^2R+M^3\right)\log\left(1-\frac{2M}{R}\right)$$
$$\left.-\frac{\mu\Omega}{4M^3}\left[2MR+2M^2+R^2\log\left(1-\frac{2M}{R}\right)\right]\right\}$$
$$\Big/\left[\frac{2}{MR}\left(3R^2-6MR+M^2\right)\right.$$
$$\left.+\frac{3}{M^2}\left(R^2-3MR+2M^2\right)\log\left(1-\frac{2M}{R}\right)\right], \qquad (2\cdot 11)$$

where R denotes the radius of the star. Consequently, using the above c_2, the induced electric field in the local frame can be written as

$$E_{(r)}=\frac{1}{2M^6r^3}\left\{c_2\left[4M^5r\left(6r^2-3Mr-M^2\right)\right.\right.$$
$$\left.+6M^4r^3\left(2r-3M\right)\ln\left(1-\frac{2M}{r}\right)\right]$$
$$-2MJ\mu\left(24r^3-12Mr^2-4M^2r-3M^3\right)$$
$$\left.-3rJ\mu\left(8r^3-12Mr^2-M^3\right)\ln\left(1-\frac{2M}{r}\right)\right\}P_2(\cos\theta), \qquad (2\cdot 12\text{a})$$

$$E_{(\theta)}=-\frac{3}{M^6r^3\sqrt{r(r-2M)}}$$
$$\times\left\{c_2\left[2M^5r^2\left(3r^2-6Mr+M^2\right)\right.\right.$$
$$\left.+3M^4r^3\left(r^2-3Mr+2M^2\right)\ln\left(1-\frac{2M}{r}\right)\right]$$
$$-MJ\mu\left(12r^4-24Mr^3+4M^2r^2-M^4\right)$$
$$\left.-6r^3J\mu\left(r^2-3Mr+2M^2\right)\ln\left(1-\frac{2M}{r}\right)\right\}\sin\theta\cos\theta. \qquad (2\cdot 12\text{b})$$

The discussion of the quantitative nature of the electromagnetic field strength is given in the next section.

§3. Comparison with results in flat space-time

In the previous section, we obtained expressions for the electromagnetic field in the general relativistic framework. However, these expressions are somewhat cumbersome. They are reduced to standard expressions given in textbooks[8)] when the gravitational terms are neglected, i.e. when we take the limits $M, J \to 0$. It is important to compare our expressions with the standard expressions and to examine the differences. From this point of view, we now derive the simpler approximate expressions by expanding in powers of $1/r$. The lowest-order forms give the standard results, and the next terms give their corrections. As an approximation, we use the radius r in the Schwarzschild coordinates as the radius in the flat space-time. The magnetic and electric fields can be expanded in the forms

$$B_{(r)} \simeq \frac{2\mu}{r^3}\left[1+\frac{3M}{2r}\right]\cos\theta, \tag{3·1a}$$

$$B_{(\theta)} \simeq \frac{\mu}{r^3}\left[1+\frac{2M}{r}\right]\sin\theta, \tag{3·1b}$$

$$E_{(r)} \simeq -\frac{2\mu R^2 \Omega}{r^4}\left[1-\left(\frac{1}{2}-\frac{8R}{3r}\right)\frac{M}{R}+\left(1-\frac{2R}{r}\right)\frac{I}{R^3}\right]P_2\left(\cos\theta\right), \tag{3·2a}$$

$$E_{(\theta)} \simeq -\frac{2\mu R^2 \Omega}{r^4}\left[1-\left(\frac{1}{6}-\frac{R}{r}\right)\frac{M}{R}+\left(1-\frac{3R}{r}\right)\frac{I}{R^3}\right]\sin\theta\cos\theta, \tag{3·2b}$$

where the terms following the first ones in each of the square brackets are the first-order corrections due to the curved space-time. In Eq.(3·2), the moment of the inertia $I = J/\Omega$ is used. These corrections can be estimated easily for stars with uniform density, in which $I \sim 2MR^2/5$ and $M/R \le 4/9$. The correction terms become larger with the relativistic factor M/R, but they are less than 1. Thus we see that the expressions obtained in the flat space-time are accurate to within a factor of 2.

