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General Relativistic Electromagnetic Fields of a Slowly Rotating Magnetized Neutron Star. I. Formulation of the equations.

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ABSTRACT
We present analytic solutions of Maxwell equations in the internal and external background spacetime of a slowly rotating magnetized neutron star. The star is considered isolated and in vacuum, with a dipolar magnetic field not aligned with the axis of rotation. With respect to a flat spacetime solution, general relativity introduces corrections related both to the monopolar and the dipolar parts of the gravitational field. In particular, we show that in the case of infinite electrical conductivity general relativistic corrections due to the dragging of reference frames are present, but only in the expression for the electric field. In the case of finite electrical conductivity, however, corrections due both to the spacetime curvature and to the dragging of reference frames are shown to be present in the induction equation. These corrections could be relevant for the evolution of the magnetic fields of pulsars and magnetars. The solutions found, while obtained through some simplifying assumption, reflect a rather general physical configuration and could therefore be used in a variety of astrophysical situations.

Key words: relativity – (magnetohydrodynamics) MHD – stars: neutron – rotation – magnetic fields

1 INTRODUCTION

The investigation of the influence of strongly curved spacetimes on the properties of electromagnetic fields has an interest of its own which is further increased when these effects could be related to a rich observable phenomenology. This coupling between general relativistic effects and electromagnetic fields is expected to be particularly important in the vicinity of neutron stars which are among the most relativistic astrophysical objects and are characterized by very intense magnetic fields (Lamb 1991, Glendenning 1996). A number of different observations indicate that in young neutron stars the surface magnetic field strengths are of the order of $10^{11} - 10^{13}$ G. In some exceptional cases, as those of magnetars, magnetic field strengths $\geq 5 \times 10^{14}$ G are considered responsible for the phenomenology observed in soft gamma-ray repeaters (Duncan & Thompson 1992, Thompson & Duncan 1995). Older neutron stars, observed as recycled pulsars and low mass X-ray binaries, show instead surface magnetic fields that are much weaker $\leq 10^{10}$ G suggesting that these are subject to a decay, even if it is still difficult to establish whether the decay is due to accretion (Geppert & Urpin, 1994; Konar & Bhattacharya, 1997) or to other processes.

In the case of isolated neutron stars, the possibility of magnetic field decay as a result of accretion does not arise, but there are still a number of different ways in which the energy stored in the magnetic can be lost. This can happen either through the emission of electromagnetic (dipole) radiation, through Ohmic decay, through ambipolar diffusion, or through more complicated effects such as "Hall cascades" (see Goldreich and Reisenneger 1992 for a review). The investigation of these scenarios requires combined efforts. On one hand, there is the search for a more precise description of the microphysics of the processes involved, some of which are still not well quantified. On the other hand, attention is paid to a more realistic description of the gravitational effects on the properties of the electromagnetic fields in highly curved spacetimes and this is also the motivation of this work.

The investigation of the general relativistic corrections to the solution of Maxwell equations in the spacetime of a relativistic star has a long history. The initial works of Ginzburg & Ozernoy (1964), Anderson & Cohen (1970) and of Petterson (1974) on the stationary electromagnetic fields in a Schwarzschild spacetime have revealed that the spacetime curvature produces magnetic fields which are generally stronger than their Newtonian counterparts (see also Wasserman & Shapiro 1983 for a subsequent derivation). Sengupta (1995) has reconsidered this problem and also looked for a general relativistic expression for the electric field in the Schwarzschild background of a neutron star. As we will discuss in Section 3.2.2 the method used in his derivation is not entirely correct and the results obtained for the electric field are not solutions of Maxwell equations. More recently, Sengupta has also considered the problem of the Ohmic decay rate in a Schwarzschild spacetime (Sengupta, 1997). His approach is strictly valid only for the region of spacetime external to the star as it does not provide a correct general relativistic description of the electromagnetic fields internal to the star. Within these approximations, however, Sengupta (1997) has pointed out that the effects of intense gravitational field seem to decrease the overall decay rate by a couple of orders of magnitude. The same problem has also been considered in more detail by Geppert, Page and Zannias (2000). Their analysis was aimed at a mathematically consistent solution of Maxwell equations also in the spacetime region internal to the star and makes therefore use of a generic metric for a non-rotating relativistic star. Their results, while confirming a decrease in the typical decay time for the magnetic field, also show that the decay time is smaller but comparable with the one found in flat spacetime.

The general relativistic effects induced by the rotation of the star were first investigated by Muslimov & Tsygan (1992) in the slow rotation approximation. A similar approach was also used by Muslimov and Harding (1997) for the electromagnetic fields external to a rotating magnetized star. Their analysis refers to a charge filled magnetosphere and represents the relativistic extension of the Goldreich-Julian model. Using a different derivation, Prasanna and Gupta (1997) have also investigated the properties of the electromagnetic fields in the magnetosphere of a relativistic rotating neutron star, with special attention being paid to the dynamics of charged test particles.

We here extend and unify all of the above investigations by considering the solution of Maxwell equations in the internal and external background spacetime of a slowly rotating magnetized relativistic star. The star is considered isolated and in vacuum, with a dipolar magnetic field which is not assumed aligned with the axis of rotation. The purpose of this paper is threefold. Firstly, we want to extend previous results to the most general case of a misaligned rotator, providing for this case also the form of the electric field. Secondly, we want to discuss the possible role played by frame dragging effects in the Ohmic decay for an isolated neutron star and estimate its importance. Thirdly, we wish to clarify a few important aspects of the solution of Maxwell equations in the gravitational field of a relativistic star that, when overlooked, have led to incorrect solutions (Sengupta 1995, Prasanna and Gupta 1997). Finally, by providing a rather general solution to the problem (although truncated at the lowest order in the expansion of the angular dependence) we offer a compact reference from which all of the previous results can be easily found in the appropriate limits and which could have practical astrophysical applications.

The paper is organized as follows: in Section 2 we write the general relativistic Maxwell equations in the metric of a slowly rotating star and the form they assume when the electromagnetic fields are those measured in the orthonormal frame of zero angular momentum observers. In Section 3 we find the stationary solutions (i.e. solutions in which the infinite conductivity of the medium prevents a variation in time of the star's magnetic moment) to Maxwell equations outside and inside the misaligned rotating star. For this we consider first the problem in Newtonian gravity and we then extend the results to general relativity within the slow rotation approximation. Section 4 is devoted to the equivalent problem, but in the case in which the magnetic field is not supposed stationary. There, we derive the basic induction equations for the evolution of the inner stellar magnetic field of a misaligned rotating star. Section 5 contains our conclusions and the prospects of future developments.

A number of appendices provide further details about some of the calculations carried out in the main part of the paper. In particular, Appendix A summarizes the components of the electromagnetic tensor in a coordinate basis and in a locally orthonormal tetrad, while Appendix B shows the derivation of the radial eigenfunctions for the electromagnetic fields in terms of Legendre's equation. Appendix C shows the explicit expressions for the surface charges and currents and, finally, Appendix D contains an alternative and equivalent derivation of the equations for the time evolution of magnetic field in terms a vector potential. Throughout, we use a space-like signature $(-,+,+,+)$ and a system of units in which $G = 1 = c$ (However, for those expressions with an astrophysical application we have written the speed of light explicitely.). Greek indices are taken to run from 0 to 3 and Latin indices from 1 to 3; covariant derivatives are denoted with a semi-colon and partial derivatives with a comma.

2 MAXWELL EQUATIONS IN A SLOWLY ROTATING SPACETIME

The difficulties of an analytic solution of the Einstein-Maxwell equations in the proximity of a rotating relativistic star inevitably force us to the use of some approximations. The first approximation comes from neglecting the influence of the electromagnetic field on the metric and by solving Maxwell equations on a given, fixed background⋆. The second approximation is in the specific form of the background metric which we choose to be that of a stationary, axially symmetric system truncated at the first order in the angular velocity Ω. In a coordinate system (ct, r, θ, ϕ), the "slow rotation metric" for a rotating relativistic star is (see, for example, Hartle 1967, Hartle & Thorne 1968, Landau & Lifshitz 1971)

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 - 2\omega(r)r^2\sin^2\theta dt d\phi + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \ , \tag{1}$$

⋆ This is indeed a very good approximation since even for very highly magnetic neutron stars the electromagnetic energy density is much smaller than the gravitational one.

where $\omega(r)$ can be interpreted as the angular velocity of a free falling (inertial) frame and is also known as the Lense-Thirring angular velocity. The radial dependence of ω in the region of spacetime internal to the star has to be found as the solution of the differential equation

$$\frac{1}{r^3}\frac{d}{dr}\left(r^4\bar{j}\frac{d\bar{\omega}}{dr}\right)+4\frac{d\bar{j}}{dr}\bar{\omega}=0\ , \tag{2}$$

where we have defined

$$\bar{j}\equiv e^{-(\Phi+\Lambda)}\ , \tag{3}$$

and where

$$\bar{\omega}\equiv\Omega-\omega\ , \tag{4}$$

is the angular velocity of the fluid as measured from the local free falling (inertial) frame. In the vacuum region of spacetime external to the star, on the other hand, $\omega(r)$ is given by the simple algebraic expression

$$\omega(r)\equiv\frac{d\phi}{dt}=-\frac{g_{0\phi}}{g_{\phi\phi}}=\frac{2J}{r^3}\ , \tag{5}$$

where $J=I(M,R)\Omega$ is the total angular momentum of metric source as measured from infinity and $I(M,R)$ its momentum of inertia (see Miller 1977 for a discussion of I and its numerical calculation). Outside the star, the metric (1) is completely known and explicit expressions for the other metric functions are given by

$$e^{2\Phi(r)}\equiv\left(1-\frac{2M}{r}\right)=e^{-2\Lambda(r)}\ , \qquad\qquad r>R\ , \tag{6}$$

where M and R are the mass and radius of the star as measured from infinity.

