Do relativistic corrections affect microlensing amplification?

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Studies of gravitational microlensing are usually based on the lens equation, which is valid only to first order in the gravitational constant G. However, the amplification factor of microlensing is a second-order quantity with respect to G. Despite this fact, conventional studies are still based on the lowest-order lens equation. Then the question naturally arises: Why are these conventional studies justified? We carefully study the relativistic correction to the amplification factor at $O(G^2)$. We show that the amplification factor for each image is corrected. However, the total amplification remains unchanged at this order.

Microlensing has been one of most successful areas of study in astrophysics over the past decade,[1,2] since microlensing due to massive astrophysical compact halo objects (MACHO) were proposed by Paczynski[3] and detected first by MACHO[4] and EROS[5] groups. The current microlensing searches are performed also in the direction of our galactic center.[6] Furthermore, there are ongoing/proposed projects for monitoring microlens events with short time intervals, say hours.[7] By such monitoring, we can quickly make follow-up observations with large telescopes, such as Keck, Subaru and the Very Large Telescope (VLT). Hence, we should be able to obtain (1) more accurate shapes of the light curve for microlens events and (2) a rapid increase in the number of events.

In this paper, taking account of expected future precise measurements of the light curve, we study relativistic corrections to amplification by microlensing. Theoretical studies of the light curve are usually based on amplification calculated using the so-called deflection angle formula $4GM/bc^2$, where M is the lens mass and b is the impact parameter of a light ray. This deflection angle is calculated at linear order, $O(G)$, in the post-Minkowskian approximation, which is an expansion scheme in terms of G. Actually, only the Newtonian potential GM/b is used. As for the amplification factor, however, it is necessary to calculate up to the second order $O(G^2)$, as shown in Eq. (7) below. Despite this fact, most studies are still based on the lens equation, which is valid only to lowest order in G. In order to examine whether these conventional studies are quantitatively correct, we study relativistic corrections to the amplification at the next order, $O(G^2)$. It is apparent that small changes could appear in the amplification. Rather surprisingly, however, we show that the total amplification remains unchanged at $O(G^2)$, while the amplifications for each image change. We use units in which $c = 1$.

First, we summarize the derivation of the total amplification for microlensing.[1]

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Denoting the source and image position angles by θ_S and θ_I, respectively, the lens equation is written as

$$\theta_S = \theta_I - \frac{D_{LS}}{D_S}\alpha, \tag{1}$$

where we have used the thin-lens approximation, and D_{LS} and D_S denote distances from the lens to the source and from the observer to the source, respectively. This gives us a mapping between the lens and source planes. We can consider the case $\theta_S \geq 0$ without loss of generality. For a point lens (generally a spherically symmetric lens), the deflection angle α becomes, at $O(G)$,

$$\alpha = \frac{4GM}{b}. \tag{2}$$

Here we have assumed that the lens moves slowly, so that the effect due to its velocity is negligible. Then, the lens equation is

$$\theta_S = \theta_I - \frac{\theta_E^2}{\theta_I}, \tag{3}$$

where θ_E is the angular radius of the Einstein ring,

$$\theta_E = \sqrt{\frac{4GMD_{LS}}{D_L D_S}}, \tag{4}$$

and D_L denotes the distance from the observer to the lens. For microlensing in our galaxy, the radius is typically

$$\theta_E \sim 10^{-4}\left(\frac{M}{0.1M_\odot}\right)^{1/2}\left(\frac{50\text{kpc}}{D_S}\right)^{1/2}\text{arcsec.}, \tag{5}$$

where we have assumed $D_L \sim D_{LS}$.

The lens equation is solved easily as

$$\theta_\pm = \frac{1}{2}(\theta_S \pm \sqrt{\theta_S^2 + 4\theta_E^2}). \tag{6}$$

From a simple geometrical argument, we find the amplification due to gravitational lensing to be

$$A = \left|\frac{\theta_I}{\theta_S}\frac{d\theta_I}{d\theta_S}\right| = \frac{1}{\left|1 - \left(\frac{\theta_E}{\theta_I}\right)^4\right|}, \tag{7}$$

which is considered a function of the image position θ_I. From Eqs. (4) and (7), it is apparent that the amplification factor A is a second-order quantity with respect to G. It should be noted that terms at first order in G disappear in Eq. (7) as a consequence of cancellation. For each image $\theta_\pm$, the amplification factor becomes