In Figs. 1 and 2, we explicitly display the results in the flat and curved space-times as functions of the radius. These figures display the normalized values of the radial parts of the magnetic and electric fields, respectively. We have adopted a polytropic stellar model with $M/R = 0.2$, which is a plausible value for neutron stars. The solid curves here denote the exact values in the curved space-time, while the dashed curves correspond to the standard results in the flat space-time. From these figures, we find that the standard expressions in the flat space-time give values deviating from the curved space-time values by 50% at most. The maximum error is roughly estimated as $2M/r$. Therefore, the standard expressions are useful for arguments within this order of the magnitude.

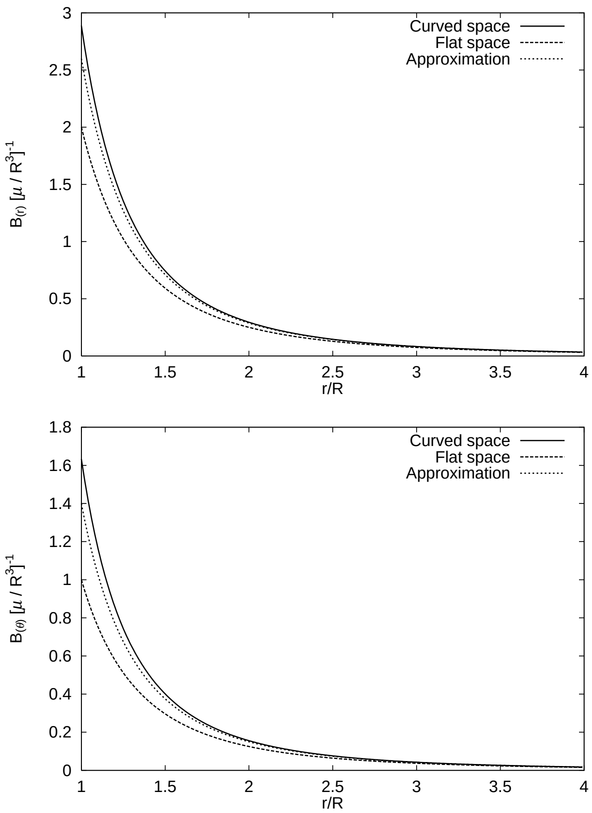

Fig. 1. Radial parts of the magnetic field components $B_{(r)}$ and $B_{(\theta)}$ are plotted as functions of the radius. The field strength is normalized by the typical value μ/R^3. The solid, dashed and dotted curves denote the curved space-time, flat space-time and approximate expressions, respectively.

§4. Implications for the acceleration of charged particles and the radiation in vacuum gaps

In this section, the results for the electromagnetic field in curved space-time are applied to analysis of the pulsar emission mechanism, that is, quantities relevant to the acceleration of charged particles and radiation in vacuum gaps above the polar caps. The gravitational force is much less than the electrostatic force, but gravity affects space-time, whose effects on the electromagnetic field are considered here. We explicitly derive the electric field along the magnetic field lines, curvature radii of the field lines, and size of polar cap regions. They are important to evaluate the available potential energy, curvature radiation, and so on. They significantly depend on the global shape of magnetic field lines, so that deviation from the standard results in flat space-time is not estimated using some local positions, although the overall error is not expected to be large.

First, we investigate the electric field component along the magnetic field lines. This plays a direct, important role in the acceleration of charged particles. The

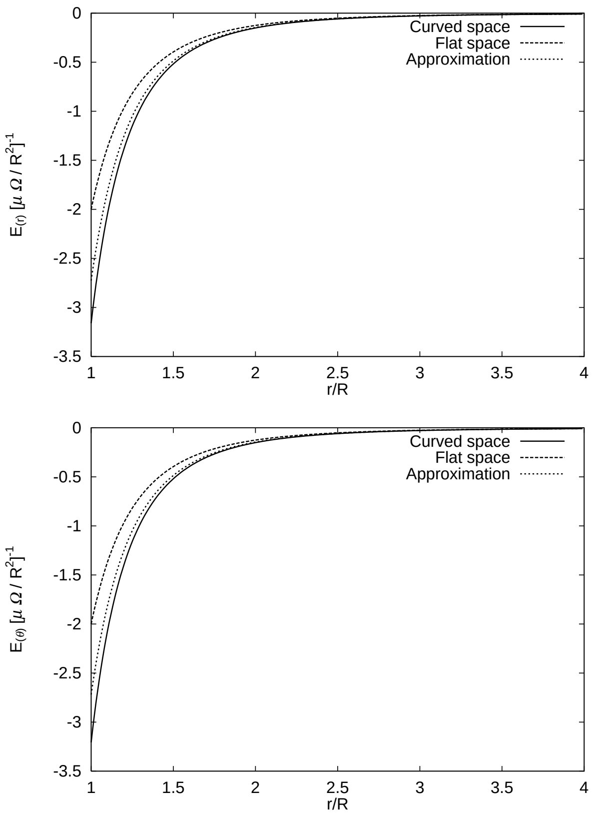

Fig. 2. Radial parts of the magnetic field components $E_{(r)}$ and $E_{(\theta)}$ are plotted as functions of the radius. The field strength is normalized by the typical value $\mu\Omega/R^2$. The solid, dashed and dotted curves denote the curved space-time, flat space-time and approximate expressions, respectively.