An important aspect, often overlooked in the literature, should now be underlined. The metric (1) is the simplest metric that provides all of the most important general relativistic corrections to the solution of the Maxwell equations in the gravitational field of a rotating relativistic star. The use of a Schwarzschild metric in place of (1) (Sengupta 1995, 1997) is potentially very dangerous. Firstly, and as pointed out by Geppert et al. (2000), a Schwarzschild metric allows for a proper treatment of the electromagnetic fields only in the spacetime region external to the star and leaves unsolved the problem of a matching of the external electromagnetic fields with the internal ones. Secondly, and despite different claims (Sengupta 1997), a Schwarzschild metric is intrinsically inadequate to describe physical systems such as pulsars in which the coupling of electromagnetic fields and rotation is a key feature. Note, on the other hand, that using the slow-rotation approximation gives rather accurate results for all pulsar periods so far observed. The metric (1) has coefficients each of which is the lowest-order term of a series expansion in ascending powers of Ω. Comparing the magnitude of the neglected higher order terms with that of the one retained in each case, gives ratios of the order $R^3\Omega^2/GM$ which is smaller than 10% even for the fastest-known millisecond pulsar PSR 1937+214.

The general form of the first pair of general relativistic Maxwell equations is given by

$$3!F_{[\alpha\beta,\gamma]}=2\left(F_{\alpha\beta,\gamma}+F_{\gamma\alpha,\beta}+F_{\beta\gamma,\alpha}\right)=0\ . \tag{7}$$

where $F_{\alpha\beta}$ is the electromagnetic field tensor expressing the strict connection between the electric and magnetic four-vector fields $E^\alpha,\ B^\alpha$. For an observer with four-velocity u^α, the covariant components of the electromagnetic tensor are given by (Lichnerowicz 1967; Ellis 1973)

$$F_{\alpha\beta}\equiv 2u_{[\alpha}E_{\beta]}+\eta_{\alpha\beta\gamma\delta}u^\gamma B^\delta\ . \tag{8}$$

where $T_{[\alpha\beta]}\equiv\frac{1}{2}(T_{\alpha\beta}-T_{\beta\alpha})$ and $\eta_{\alpha\beta\gamma\delta}$ is the pseudo-tensorial expression for the Levi-Civita symbol $\epsilon_{\alpha\beta\gamma\delta}$ (Stephani 1990)

$$\eta^{\alpha\beta\gamma\delta}=-\frac{1}{\sqrt{-g}}\epsilon_{\alpha\beta\gamma\delta}\ , \qquad\qquad \eta_{\alpha\beta\gamma\delta}=\sqrt{-g}\epsilon_{\alpha\beta\gamma\delta}\ , \tag{9}$$

with $g\equiv\det|g_{\alpha\beta}|=-e^{2(\Phi+\Lambda)}r^4\sin^2\theta$ for the metric (1). A useful class of observers is represented by the "zero angular momentum observers" or ZAMOs (Bardeen, Press & Teukolsky 1972). These are observers that are locally stationary (i.e. at fixed values of r and θ) but who are "dragged" into rotation with respect to a reference frame fixed with respect to distant observers. At first order in Ω they have four-velocity components given by

$$(u^\alpha)_{\rm ZAMO}\equiv e^{-\Phi(r)}\Big(1,0,0,\omega\Big)\ ; \qquad\qquad (u_\alpha)_{\rm ZAMO}\equiv e^{\Phi(r)}\Big(-1,0,0,0\Big)\ . \tag{10}$$

In the coordinate system (ct,r,θ,ϕ) and with the definition (8) referred to the observers (10), the first pair of Maxwell equations (7) take then the form (see Appendix A for the explicit expressions of the electromagnetic tensor)

$$\left(e^\Lambda r^2\sin\theta B^i\right)_{,i}=0\ , \tag{11}$$

$$\left(e^\Lambda r^2\sin\theta\right)\frac{\partial B^r}{\partial t}=e^\Phi\left(E_{\theta,\phi}-E_{\phi,\theta}\right)-\left(\omega e^\Lambda r^2\sin\theta\right)B^r_{\ ,\phi}\ , \tag{12}$$

$$\left(e^\Lambda r^2\sin\theta\right)\frac{\partial B^\theta}{\partial t}=\left(E_\phi\, e^\Phi\right)_{,r}-e^\Phi E_{r,\phi}-\left(\omega e^\Lambda r^2\sin\theta\right)B^\theta_{\ ,\phi}\ , \tag{13}$$

$$\left(e^\Lambda r^2\sin\theta\right)\frac{\partial B^\phi}{\partial t}=-\left(E_\theta\, e^\Phi\right)_{,r}+e^\Phi E_{r,\theta}+\sin\theta\left(\omega e^\Lambda r^2 B^r\right)_{,r}+\omega e^\Lambda r^2\left(\sin\theta B^\theta\right)_{,\theta}\ . \tag{14}$$

The general form of the second pair of Maxwell equations is given by

$$F^{\alpha\beta}{}_{;\beta} = 4\pi J^{\alpha} \tag{15}$$

where the four-current J^{α} is a sum of convection and conduction currents

$$J^{\alpha} = \rho_e w^{\alpha} + j^{\alpha} \ , \qquad\qquad j^{\alpha} w_{\alpha} \equiv 0 \ , \tag{16}$$

with $\boldsymbol{w}$ being the conductor four-velocity and ρ_e the proper charge density. If the conduction current j^{α} is carried by the electrons† with electrical conductivity σ, Ohm's law can then be written as

$$j_{\alpha} = \sigma F_{\alpha\beta} w^{\beta} \ , \tag{17}$$

while a more general expression can be found in Ahmedov (1999). We can now rewrite the second pair of Maxwell equations as

$$\left(e^{\Lambda} r^2 \sin\theta E^i\right)_{,i} = 4\pi e^{\Phi+\Lambda} r^2 \sin\theta J^0 \ , \tag{18}$$

$$e^{\Phi}\left(B_{\phi\ ,\theta} - B_{\theta\ ,\phi}\right) - \left(\omega e^{\Lambda} r^2 \sin\theta\right) E^r{}_{,\phi} = \left(e^{\Lambda} r^2 \sin\theta\right) \frac{\partial E^r}{\partial t} + 4\pi e^{\Phi+\Lambda} r^2 \sin\theta J^r \ , \tag{19}$$

$$e^{\Phi} B_{r,\phi} - \left(e^{\Phi} B_{\phi}\right)_{,r} - \left(\omega e^{\Lambda} r^2 \sin\theta\right) E^{\theta}{}_{,\phi} = \left(e^{\Lambda} r^2 \sin\theta\right) \frac{\partial E^{\theta}}{\partial t} + 4\pi e^{\Phi+\Lambda} r^2 \sin\theta J^{\theta} \ , \tag{20}$$

$$\left(e^{\Phi} B_{\theta}\right)_{,r} - e^{\Phi} B_{r\ ,\theta} + \sin\theta \left(\omega e^{\Lambda} r^2 E^r\right)_{,r} + \omega e^{\Lambda} r^2 \left(\sin\theta E^{\theta}\right)_{,\theta} = \left(e^{\Lambda} r^2 \sin\theta\right) \frac{\partial E^{\phi}}{\partial t} + 4\pi e^{\Phi+\Lambda} r^2 \sin\theta J^{\phi} \ . \tag{21}$$

Maxwell equations assume a familiar flat-spacetime form when projected onto a locally orthonormal tetrad. In principle such tetrad is arbitrary, but in the case of a relativistic rotating metric source a "natural" choice is offered by the tetrad carried by the ZAMOs. Using (10) we find that the components of the tetrad $\{\boldsymbol{e}_{\hat{\mu}}\} = (\boldsymbol{e}_{\hat{0}}, \boldsymbol{e}_{\hat{r}}, \boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\phi}})$ carried by a ZAMO observer are

$$\boldsymbol{e}_{\hat{0}}^{\alpha} = e^{-\Phi}\Big(1, 0, 0, \omega\Big) \ , \tag{22}$$

$$\boldsymbol{e}_{\hat{r}}^{\alpha} = e^{-\Lambda}\Big(0, 1, 0, 0\Big) \ , \tag{23}$$

$$\boldsymbol{e}_{\hat{\theta}}^{\alpha} = \frac{1}{r}\Big(0, 0, 1, 0\Big) \ , \tag{24}$$

$$\boldsymbol{e}_{\hat{\phi}}^{\alpha} = \frac{1}{r\sin\theta}\Big(0, 0, 0, 1\Big) \ . \tag{25}$$

The 1-forms $\{\boldsymbol{\omega}^{\hat{\mu}}\} = (\boldsymbol{\omega}^{\hat{0}}, \boldsymbol{\omega}^{\hat{r}}, \boldsymbol{\omega}^{\hat{\theta}}, \boldsymbol{\omega}^{\hat{\phi}})$, corresponding to this tetrad have instead components

$$\boldsymbol{\omega}_{\alpha}^{\hat{0}} = e^{\Phi}\Big(1, 0, 0, 0\Big) \ , \tag{26}$$

$$\boldsymbol{\omega}_{\alpha}^{\hat{r}} = e^{\Lambda}\Big(0, 1, 0, 0\Big) \ , \tag{27}$$

$$\boldsymbol{\omega}_{\alpha}^{\hat{\theta}} = r\Big(0, 0, 1, 0\Big) \ , \tag{28}$$

$$\boldsymbol{\omega}_{\alpha}^{\hat{\phi}} = r\sin\theta\Big(-\omega, 0, 0, 1\Big) \ . \tag{29}$$