$$A_\pm = \frac{u^2 + 2}{2u\sqrt{u^2 + 4}} \pm \frac{1}{2}, \tag{8}$$

where u denotes θ_S/θ_E, the source position in units of the Einstein ring radius. In typical microlensing events, the angular separation between the images is so small that all we can measure is the amplification of the total flux,

$$A_{\text{total}} = A_+ + A_- = \frac{u^2 + 2}{u\sqrt{u^2 + 4}}. \tag{9}$$

Now, we are in the position to study the relativistic correction at $O(G^2)$. To this order, the correction to the deflection angle is[8]

$$\Delta\alpha = \frac{15\pi G^2 M^2}{4b^2}. \tag{10}$$

Since the relativistic correction is generally small, it is convenient to treat the correction as a perturbation around the well-known results at $O(G)$. In order to do so, we define the angle in units of the Einstein ring radius as

$$\tilde{\theta} = \frac{\theta}{\theta_E}. \tag{11}$$

Then, by noting that $\tilde{\theta}_I$ can be negative, we obtain the lens equation up to $O(G^2)$,

$$\tilde{\theta}_S = \tilde{\theta}_I - \frac{1}{\tilde{\theta}_I}\left(1 + \frac{\lambda}{|\tilde{\theta}_I|}\right), \tag{12}$$

where we have defined the dimensionless parameter λ as

$$\lambda = \frac{15\pi D_S \theta_E}{64 D_{LS}}. \tag{13}$$

This lens equation can be rewritten as

$$\tilde{\theta}_I^3 - \tilde{\theta}_S \tilde{\theta}_I^2 - \tilde{\theta}_I - \lambda\frac{\tilde{\theta}_I}{|\tilde{\theta}_I|} = 0. \tag{14}$$

Since θ_E is sufficiently small in most astronomical situations, we can take λ as an expansion parameter.

The lens equation (14) is solved iteratively in terms of λ: For $\tilde{\theta}_I \geq 0$, the lens equation is rewritten as

$$\tilde{\theta}_I^3 - \tilde{\theta}_S \tilde{\theta}_I^2 - \tilde{\theta}_I - \lambda = 0. \tag{15}$$

Substituting the form $\tilde{\theta}_+ + \lambda\phi + O(\lambda^2)$ for $\tilde{\theta}_I$ in this equation, we find the solution as

$$\tilde{\theta}'_+ = \tilde{\theta}_+ + \frac{\lambda}{\tilde{\theta}_+\sqrt{\tilde{\theta}_S^2 + 4}} + O(\lambda^2), \tag{16}$$

where the prime denotes a quantity with relativistic corrections. For $\tilde{\theta}_I < 0$, the lens equation becomes

$$\tilde{\theta}_I^3 - \tilde{\theta}_S \tilde{\theta}_I^2 - \tilde{\theta}_I + \lambda = 0. \tag{17}$$

which is solved up to $O(\lambda)$ as

$$\tilde{\theta}'_- = \tilde{\theta}_- + \frac{\lambda}{\tilde{\theta}_-\sqrt{\tilde{\theta}_S^2 + 4}} + O(\lambda^2). \tag{18}$$

The amplification for each image becomes

$$A'_\pm = \frac{u^2 + 2}{2u\sqrt{u^2 + 4}} \pm \left(\frac{1}{2} - \lambda(u^2 + 4)^{-3/2}\right) + O(\lambda^2). \tag{19}$$

Hence, relativistic corrections at $O(G^2)$ change the amplification for each image. We find, however, the amplification for the total flux as

$$A'_{\text{total}} = \frac{u^2 + 2}{u\sqrt{u^2 + 4}} + O(\lambda^2), \tag{20}$$

which is the same as Eq. (9). Hence, in the microlensing, the total amplification remains unchanged, up to $O(G^2)$.

For higher order relativistic corrections, terms of inverse powers in $\tilde{\theta}$ appear in Eq. (12). In a similar manner to the present case of $O(G^2)$, we may find that for higher orders in relativistic corrections, only terms even in λ in A_{total} may appear, with every odd terms vanishing.

Before closing this paper, we give a remark on θ_E. The Einstein ring radius given by Eq. (4) should be corrected at higher orders of G. Then, we may use the corrected radius θ'_E in the analysis at $O(G^2)$. Only the ratio between the Einstein ring radius and the impact parameter determines the maximal amplification in light curves due to microlensing. However, since the impact parameter is not observable, a difference between θ_E and θ'_E can be never observed. Therefore, the difference does not change our conclusion.

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