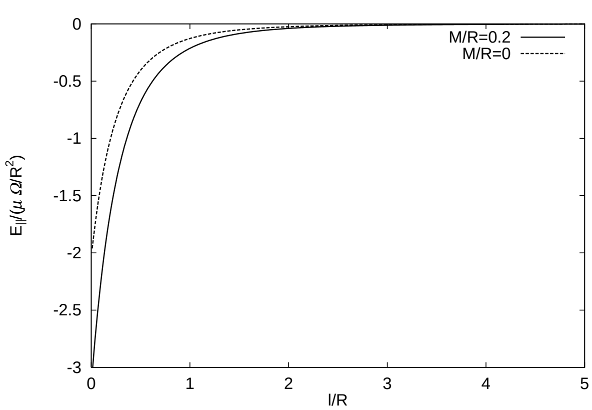

Fig. 3. The electric field component along a magnetic field line that flows from the stellar surface with $\theta = 1^\circ$. The field strength, which is normalized by the typical value $\mu\Omega/R^2$, is calculated for the Minkowskian case $M/R = 0$ (dashed) and the relativistic case $M/R = 0.2$ (solid). The proper distance l from the stellar surface is normalized by R.

component is derived from Eqs. (2·8) and (2·12) as

$$E_{||} = \frac{E_{(r)}B_{(r)} + E_{(\theta)}B_{(\theta)}}{\sqrt{B_{(r)}^2 + B_{(\theta)}^2}}. \tag{4·1}$$

Figure 3 displays $E_{||}$ normalized by the typical value $\mu\Omega/R^2$ as a function of the proper distance l from the stellar surface along a field line. The dashed curve denotes the Minkowskian case, and the solid curve denotes the general relativistic case of $M/R = 0.2$. This figure shows that the electric field component is strengthened by the general relativistic effect with respect to the same value of $\mu\Omega/R^2$. The result in the curved case is about 1.5 times as large as that in the flat case near the surface. A similar kind of enhancement can be seen in the stellar interior due to the general relativistic effect.[10] These enhancements may be regarded as having a common origin.

The configurations of the magnetic field lines are also modified by the general relativistic effect. In general, a magnetic field line is described by an ordinary differential equation:[13]

$$\frac{dr}{d\theta} = \frac{B_r}{B_\theta}. \tag{4·2}$$

The solution of this equation is

$$A_\phi = \mathrm{const}\,(\equiv \tilde{c})\,. \tag{4·3}$$

Each field line is labeled by a constant $\tilde{c}$. Figure 4 displays the magnetic field lines embedded in the z-x plane, where $(z,x) = (r\cos\theta, r\sin\theta)$, both in the Minkowskian case and in the general relativistic case. As easily seen from this figure, the magnetic field lines are moderately modified by the general relativistic effect. Owing to this change, curvature radii of the field lines are also modified by the general relativistic effect.

Mathematically, the radius is defined as

$$\tilde{\rho} = \left(\frac{d\theta}{dl}\right)^{-1}, \tag{4·4}$$

where l denotes the proper distance along a field line. In the Minkowskian case, the field line is simply specified as $\tilde{c}r = \sin^2\theta$, so that the curvature radius along the line labeled by $\tilde{c}$ is given by

$$\tilde{\rho} = \frac{\sin\theta}{\tilde{c}}\sqrt{1 + 3\cos^2\theta}. \tag{4·5}$$

The general relativistic counterpart should be obtained numerically. Figure 5 displays the curvature radii $\tilde{\rho}$ of magnetic field lines which start from the stellar surface with an angle of $\theta = 1°$. From Fig. 5, we find that the general relativistic effect causes the curvature radius to become smaller for a fixed magnetic moment. The curvature radiation is produced by charged particles moving along the magnetic field lines. The resulting curvature radiation photon energy is proportional to $\tilde{\rho}^{-1}$. The