We can now rewrite Maxwell equations (11)–(14) and (18)–(21) in the ZAMO reference frame by contracting (7) and (15) with (22)–(25) and (26)–(29). After some lengthy but straightforward algebra, we obtain Maxwell equations in the more useful form

$$\sin\theta\left(r^2 B^{\hat{r}}\right)_{,r} + e^{\Lambda} r \left(\sin\theta B^{\hat{\theta}}\right)_{,\theta} + e^{\Lambda} r B^{\hat{\phi}}_{,\phi} = 0 \ , \tag{30}$$

$$(r\sin\theta)\frac{\partial B^{\hat{r}}}{\partial t} = e^{\Phi}\left[E^{\hat{\theta}}_{,\phi} - \left(\sin\theta E^{\hat{\phi}}\right)_{,\theta}\right] - (\omega r \sin\theta) B^{\hat{r}}_{,\phi} \ , \tag{31}$$

$$\left(e^{\Lambda} r\sin\theta\right)\frac{\partial B^{\hat{\theta}}}{\partial t} = -e^{\Phi+\Lambda} E^{\hat{r}}_{,\phi} + \sin\theta\left(r e^{\Phi} E^{\hat{\phi}}\right)_{,r} - \left(\omega e^{\Lambda} r\sin\theta\right) B^{\hat{\theta}}_{,\phi} \ , \tag{32}$$

$$\left(e^{\Lambda} r\right)\frac{\partial B^{\hat{\phi}}}{\partial t} = -\left(r e^{\Phi} E^{\hat{\theta}}\right)_{,r} + e^{\Phi+\Lambda} E^{\hat{r}}_{,\theta} + \sin\theta\left(\omega r^2 B^{\hat{r}}\right)_{,r} + \omega e^{\Lambda} r\left(\sin\theta B^{\hat{\theta}}\right)_{,\theta} \tag{33}$$

† This is a reasonable assumption if the neutron star has a temperature such that the atomic nuclei are frozen into a lattice and the electrons form a completely relativistic, and degenerate gas.

and

$$\begin{aligned}
\sin\theta\left(r^2 E^{\hat r}\right)_{,r} + e^{\Lambda} r\left(\sin\theta E^{\hat\theta}\right)_{,\theta} + e^{\Lambda} r E^{\hat\phi}_{,\phi} &= 4\pi e^{\Lambda} r^2 \sin\theta J^{\hat t} \ , && (34)\\
e^{\Phi}\left[\left(\sin\theta B^{\hat\phi}\right)_{,\theta} - B^{\hat\theta}_{,\phi}\right] - (\omega r\sin\theta)\, E^{\hat r}_{,\phi} &= (r\sin\theta)\,\frac{\partial E^{\hat r}}{\partial t} + 4\pi e^{\Phi} r\sin\theta J^{\hat r} \ , && (35)\\
e^{\Phi+\Lambda} B^{\hat r}_{,\phi} - \sin\theta\left(r\, e^{\Phi} B^{\hat\phi}\right)_{,r} - \left(\omega e^{\Lambda} r\sin\theta\right) E^{\hat\theta}_{,\phi} &= \left(e^{\Lambda} r\sin\theta\right)\frac{\partial E^{\hat\theta}}{\partial t} + 4\pi e^{\Phi+\Lambda} r\sin\theta J^{\hat\theta} \ , && (36)\\
\left(e^{\Phi} r B^{\hat\theta}\right)_{,r} - e^{\Phi+\Lambda} B^{\hat r}_{,\theta} + \sin\theta\left(\omega r^2 E^{\hat r}\right)_{,r} + \omega e^{\Lambda} r\left(\sin\theta E^{\hat\theta}\right)_{,\theta} &= \left(e^{\Lambda} r\right)\frac{\partial E^{\hat\phi}}{\partial t} + 4\pi e^{\Phi+\Lambda} r J^{\hat\phi} + 4\pi e^{\Lambda}\omega r^2 \sin\theta J^{\hat t} \ . && (37)
\end{aligned}$$

Equations (35)–(37) can now be rewritten in a more convenient form. Taking our conductor to be the star with four-velocity components

$$w^{\alpha} \equiv e^{-\Phi(r)}\Big(1,0,0,\Omega\Big) \ , \qquad\qquad w_{\alpha} \equiv e^{\Phi(r)}\left(-1,0,0,\frac{\bar\omega r^2\sin^2\theta}{e^{2\Phi(r)}}\right) \ , \tag{38}$$

we can use Ohm's law (17) to derive the following explicit components of $J^{\hat\alpha}$ in the ZAMO frame

$$J^{\hat t} = \rho_e + \sigma\frac{\bar\omega r\sin\theta}{e^{\Phi}} E^{\hat\phi} \ , \tag{39}$$

$$J^{\hat r} = \sigma\left(E^{\hat r} - \frac{\bar\omega r\sin\theta}{e^{\Phi}} B^{\hat\theta}\right) \ , \tag{40}$$

$$J^{\hat\theta} = \sigma\left(E^{\hat\theta} + \frac{\bar\omega r\sin\theta}{e^{\Phi}} B^{\hat r}\right) \ , \tag{41}$$

$$J^{\hat\phi} = \sigma E^{\hat\phi} + \frac{\bar\omega r\sin\theta}{e^{\Phi}}\rho_e \ . \tag{42}$$

Next, we discuss a few assumptions that are going to be used hereafter. Firstly, we assume there is no matter outside the star so that the conductivity $\sigma = 0$ for $r > R$ and that $\sigma \neq 0$ only in a shell with $R_{IN} \le r \le R$ (e.g. the neutron star crust). Secondly, we consider σ to be uniform within this shell (Note that this might be incorrect in the outermost layers of the neutron star but is a rather good approximation on the crust as a whole.). Thirdly, we ignore the contributions coming from displacement currents. The latter could, in principle, be relevant in the evolution of the electromagnetic fields, but their effects are negligible on timescales that are long as compared with the electromagnetic waves crossing time. In view of this, we will neglect in (35)–(37) all terms involving time derivatives of the electric field and use Ohm's law to rewrite equations (35) and (36) as

$$r\sin\theta E^{\hat r} = \frac{1}{4\pi\sigma}\left[\left(\sin\theta B^{\hat\phi}\right)_{,\theta} - B^{\hat\theta}_{,\phi}\right] + \mathcal{O}(\Omega) \ , \tag{43}$$

$$e^{\Lambda} r\sin\theta E^{\hat\theta} = \frac{e^{-\Phi}}{4\pi\sigma}\left[e^{\Phi+\Lambda} B^{\hat r}_{,\phi} - \sin\theta\left(r e^{\Phi} B^{\hat\phi}\right)_{,r}\right] + \mathcal{O}(\Omega) \ . \tag{44}$$

Substituting (43) and (44) in the left hand sides of equations (35)–(37) eliminates the dependence from the electric field and yields

$$\begin{aligned}
\left[\left(\sin\theta B^{\hat\phi}\right)_{,\theta} - B^{\hat\theta}_{,\phi}\right] - \frac{\omega e^{-\Phi}}{4\pi\sigma}\left[\left(\sin\theta B^{\hat\phi}\right)_{,\theta} - B^{\hat\theta}_{,\phi}\right]_{,\phi} &= 4\pi r\sin\theta J^{\hat r} \ , && (45)\\
e^{\Phi+\Lambda} B^{\hat r}_{,\phi} - \sin\theta\left(e^{\Phi} r B^{\hat\phi}\right)_{,r} - \frac{\omega e^{-\Phi}}{4\pi\sigma}\left[e^{\Phi+\Lambda} B^{\hat r}_{,\phi} - \sin\theta\left(e^{\Phi} r B^{\hat\phi}\right)_{,r}\right]_{,\phi} &= 4\pi e^{\Phi+\Lambda} r\sin\theta J^{\hat\theta} \ , && (46)\\
\left(e^{\Phi} r B^{\hat\theta}\right)_{,r} - e^{\Phi+\Lambda} B^{\hat r}_{,\theta} + \frac{1}{4\pi}\left\{\frac{\omega r}{\sigma}\left[\left(\sin\theta B^{\hat\phi}\right)_{,\theta} - B^{\hat\theta}_{,\phi}\right]\right\}_{,r} + \qquad & && \\
\frac{\omega e^{-\Phi}}{4\pi\sigma}\left[e^{\Phi+\Lambda} B^{\hat r}_{,\phi} - \sin\theta\left(e^{\Phi} r B^{\hat\phi}\right)_{,r}\right]_{,\theta} &= 4\pi e^{\Phi+\Lambda} r J^{\hat\phi} + 4\pi e^{\Lambda}\omega r^2 \sin\theta \rho_e \ . && (47)
\end{aligned}$$

3 STATIONARY SOLUTIONS TO MAXWELL EQUATIONS

In this Section we will look for stationary solutions of the Maxwell equation, i.e. for solutions in which we assume that the magnetic moment of the magnetic star does not vary in time as a result of the infinite conductivity of the medium. Note that this does not mean the electromagnetic fields are independent of time. As a result of the misalignment between the magnetic dipole $\boldsymbol{\mu}$ and the angular velocity vector $\boldsymbol{\Omega}$, in fact, both the magnetic and the electric fields will posses a *periodic* time dependence produced by the precession of $\boldsymbol{\mu}$ around $\boldsymbol{\Omega}$ (see Fig. 1).

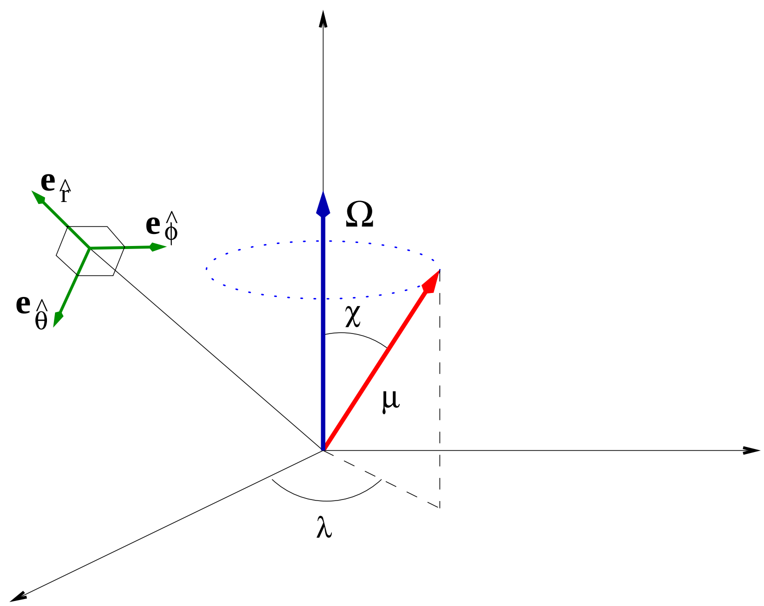

Figure 1. Schematic representation of a misaligned rotator. Here $(\boldsymbol{e}_{\hat{0}}, \boldsymbol{e}_{\hat{r}}, \boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\phi}})$ is a local orthonormal frame, $\boldsymbol{\mu}$ is the magnetic dipole moment of the star, χ is the inclination angle relative to the rotation axis, and λ the instantaneous azimuthal position.

3.1 Rotating Magnetized Conductor in a Minkowski Spacetime

Before looking at the problem of a magnetized rotating conductor in a rotating spacetime, it is useful to start with a simpler analogous configuration: that of a rotating magnetized conductor in a Minkowski (flat) spacetime. This will provide important insight for the search of general relativistic solutions and useful limits against which match the fully relativistic solutions.

Consider therefore a conducting magnetized sphere of radius R rotating at angular velocity Ω, and with the magnetic four-vector field $\boldsymbol{B}$ being uniform (in radius) inside the sphere and dipolar outside (This is a simple but instructive example.). Because of discontinuities in the fields across the surface of the sphere we will refer to as *interior solutions* those solutions valid within the radial range $R_{_{IN}} \leq r \leq R$, and to as *exterior solutions* those valid in the range $R < r \leq \infty$.

3.1.1 Interior Solution

The interior solution for the electromagnetic fields of a magnetized sphere with magnetic moment aligned with the rotation axis was found by Ruffini and Treves in 1973 (Ruffini & Treves 1973). Extending it to the case of a misaligned rotator we obtain

$$B^{\hat{r}} = \frac{2\mu}{R^3}\left(\cos\chi\cos\theta + \sin\chi\sin\theta\cos\lambda\right) \ , \tag{48}$$

$$B^{\hat{\theta}} = -\frac{2\mu}{R^3}\left(\cos\chi\sin\theta - \sin\chi\cos\theta\cos\lambda\right) \ , \tag{49}$$

$$B^{\hat{\phi}} = -\frac{2\mu}{R^3}\sin\chi\sin\lambda \ , \tag{50}$$

where $\boldsymbol{\mu}$ is the magnetic dipole moment of the star, χ is the inclination angle of the magnetic moment relative to the rotation axis and $\lambda(t) \equiv \phi - \Omega t$ is the instantaneous azimuthal position (see Fig. 1).

The expressions for the components of the electric field are very simple to derive when one assumes that the sphere is a "perfect conductor" (i.e. $\sigma \to \infty$) and there are no conduction currents inside the sphere. In this case, Ohm's law can be used to obtain

$$E^{\hat{r}} = \frac{\Omega r\sin\theta}{c}B^{\hat{\theta}} = -\frac{2\mu\Omega r\sin\theta}{cR^3}\left(\cos\chi\sin\theta - \sin\chi\cos\theta\cos\lambda\right) \ , \tag{51}$$

$$E^{\hat{\theta}} = -\frac{\Omega r\sin\theta}{c}B^{\hat{r}} = -\frac{2\mu\Omega r\sin\theta}{cR^3}\left(\cos\chi\cos\theta + \sin\chi\sin\theta\cos\lambda\right) \ , \tag{52}$$

$$E^{\hat{\phi}} = 0 \ . \tag{53}$$

3.1.2 Exterior Solution

The solution to this problem, i.e. to the form of the electromagnetic fields external to a misaligned rotating magnetized sphere, was found in 1955 by Deutsch (Deutsch, 1955). The full solutions are complicated expressions involving spherical Bessel functions of the third kind, but they become much simpler when truncated at the lowest order. The magnetic field components, in particular, have the form

$$B^{\hat{r}} = \frac{2\mu}{r^3}\left(\cos\chi\cos\theta + \sin\chi\sin\theta\cos\lambda\right)\ , \tag{54}$$

$$B^{\hat{\theta}} = \frac{\mu}{r^3}\left(\cos\chi\sin\theta - \sin\chi\cos\theta\cos\lambda\right)\ , \tag{55}$$

$$B^{\hat{\phi}} = \frac{\mu}{r^3}\sin\chi\sin\lambda\ , \tag{56}$$

while the corresponding electric fields are given by

$$E^{\hat{r}} = -\frac{\mu\Omega R^2}{cr^4}\left[\cos\chi(3\cos^2\theta - 1) + 3\sin\chi\cos\lambda\sin\theta\cos\theta\right]\ , \tag{57}$$

$$E^{\hat{\theta}} = -\frac{\mu\Omega}{cr^2}\left\{\frac{2R^2}{r^2}\cos\chi\sin\theta\cos\theta + \sin\chi\left[1 - \frac{R^2}{r^2}(\cos^2\theta - \sin^2\theta)\right]\cos\lambda\right\}\ , \tag{58}$$

$$E^{\hat{\phi}} = \frac{\mu\Omega}{cr^2}\sin\chi\cos\theta\sin\lambda\left(1 - \frac{R^2}{r^2}\right)\ . \tag{59}$$

Three interesting features of solutions (54)–(56) and (57)–(59) should be noticed. The first one is given by the periodic time modulation introduced by the precession of the magnetic moment and which disappears when the dipole is aligned, i.e. for $\chi = 0$. The second feature is that, as one might have expected on the basis of symmetry considerations, the toroidal components of the external electromagnetic fields are just a by-product of the misalignment between the rotation axis and the magnetic dipole and again disappear when $\chi = 0$. Finally, the third relevant feature is the appearance of an electric field of $\mathcal{O}(\Omega)$ introduced by the rotation of the sphere and whose quadrupolar part [i.e. $\propto (3\cos^2\theta - 1)$] is present also in the case of an aligned rotator. As we will see in Section 3.2.2, where we study the analogous problem in a slowly rotating spacetime, an additional contribution of $\mathcal{O}(\omega)$[‡] to the form of the external electric field will be introduced by the general relativistic frame dragging effect[§].

3.2 Rotating Magnetized Conductor in a Slowly Rotating Spacetime

We now consider the general relativistic analogue of the problem in Section 3.1 and look for a solution of Maxwell equations (30)–(33) and (34)–(37) assuming that magnetic field of the star is dipolar. To simply the search for a solution we look for separable solutions of Maxwell equations in the form

$$B^{\hat{r}}(r,\theta,\phi,\chi,t) = F(r)\Psi_1(\theta,\phi,\chi,t)\ , \tag{60}$$

$$B^{\hat{\theta}}(r,\theta,\phi,\chi,t) = G(r)\Psi_2(\theta,\phi,\chi,t)\ , \tag{61}$$

$$B^{\hat{\phi}}(r,\theta,\phi,\chi,t) = H(r)\Psi_3(\theta,\phi,\chi,t)\ , \tag{62}$$

where $F(r), G(r)$, and $H(r)$ will account for the relativistic corrections due to a curved background spacetime.

A considerable simplification comes from the fact that, at first order in Ω, the solutions for the electromagnetic fields will not acquire general relativistic corrections to their angular dependence. We therefore expect that, as for the case of the Deutsch solution, the general expressions for the angular eigenfunctions Ψ_i, with $i = 1,\ldots,3$, will have a complicated angular dependence expressed in terms of spherical Bessel functions of the third kind. This is however over-complicated and for most of the astrophysical applications it would sufficient to know the form at the lowest order which can be known by requiring that the solutions match the lowest order solution for a misaligned rotating

‡ Hereafter we will refer to as $\mathcal{O}(\omega)$ any quantity that is the result of the dragging of reference frames and that is therefore $\propto g_{0\phi}$.