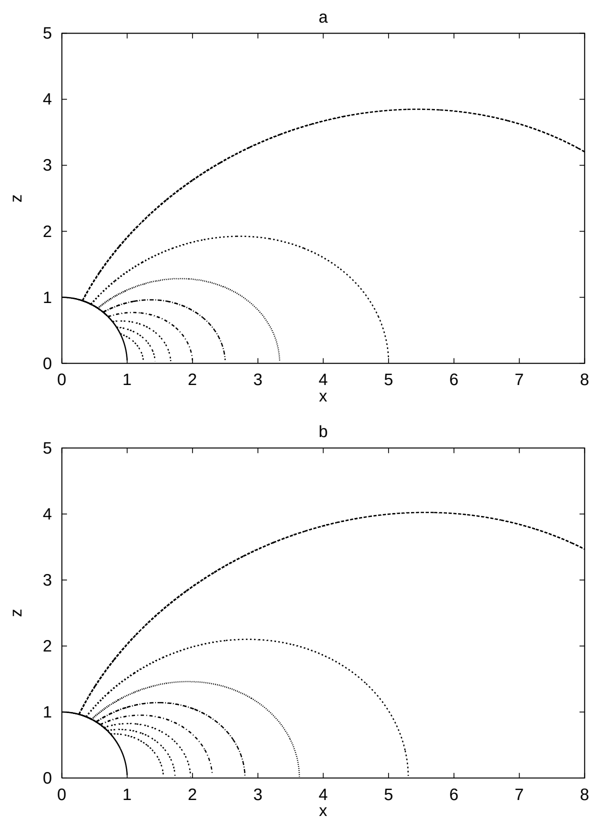

Fig. 4. Magnetic field lines for the Minkowskian case ($M/R = 0$) (a) and the general relativistic case with $M/R = 0.2$ (b), plotted in the z-x plane, where $(z, x) = (r \sin\theta, r \cos\theta)$. Both cases have the same magnetic moment. The surface of the star is denoted by the circle of radius 1.

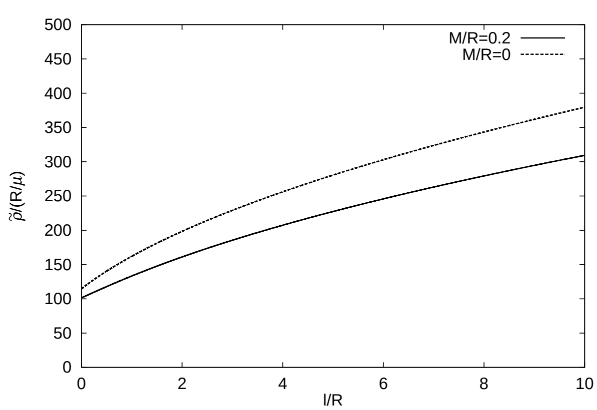

Fig. 5. Curvature radii of magnetic field lines that flow from the stellar surface with $\theta = 1^\circ$. The radii are plotted as functions of the proper distance l along the field line. The solid line denotes the general relativistic case with $M/R = 0.2$, while the dashed line denotes the Minkowskian case $M/R = 0$. The radii $\tilde{\rho}$ and the proper distance l are normalized by R/μ and R, respectively.

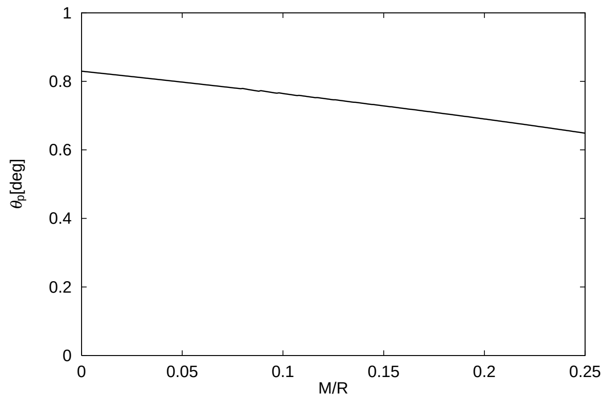

Fig. 6. Polar cap angles plotted as functions of the general relativistic factor M/R for $R_L \simeq 5 \times 10^3 R$.

correct treatment in curved space-time implies an increase of the photon energy. Although we have displayed only one comparison between the flat and curved cases, almost the same results were obtained for all small values of θ.