§ Note that an electric field induced by the rotation of the star must appear also in the general relativistic case. This is not present in the solution proposed by Prasanna and Gupta 1997, where the external electric field is only of $\mathcal{O}(\omega)$ and the radial dependence does not contain higher order terms in M/r.

dipole in flat spacetime. In this case, then, we obtain

$$\Psi_1(\theta,\phi,\chi,t) = \cos\chi\cos\theta + \sin\chi\sin\theta\cos\lambda(t)\ , \tag{63}$$

$$\Psi_2(\theta,\phi,\chi,t) = \cos\chi\sin\theta - \sin\chi\cos\theta\cos\lambda(t)\ , \tag{64}$$

$$\Psi_3(\phi,\chi,t) = \sin\chi\sin\lambda(t)\ , \tag{65}$$

which also satisfy the following useful relations

$$\Psi_{1,\theta} = -\Psi_2\ , \qquad \Psi_{1,\phi} = -\Psi_3\sin\theta\ , \qquad \Psi_{2,\theta} = \Psi_1\ , \qquad \Psi_{2,\phi} = \Psi_3\cos\theta\ . \tag{66}$$

Maxwell equations (30), (45)–(47) with the ansatz (60)–(62), yield the following set of equations

$$\left[\left(r^2F\right)_{,r} + 2e^{\Lambda}rG\right]\sin\theta\left(\cos\chi\cos\theta + \sin\chi\sin\theta\cos\lambda\right) + e^{\Lambda}r\left(H-G\right)\sin\chi\cos\lambda = 0\ , \tag{67}$$

$$(H-G)\cos\theta\sin\chi\left[\sin\lambda - \frac{\omega e^{-\Phi}}{4\pi\sigma}\cos\lambda\right] = 4\pi r\sin\theta J^{\hat{r}}\ , \tag{68}$$

$$\left[\left(re^{\Phi}H\right)_{,r} + e^{\Phi+\Lambda}F\right]\sin\theta\sin\chi\left[\sin\lambda - \frac{\omega e^{-\Phi}}{4\pi\sigma}\cos\lambda\right] = -4\pi e^{\Phi+\Lambda}r\sin\theta J^{\hat{\theta}}\ , \tag{69}$$

$$\begin{aligned}\left\{\left[\frac{\omega r}{4\pi\sigma}\left(H-G\right)\right]_{,r} - \frac{\omega r}{4\pi\sigma}\Phi_{,r}\left(G-H\right) - \frac{\omega e^{-\Phi}}{4\pi\sigma}\left[\left(re^{\Phi}H\right)_{,r} + e^{\Phi+\Lambda}F\right]\right\}\cos\theta\sin\chi\sin\lambda & \\ + \left[\left(re^{\Phi}G\right)_{,r} + e^{\Phi+\Lambda}F\right]\left(\cos\chi\sin\theta - \sin\chi\cos\theta\cos\lambda\right) &= 4\pi e^{\Phi+\Lambda}rJ^{\hat{\phi}}\ .\end{aligned} \tag{70}$$

Next, we will distinguish between an external vacuum solution to Maxwell equations (for which fully analytic solution can be given) from the interior non-vacuum solution. Since we are treating the interior of the star as a perfect conductor and the exterior of the star as vacuum, we can impose $J^{\hat{r}} = J^{\hat{\theta}} = J^{\hat{\phi}} = 0$ in (67)–(70) and obtain as Maxwell equations for the radial part of the magnetic field

$$\left(r^2F\right)_{,r} + 2e^{\Lambda}rG = 0\ , \tag{71}$$

$$\left(re^{\Phi}H\right)_{,r} + e^{\Phi+\Lambda}F = 0\ , \tag{72}$$

$$H - G = 0\ . \tag{73}$$

Note a first important result in the system of equations (71)–(73). In the case of stationary electromagnetic fields, the general relativistic frame dragging effect does not introduce a correction to the radial eigenfunctions of the magnetic fields. In other words, in the case of infinite conductivity and as far as the magnetic field is concerned, the study of Maxwell equations in a slow rotation metric provides no additional information with respect to a non-rotating metric. The frame dragging effects are therefore expected to appear at $\mathcal{O}(\omega^2)$.

3.2.1 Interior solution

Limiting the solution to an inner radius R_{IN} removes the problem of suitable boundary conditions for $r \to 0$, and reflects the basic ignorance of the properties of magnetic fields in the interior regions of neutron stars.

It is important to notice how the system of equations (71)–(73) combines information about the structure and physics of the star (through the metric functions Φ and Λ) with information about the microphysics of the magnetic field (through the radial eigenfunctions F and G). As a result, a relativistic solution for the interior electromagnetic field cannot be given independently of a self-consistent solution of Einstein equations for the structure of the star. In practice, to calculate a generic solution to (71)–(73), it is necessary to start with a (realistic) equation of state and obtain a full solution of the relativistic star. Once the latter is known, the system of equations (71)–(73) can be solved for a magnetic field which is consistent with the star's structure and corresponds to a magnetic configuration of some astrophysical interest. (This is what done, for instance, by Gupta et al. 1998 in the case of an internal dipolar magnetic field).

Alternatively, one might specify a magnetic field configuration and look for a compatible equation of state for the stellar structure (This is a less satisfactory way to proceed but one which is useful to get insight into the problem.). In this case, the simplest possible solution to the system (71)–(73) is one in which the magnetic field is constant throughout the region of the star of interest and is therefore the general relativistic analogue of the solution presented in 3.1.1. In this case, then

$$F = \frac{C_1}{R^3}\mu\ , \qquad G = -\frac{e^{-\Lambda}C_1}{R^3}\mu = -e^{-\Lambda}F\ , \tag{74}$$

where C_1 is an arbitrary constant whose value can be determined after imposing the continuity across the star surface of $B^{\hat{r}}$.

We can now check whether the solution (74) is physically possible. Using (74) in the system of equations (71)–(73) requires that the metric functions satisfy the condition

$$\left(re^{\Phi-\Lambda}\right)_{,r} - e^{\Phi+\Lambda} = 0 \,. \tag{75}$$

Recalling now that Einstein equations for a spherical star yield

$$e^{2\Lambda(r)} = \left(1 - \frac{2m(r)}{r}\right)^{-1} , \qquad r \le R \,, \tag{76}$$

with $m(r) = 4\pi \int_0^R r^2 \rho(r) dr$, and $\rho(r)$ being the total energy density, we can rewrite (75) as

$$\Phi_{,r} = e^{2\Lambda}\left(\frac{m + rm_{,r}}{r^2}\right) \,. \tag{77}$$

On other hand, the solution of the Einstein equations for the interior of a relativistic spherical star (i.e. the solution of the Tolmann-Oppenheimer-Volkoff equations; Tolmann, 1939; Oppenheimer & Volkoff, 1939) requires that

$$\Phi_{,r} = e^{2\Lambda}\left(\frac{m + 4\pi r^3 P}{r^2}\right) , \tag{78}$$

where P is the isotropic pressure. The comparison of (78) with (77) shows that the general relativistic uniform magnetic field solution

$$B^{\hat{r}} = \frac{1}{R^3}\left(\cos\chi\cos\theta + \sin\chi\sin\theta\cos\lambda\right) C_1\mu \,, \tag{79}$$

$$B^{\hat{\theta}} = -\frac{e^{-\Lambda}}{R^3}\left(\cos\chi\sin\theta - \sin\chi\cos\theta\cos\lambda\right) C_1\mu \,, \tag{80}$$

$$B^{\hat{\phi}} = -\frac{e^{-\Lambda}}{R^3}\left(\sin\chi\sin\lambda\right) C_1\mu \,, \tag{81}$$

is possible only for the (unrealistic) case of the "stiff matter" equation of state $P = \rho$.

The corresponding form of the internal electric field is also straightforward to derive in the case of no conduction currents. In this case, in fact, Ohm's law (39) and (40) yield the simple expressions

$$E^{\hat{r}} = \frac{\bar{\omega} r\sin\theta}{ce^{\Phi}} B^{\hat{\theta}} = -\frac{e^{-(\Phi+\Lambda)} r\sin\theta}{cR^3} \bar{\omega}\left(\cos\chi\sin\theta - \sin\chi\cos\theta\cos\lambda\right) C_1\mu \,, \tag{82}$$

$$E^{\hat{\theta}} = -\frac{\bar{\omega} r\sin\theta}{ce^{\Phi}} B^{\hat{r}} = -\frac{e^{-\Phi} r\sin\theta}{cR^3} \bar{\omega}\left(\cos\chi\cos\theta + \sin\chi\sin\theta\cos\lambda\right) C_1\mu \,, \tag{83}$$

$$E^{\hat{\phi}} = 0 \,, \tag{84}$$

where we have taken into account that $\rho_e = \mathcal{O}(\omega)$ and that the contribution proportional to $\bar{\omega}\rho_e$ is therefore of higher order¶. Note that, apart for red-shift correction proportional to $e^{-\Phi}$, equations (82)–(84) are the same as (51)–(53) with Ω being replaced by the effective fluid velocity measured by a free falling observer $\bar{\omega}$. The internal charge density corresponding to the electrical field (82)–(84) can be calculated after imposing that [cf. eq. (34)]

$$\rho_e = \frac{1}{4\pi}\left[\frac{e^{-\Lambda}}{r^2}\left(r^2 E^{\hat{r}}\right)_{,r} + \frac{1}{r\sin\theta}\left(\sin\theta E^{\hat{\theta}}\right)_{,\theta}\right] \,. \tag{85}$$

Using now expressions (82)–(84), we easily obtain

$$\rho_e = \frac{1}{4\pi}\left\{\left[3e^{-\Phi}\bar{\omega} - \frac{e^{-\Lambda}}{r^2}\left(e^{-(\Phi+\Lambda)}\bar{\omega} r^3\right)_{,r}\right]\frac{\sin\theta}{cR^3}\left(\cos\chi\sin\theta - \sin\chi\cos\theta\cos\lambda\right) - \frac{2e^{-\Phi}}{cR^3}\bar{\omega}\cos\chi\right\} C_1\mu \,. \tag{86}$$

3.2.2 *Exterior Solution*

The exterior solution for the magnetic field is simplified by the knowledge of explicit analytic expressions for the metric functions Φ and Λ. In particular, after defining $N \equiv e^{\Phi} = e^{-\Lambda} = (1 - 2M/r)^{1/2}$, the system (71)–(73) can be written as a single, second-order ordinary differential equation for the unknown function F

$$\frac{d}{dr}\left[\left(1 - \frac{2M}{r}\right)\frac{d}{dr}\left(r^2 F\right)\right] - 2F = 0 \,. \tag{87}$$

¶ Prasanna and Gupta (1997) have used the assumption of infinite conductivity also for the matter outside the neutron star. We note that their expressions for the electric fields do not contain the (important) contribution of $O(\Omega)$ and the radial component does not seem to satisfy Ohm's law.

Introducing now the new variable $x \equiv 1 - r/M$, equation (87) can be written as

$$\frac{d}{dx}\left\{\left(\frac{1+x}{1-x}\right)\frac{d}{dx}\left[(1-x)^2 F\right]\right\} + 2F = 0 \,. \tag{88}$$

Equation (88) is an example from a class of equations which can be solved in terms of the Legendre functions of the second kind Q_ℓ (see Appendix B for details on the derivation of the solution). In the case of equation (88) we have $\ell = 1$, and (Jeffrey 1995)

$$Q_1 = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1 \,. \tag{89}$$

The radial eigenfunctions $F(r),\ G(r)$, and $H(r)$, are then given by

$$F(r) = -\frac{3}{4M^3}\left[\ln N^2 + \frac{2M}{r}\left(1+\frac{M}{r}\right)\right]\mu \,, \tag{90}$$

$$G(r) = \frac{3N}{4M^2 r}\left[\frac{r}{M}\ln N^2 + \frac{1}{N^2} + 1\right]\mu \,, \tag{91}$$

$$H(r) = G(r) \,, \tag{92}$$

and satisfy the following boundary conditions:

(i) refer to a vanishing field at infinity, i.e.

$$\lim_{r\to\infty} F(r) = 0 \,, \qquad \lim_{r\to\infty} G(r) = 0 \,; \tag{93}$$

(ii) reduce to a flat spacetime solution for a dipole, i.e.

$$\lim_{M/r\to 0} F(r) = \frac{2\mu}{r^3} \,, \qquad \lim_{M/r\to 0} G(r) = \frac{\mu}{r^3} \,. \tag{94}$$

(iii) coincide with the corresponding radial eigenfunctions found for a Schwarzschild spacetime (Ginzburg & Ozernoy, 1964; Anderson & Cohen, 1970). This is what we expected since there are no first order contributions due to the rotation of the spacetime.

Using expressions (90)–(92) we can now determine the value of the matching constant C_1 by requiring that the radial magnetic field is continuous across the star surface, i.e. that $[B^{\hat r}(r=R)]_{IN} = [B^{\hat r}(r=R)]_{EXT}$. As a result, we obtain

$$C_1 = -\frac{3R^3}{4M^3}\left[\ln\left(1-\frac{2M}{R}\right) + \frac{2M}{R}\left(1+\frac{M}{R}\right)\right] = \frac{F(R)R^3}{\mu} \,, \tag{95}$$

whose flat spacetime limit is

$$\lim_{M/R\to 0} C_1 = 2 \,. \tag{96}$$

Collecting all the expressions for the radial eigenfunctions, the stationary vacuum magnetic field external to a misaligned magnetized relativistic star is given by

$$B^{\hat r} = -\frac{3}{4M^3}\left[\ln N^2 + \frac{2M}{r}\left(1+\frac{M}{r}\right)\right](\cos\chi\cos\theta + \sin\chi\sin\theta\cos\lambda)\mu \,, \tag{97}$$

$$B^{\hat\theta} = \frac{3N}{4M^2 r}\left[\frac{r}{M}\ln N^2 + \frac{1}{N^2} + 1\right](\cos\chi\sin\theta - \sin\chi\cos\theta\cos\lambda)\mu \,, \tag{98}$$

$$B^{\hat\phi} = \frac{3N}{4M^2 r}\left[\frac{r}{M}\ln N^2 + \frac{1}{N^2} + 1\right](\sin\chi\sin\lambda)\mu \,. \tag{99}$$

The search for the form of the electric field is much more involved than for the magnetic field. However, hereafter we will make use of the insight gained in Section 3.1.2 as a guide and start the derivation of the solution by rewriting vacuum Maxwell equations (31)–(33) and (34) as

$$\frac{3\bar\omega r}{4M^3 N}\mu\left[\ln N^2 + \frac{2M}{r}\left(1+\frac{M}{r}\right)\right]\sin\chi\sin^2\theta\sin\lambda = \left(\sin\theta E^{\hat\phi}\right)_{,\theta} - E^{\hat\theta}_{,\phi} \,, \tag{100}$$

$$\frac{3\bar\omega r}{4M^2}\mu\left[\frac{r}{M}\ln N^2 + \frac{1}{N^2} + 1\right]\sin\chi\sin\theta\cos\theta\sin\lambda = E^{\hat r}_{,\phi} - \sin\theta\left(rNE^{\hat\phi}\right)_{,r} \,, \tag{101}$$

$$\frac{9\omega r}{4M^3}\mu\left[\ln N^2 + \frac{2M}{r}\left(1+\frac{M}{r}\right)\right](\cos\chi\cos\theta + \sin\chi\sin\theta\cos\lambda)\sin\theta + \frac{3\bar\omega}{4M^2}\mu\left[\frac{r}{M}\ln N^2 + \frac{1}{N^2} + 1\right]\sin\chi\cos\lambda = \left(rNE^{\hat\theta}\right)_{,r} - E^{\hat r}_{,\theta} \,, \tag{102}$$

$$N\sin\theta\left(r^2 E^{\hat r}\right)_{,r} + r\left(\sin\theta E^{\hat\theta}\right)_{,\theta} + rE^{\hat\phi}_{,\phi} = 0 \,, \tag{103}$$

and which already indicate that the dragging of inertial frames with angular velocity ω introduces electric fields in the surrounding space when magnetic fields are present. Using as a reference the solutions (57), (58), and (59) for a misaligned rotating sphere in Minkowski spacetime, we look for the simplest solutions of vacuum Maxwell equations in the form

$$E^{\hat{r}} = (f_1 + f_3)\cos\chi(3\cos^2\theta - 1) + (g_1 + g_3)\, 3\sin\chi\cos\lambda\sin\theta\cos\theta \ , \tag{104}$$

$$E^{\hat{\theta}} = (f_2 + f_4)\cos\chi\sin\theta\cos\theta + (g_2 + g_4)\sin\chi\cos\lambda - (g_5 + g_6)\left(\cos^2\theta - \sin^2\theta\right)\sin\chi\cos\lambda \ , \tag{105}$$

$$E^{\hat{\phi}} = [g_5 + g_6 - (g_2 + g_4)]\sin\chi\cos\theta\sin\lambda \ , \tag{106}$$

where the unknown eigenfunctions $f_1 - f_4$, and $g_1 - g_6$ can be found as solutions to vacuum Maxwell equations and have radial dependence only. Substituting (104)–(106) in (100)–(102) we obtain the following set of linear differential equations

$$N\left(r^2 f_1\right)_{,r} + r f_2 = 0 \ , \tag{107}$$

$$(rNf_2)_{,r} + 6f_1 = 0 \ , \tag{108}$$

$$N\left(r^2 f_3\right)_{,r} + r f_4 = 0 \ , \tag{109}$$

$$(rNf_4)_{,r} + 6f_3 - \frac{9\omega r}{4M^3}\mu\left[\ln N^2 + \frac{2M}{r}\left(1 + \frac{M}{r}\right)\right] = 0 \ , \tag{110}$$

$$N\left(r^2 g_1\right)_{,r} + 2r g_5 = 0 \ , \tag{111}$$

$$\left(rNg_5\right)_{,r} + 3g_1 = 0 \ , \tag{112}$$

$$N\left(r^2 g_3\right)_{,r} + 2r g_6 = 0 \ , \tag{113}$$

$$\left(rNg_6\right)_{,r} + 3g_3 - \frac{9\omega r}{8M^3}\mu\left[\ln N^2 + \frac{2M}{r}\left(1 + \frac{M}{r}\right)\right] = 0 \ . \tag{114}$$

Note that both the sets of radial eigenfunctions $f_1 - f_4$, and g_1, g_3, g_5, g_6 are linearly independent, but that relations can be written between the two sets. In particular, the comparison of equation (107) with (111) and of equation (109) with (113) indicates that

$$g_1 = f_1 \ , \qquad g_3 = f_3 \ , \qquad g_5 = \frac{f_2}{2} \ , \qquad g_6 = \frac{f_4}{2} \ . \tag{115}$$