A modification of the field lines, further, leads to a change of the polar cap radius. The polar cap angle θ_p is given by

$$\theta_p = \sin^{-1}\sqrt{\frac{a_\phi(R_L)}{a_\phi(R)}}, \tag{4·6}$$

where R_L is the radius of the light cylinder. To derive the polar cap angle explicitly, we have assumed

$$R_L = \frac{c}{\Omega} \simeq 5 \times 10^3 R \tag{4·7}$$

for any value of M/R. Figure 6 displays the dependence of the polar cap angle θ_p on the general relativistic factor M/R. From this figure, we see that the polar cap angle is reduced by about 15% due to the curved nature.

§5. Discussion

Recent observations of compact stars have given remarkable results that demand the refinement of theoretical models. Inspired by this, we have reconsidered an exterior electromagnetic field surrounding a rotating star endowed with an aligned dipole magnetic field in the context of general relativity. The electromagnetic fields were derived in analytic and approximate forms. We found that the expressions calculated in the flat space-time are accurate within a factor of approximately 2. We have not calculated the emission and propagation of radiation from polar caps of pulsars, but rather have discussed the implications for the underlining physical processes. We have found that the general relativistic effects increase the strength of electric fields and decrease the curvature radii of the magnetic field lines. Both of these factors contribute to increase the photon energy emitted from charged particles. The magnitude of the correction is of order M/R. Another important general relativistic effect, which has not been considered here, is the redshift factor. The observed energy is shifted to a lower value by a factor of M/R. It is not clear whether or not all these general relativistic effects are canceled. It is important to construct detailed models of pulsar radiation, taking these factors into account.

Although we have restricted our investigation to a rotating star in a vacuum, it seems that actual neutron stars are surrounded by plasma. Hence, it is important to investigate the acceleration of charged particles and the radiation taking into account the plasma distributions around stars. A general relativistic analysis using a certain pulsar model which specifies the plasma distribution has been given by Muslimov and Tsygan.[14] The general relativistic effects in pulsar models are not yet clear, since the magnitude of the effects significantly depends on the plasma distribution. It is necessary to discuss the effects in a more general framework. This will be the subject of future investigation.

Acknowledgements

This work was supported in part by a Grant-in-Aid for Scientific Research Fellowship of the Ministry of Education, Science, Sports and Culture of Japan (No. 12001146).

References

[1] W. Becker and J. Trümper, Astron. Astrophys. **341** (1999), 803.
[2] D. J. Thompson et al., Astrophys. J. **516** (1999), 297.
[3] See, e.g., C. Kouveliotou et al., Nature **393** (1998), 235.
C. Kouveliotou et al., Astrophys. J. **510** (1999), L115.
S. Mereghetti and L. Stella, Astrophys. J. **442** (1995), L17.
[4] R. C. Duncan and C. Thompson, Astrophys. J. **392** (1992), L9.
[5] See, e.g., S. J. Sturner, C. D. Dermer and F. C. Michel, Astrophys. J. **445** (1995), 736.
J. K. Daugherty and A. K. Harding, Astrophys. J. **458** (1996), 278.
B. Zhang and A. K. Harding, Astrophys. J. **532** (2000), 1150.
[6] See, e.g., K. S. Cheng, C. Ho and M. Ruderman, Astrophys. J. **300** (1986), 500; **300** (1986), 522.
R. W. Romani, Astrophys. J. **470** (1996), 469.
[7] P. L. Gonthier and A. K. Harding, Astrophys. J. **425** (1994), 767.
[8] P. Mészáros, *High-Energy Radiation from Magnetized Neutron Stars* (The University of Chicago Press, USA, 1992).
[9] A. G. Muslimov and A. I. Tsygan, Sov. Astron. **30** (1986), 567.
[10] K. Konno, T. Obata and Y. Kojima, Astron. Astrophys. **352** (1999), 211; **356** (2000), 234.
[11] S. Bonazzola, E. Gourgoulhon, M. Salgado and J. A. Marck, Astron. Astrophys. **278** (1993), 421.
[12] V. L. Ginzburg and L. M. Ozernoĭ, Sov. Phys. -JETP **20** (1965), 689.
[13] M. Dovčiak, V. Karas and A. Lanza, astro-ph/0005216.
[14] A. G. Muslimov and A. I. Tsygan, Sov. Astron. **34** (1990), 133; Mon. Not. R. Astron. Sci. **255** (1992), 61.
A. I. Tsygan, Astron. Lett. **19** (1993), 268.