We start the search for explicit expressions for the radial eigenfunctions by combining equations (107) and (108) to obtain a single differential equation of second order for the unknown function f_1

$$\frac{d}{dr}\left[\left(1 - \frac{2M}{r}\right)\frac{d}{dr}\left(r^2 f_1\right)\right] - 6f_1 = 0 \ , \tag{116}$$

and which can again be recast in a form similar to equation (88). Proceeding in a way analogous to what done for the magnetic field (see Appendix B for details) it is possible to realize that the solution should be expressed in terms of a Legendre function of the second kind and of order $\ell = 2$. Recalling now that (Jeffrey 1995)

$$Q_2(x) = \frac{1}{4}\left(3x^2 - 1\right)\ln\left(\frac{x+1}{x-1}\right) - \frac{3x}{2} \ , \tag{117}$$

we obtain, as solution to (116) at the $\ell = 2$ order in the expansion

$$f_1 = \frac{\Omega}{6cR^2}C_1 C_2\left[\left(3 - \frac{2r}{M}\right)\ln N^2 + \frac{2M^2}{3r^2} + \frac{2M}{r} - 4\right]\mu \ , \tag{118}$$

where C_2 is an arbitrary constant to be determined through the imposition of boundary conditions. Making now use of the equation (107) we also obtain that

$$f_2 = -\frac{\Omega}{cR^2}C_1 C_2 N\left[\left(1 - \frac{r}{M}\right)\ln N^2 - 2 - \frac{2M^2}{3r^2N^2}\right]\mu \ . \tag{119}$$

In a similar way, the solutions to equations (109) and (110) are found to be

$$f_3 = \frac{15\omega r^3}{16M^5 c}\left\{C_3\left[\left(3-\frac{2r}{M}\right)\ln N^2 + \frac{2M^2}{3r^2} + \frac{2M}{r} - 4\right] + \frac{2M^2}{5r^2}\ln N^2 + \frac{4M^3}{5r^3}\right\}\mu\,, \tag{120}$$

$$f_4 = -\frac{45\omega r^3}{8M^5 c}N\left\{C_3\left[\left(1-\frac{r}{M}\right)\ln N^2 - 2 - \frac{2M^2}{3r^2N^2}\right] + \frac{4M^4}{15r^4N^2}\right\}\mu\,, \tag{121}$$

where again C_3 is an arbitrary constant determined through the boundary conditions. Finally, the functions g_3 and g_4 are given by

$$g_2 = \frac{3\Omega r}{8M^3 cN}\left[\ln N^2 + \frac{2M}{r}\left(1+\frac{M}{r}\right)\right]\mu\,, \tag{122}$$

$$g_4 = -\frac{\omega}{\Omega}g_2 = -\frac{3\omega r}{8M^3 cN}\left[\ln N^2 + \frac{2M}{r}\left(1+\frac{M}{r}\right)\right]\mu\,, \tag{123}$$

so that $g_2 + g_4 = (\bar{\omega}/\Omega)g_2$.

Collecting again all the expressions for the radial eigenfunctions, the stationary vacuum electric field external to a misaligned magnetized relativistic star is given by

$$\begin{aligned} E^{\hat{r}} = &\left\{\frac{15\omega r^3}{16M^5 c}\left\{C_3\left[\left(3-\frac{2r}{M}\right)\ln N^2 + \frac{2M^2}{3r^2} + \frac{2M}{r} - 4\right] + \frac{2M^2}{5r^2}\ln N^2 + \frac{4M^3}{5r^3}\right\}\right. \\ &\left. + \frac{\Omega}{6cR^2}C_1C_2\left[\left(3-\frac{2r}{M}\right)\ln N^2 + \frac{2M^2}{3r^2} + \frac{2M}{r} - 4\right]\right\}\left[\cos\chi(3\cos^2\theta - 1) + 3\sin\chi\cos\lambda\sin\theta\cos\theta\right]\mu\,, \end{aligned} \tag{124}$$

$$\begin{aligned} E^{\hat{\theta}} = &-\left\{\frac{45\omega r^3}{16M^5 c}N\left\{C_3\left[\left(1-\frac{r}{M}\right)\ln N^2 - 2 - \frac{2M^2}{3r^2N^2}\right] + \frac{4M^4}{15r^4N^2}\right\}\right. \\ &\left. + \frac{\Omega}{2cR^2}C_1C_2N\left[\left(1-\frac{r}{M}\right)\ln N^2 - 2 - \frac{2M^2}{3r^2N^2}\right]\right\}\left[2\cos\chi\sin\theta\cos\theta - \left(\cos^2\theta - \sin^2\theta\right)\sin\chi\cos\lambda\right]\mu \\ &+ \frac{3\bar{\omega}r}{8M^3 cN}\left[\ln N^2 + \frac{2M}{r}\left(1+\frac{M}{r}\right)\right](\sin\chi\cos\lambda)\mu\,, \end{aligned} \tag{125}$$

$$\begin{aligned} E^{\hat{\phi}} = &-\left\{\frac{45\omega r^3}{16M^5 c}N\left\{C_3\left[\left(1-\frac{r}{M}\right)\ln N^2 - 2 - \frac{2M^2}{3r^2N^2}\right] + \frac{4M^4}{15r^4N^2}\right\} + \frac{\Omega}{2cR^2}C_1C_2N\left[\left(1-\frac{r}{M}\right)\ln N^2 - 2 - \frac{2M^2}{3r^2N^2}\right]\right. \\ &\left. - \frac{3\bar{\omega}r}{8M^3 cN}\left[\ln N^2 + \frac{2M}{r}\left(1+\frac{M}{r}\right)\right]\right\}(\sin\chi\cos\theta\sin\lambda)\mu\,. \end{aligned} \tag{126}$$

As anticipated in Section 3.1.2, expressions (124)–(126) confirm that the general relativistic dragging of reference frames introduces a new contribution to the form of the electric field which does not have a flat spacetime analogue. This effect is $\mathcal{O}(\omega)$ and therefore present already in a slow rotation approximation. This is in contrast with what happens for the magnetic fields, where higher order approximations of the form of the metric are necessary for frame dragging corrections to appear.

The values of the arbitrary constants C_2 and C_3 can now be found after imposing the continuity of the tangential electric field across the star surface. Using then (82)–(84) as solutions for the internal electric field and imposing that $[E^{\hat{\theta}}(r=R)]_{IN} = [E^{\hat{\theta}}(r=R)]_{EXT}$ as well as $[E^{\hat{\phi}}(r=R)]_{IN} = [E^{\hat{\phi}}(r=R)]_{EXT}$, yields

$$C_2 = \frac{1}{N_R^2}\left[\left(1-\frac{R}{M}\right)\ln N_R^2 - 2 - \frac{2M^2}{3R^2N_R^2}\right]^{-1}\,, \tag{127}$$

$$C_3 = \frac{2M^2}{15R^2}C_2\left[\ln N_R^2 + \frac{2M}{R}\right]\,, \tag{128}$$

with $N_R^2 \equiv N^2(r=R) = 1 - 2M/R$. It is now also possible to calculate the surface charge distribution σ_s resulting from the discontinuity across the star's surface of the radial electric field. Explicit expressions for this, as well as for the surface currents corresponding to the discontinuities across the surface of $B^{\hat{\theta}}$ and $B^{\hat{\phi}}$, will not be given here but can be found in Appendix C.

Before concluding this Section on stationary solutions we will comment on the relevant limits of equations (124)–(126). Firstly, we verify that they reduce to the Deutsch solutions (57)–(59) in the limit $\omega = 0$ and $M/r,\ M/R \to 0$. In this case, in fact

$$\lim_{M/r,\ M/R\to 0} f_1(r) = -\frac{\mu\Omega R^2}{cr^4} = \lim_{M/r,\ M/R\to 0} g_1(r) = \lim_{M/r,\ M/R\to 0} g_5(r)\ , \tag{129}$$

$$\lim_{M/r,\ M/R\to 0} f_2(r) = -\frac{2\mu\Omega R^2}{cr^4}\ , \tag{130}$$

$$\lim_{M/r,\ M/R\to 0} g_2(r) = -\frac{\mu\Omega}{cr^2}\ , \tag{131}$$

$$\lim_{M/r,\ M/R\to 0} g_4(r) = \frac{\mu\omega}{cr^2}\ , \tag{132}$$

$$\lim_{M/r,\ M/R\to 0} f_3(r) = 0 = \lim_{M/r,\ M/R\to 0} f_4(r) = \lim_{M/r,\ M/R\to 0} g_3(r) = \lim_{M/r,\ M/R\to 0} g_6(r)\ . \tag{133}$$

Secondly, in the limit of an aligned dipole in a Schwarzschild spacetime, $\chi = 0 = \omega$, and equations (124)–(126) reduce to

$$E^{\hat r} = -\frac{\Omega R}{4M^3cN_R^2}\left[\ln N_R^2 + \frac{2M}{R}\left(1+\frac{M}{R}\right)\right]\left[\left(1-\frac{R}{M}\right)\ln N_R^2 - 2 - \frac{2M^2}{3R^2N_R^2}\right]^{-1} \left[\left(3-\frac{2r}{M}\right)\ln N^2 + \frac{2M^2}{3r^2} + \frac{2M}{r} - 4\right](3\cos^2\theta - 1)\mu\ , \tag{134}$$

$$E^{\hat\theta} = \frac{3\Omega R}{4M^3cN_R^2}\left[\ln N_R^2 + \frac{2M}{R}\left(1+\frac{M}{R}\right)\right]\left[\left(1-\frac{R}{M}\right)\ln N_R^2 - 2 - \frac{2M^2}{3R^2N_R^2}\right]^{-1} N\left[\left(1-\frac{r}{M}\right)\ln N^2 - 2 - \frac{2M^2}{3r^2N^2}\right](\sin\theta\cos\theta)\mu\ , \tag{135}$$

$$E^{\hat\phi} = 0\ . \tag{136}$$

Note that (134)–(136) do not coincide with the corresponding expressions found by Sengupta (1995). A straightforward calculation would show that his suggested expressions, while reducing to the Deutsch solution in the flat spacetime limit, do not satisfy Maxwell equations. A possible explanation for the disagreement could be found in the method followed by Sengupta in his derivation which is not based on the explicit solution of Maxwell equations. Because of this, subsequent results obtained on the basis of Sengupta's expressions for the external electric field (e.g. De Paolis et al., 1999) should be revisited in terms of expressions (134)–(136).

4 NON-STATIONARY SOLUTIONS TO MAXWELL EQUATIONS

In this Section we will drop the assumption of infinite conductivity which prevented the variation of the star's magnetic moment and led to the stationary electromagnetic fields presented in the previous Sections. Here, on the contrary, we are interested in time evolving electromagnetic fields and, in particular, in establishing the general relativistic corrections to the induction equation. A direct consequence of a finite conductivity is, in fact, the generation of a time varying charge density and conduction currents which will be then responsible for the Ohmic decay. Using Maxwell equations (34) and Ohm's laws (39)–(42), we find that the space charge density $\rho_e = \rho_e(t, r, \theta, \phi)$ inside the star has a zeroth-order contribution given by

$$\rho_e = \frac{ce^{-\Lambda}}{16\pi^2\sigma r^2\sin\theta}\Bigg\{\left[r\left(\sin\theta B^{\hat\phi}\right)_{,\theta} - rB^{\hat\theta}_{,\phi}\right]_{,r} + \left[e^{\Phi+\Lambda}B^{\hat r}_{,\phi} - \sin\theta\left(e^{\Phi}rB^{\hat\phi}\right)_{,r}\right]_{,\theta} e^{-\Phi} + \left[\left(e^{\Phi}rB^{\hat\theta}\right)_{,r} - e^{\Phi+\Lambda}B^{\hat r}_{,\theta}\right]_{,\phi} e^{-\Phi}\Bigg\} + \mathcal{O}(\Omega)\ ; \tag{137}$$

$$= \frac{ce^{-\Lambda}}{16\pi^2\sigma}\left(\frac{\cos\theta\sin\chi\sin\lambda}{\sin\theta}\right)\frac{(G-H)}{r}\Phi_{,r} + \mathcal{O}(\Omega)\ . \tag{138}$$

where the second expression is the one obtained after using the ansatz (60)–(62). It follows from equations (143) and (144) that the zero-order term in equation (138) vanishes, so that the leading contribution is at first order in Ω. Using now equations (31), (46), (47), (137) and Ohm's laws (41), (42), we obtain the evolution equation for the radial component of magnetic field

$$
\begin{aligned}
\frac{\partial B^{\hat r}}{\partial t} &= \frac{c^2 e^{-\Lambda}}{4\pi\sigma r^2\sin\theta}\Bigg\{\frac{1}{\sin\theta}\left[e^{\Phi+\Lambda}B^{\hat r}{}_{,\phi}-\sin\theta\left(e^{\Phi}rB^{\hat\phi}\right)_{,r}\right]_{,\phi}-\frac{\omega e^{-\Phi}}{4\pi\sigma\sin\theta}\left[e^{\Phi+\Lambda}B^{\hat r}{}_{,\phi}-\sin\theta\left(e^{\Phi}rB^{\hat\phi}\right)_{,r}\right]_{,\phi\phi}\\
&-\left\{\sin\theta\left[\left(e^{\Phi}rB^{\hat\theta}\right)_{,r}-e^{\Phi+\Lambda}B^{\hat r}{}_{,\theta}\right]\right\}_{,\theta}-\left\{\sin\theta\left[\frac{\omega r}{4\pi\sigma}\left[\left(\sin\theta B^{\hat\phi}\right)_{,\theta}-B^{\hat\theta}{}_{,\phi}\right]\right]_{,r}\right\}_{,\theta}\\
&-\frac{\omega e^{-\Phi}}{4\pi\sigma}\left\{\sin\theta\left[e^{\Phi+\Lambda}B^{\hat r}{}_{,\phi}-\sin\theta\left(e^{\Phi}rB^{\hat\phi}\right)_{,r}\right]_{,\theta}\right\}_{,\theta}+\frac{\Omega e^{-\Phi}}{4\pi\sigma}\Bigg\{e^{\Phi}\left\{\sin\theta\left[r\left(\sin\theta B^{\hat\phi}\right)_{,\theta}-rB^{\hat\theta}{}_{,\phi}\right]_{,r}\right\}_{,\theta}\\
&+\left\{\sin\theta\left[e^{\Phi+\Lambda}B^{\hat r}{}_{,\phi}-\sin\theta\left(e^{\Phi}rB^{\hat\phi}\right)_{,r}\right]_{,\theta}\right\}_{,\theta}+\left\{\sin\theta\left[\left(e^{\Phi}rB^{\hat\theta}\right)_{,r}-e^{\Phi+\Lambda}B^{\hat r}{}_{,\theta}\right]_{,\phi}\right\}_{,\theta}\Bigg\}\Bigg\}-\Omega B^{\hat r}{}_{,\phi}\,. \qquad (139)
\end{aligned}
$$

Similarly, using equations (32), (45), (47), (137) and Ohm's laws (40), (42), we obtain the evolution equation for the polar component of magnetic field

$$
\begin{aligned}
\frac{\partial B^{\hat\theta}}{\partial t} &= \frac{c^2 e^{-\Lambda}}{4\pi\sigma r}\Bigg\{\left\{e^{-\Lambda}\left[\left(e^{\Phi}rB^{\hat\theta}\right)_{,r}-e^{\Phi+\Lambda}B^{\hat r}{}_{,\theta}\right]\right\}_{,r}+\left\{e^{-\Lambda}\left[\frac{\omega r}{4\pi\sigma}\left[\left(\sin\theta B^{\hat\phi}\right)_{,\theta}-B^{\hat\theta}{}_{,\phi}\right]\right]_{,r}\right\}_{,r}\\
&+\frac{1}{4\pi\sigma}\left\{\omega e^{-(\Phi+\Lambda)}\left[e^{\Phi+\Lambda}B^{\hat r}{}_{,\phi}-\sin\theta\left(e^{\Phi}rB^{\hat\phi}\right)_{,r}\right]_{,\theta}\right\}_{,r}-\frac{\Omega}{4\pi\sigma}\Bigg\{\left\{e^{-\Lambda}\left[r\left(\sin\theta B^{\hat\phi}\right)_{,\theta}-rB^{\hat\theta}{}_{,\phi}\right]_{,r}\right\}_{,r}\\
&+\left\{e^{-(\Phi+\Lambda)}\left[e^{\Phi+\Lambda}B^{\hat r}{}_{,\phi}-\sin\theta\left(e^{\Phi}rB^{\hat\phi}\right)_{,r}\right]_{,\theta}\right\}_{,r}+\left\{e^{-(\Phi+\Lambda)}\left[\left(e^{\Phi}rB^{\hat\theta}\right)_{,r}-e^{\Phi+\Lambda}B^{\hat r}{}_{,\theta}\right]_{,\phi}\right\}_{,r}\Bigg\}\Bigg\}\\
&-\frac{c^2 e^{\Phi}}{4\pi\sigma r^2\sin^2\theta}\left\{\left[\left(\sin\theta B^{\hat\phi}\right)_{,\theta}-B^{\hat\theta}{}_{,\phi}\right]_{,\phi}-\frac{\omega e^{-\Phi}}{4\pi\sigma}\left[\left(\sin\theta B^{\hat\phi}\right)_{,\theta}-B^{\hat\theta}{}_{,\phi}\right]_{,\phi\phi}\right\}-\Omega B^{\hat\theta}{}_{,\phi}\,. \qquad (140)
\end{aligned}
$$

Finally, equations (33), (45), (46) and the Ohm's law (40), (41) yield the evolution equation for the toroidal part of the magnetic field

$$
\begin{aligned}
\frac{\partial B^{\hat\phi}}{\partial t} &= -\frac{c^2 e^{-\Lambda}}{4\pi\sigma r\sin\theta}\left\{\left\{e^{-\Lambda}\left[e^{\Phi+\Lambda}B^{\hat r}{}_{,\phi}-\sin\theta\left(e^{\Phi}rB^{\hat\phi}\right)_{,r}\right]\right\}_{,r}-\left\{\frac{\omega e^{-(\Phi+\Lambda)}}{4\pi\sigma}\left[e^{\Phi+\Lambda}B^{\hat r}{}_{,\phi}-\sin\theta\left(re^{\Phi}B^{\hat\phi}\right)_{,r}\right]_{,\phi}\right\}_{,r}\right\}\\
&+\frac{c^2 e^{\Phi}}{4\pi\sigma r^2}\left\{\left\{\frac{1}{\sin\theta}\left[\left(\sin\theta B^{\hat\phi}\right)_{,\theta}-B^{\hat\theta}{}_{,\phi}\right]\right\}_{,\theta}-\frac{\omega e^{-\Phi}}{4\pi\sigma}\left\{\frac{1}{\sin\theta}\left[\left(\sin\theta B^{\hat\phi}\right)_{,\theta}-B^{\hat\theta}{}_{,\phi}\right]_{,\phi}\right\}_{,\theta}\right\}\\
&+\frac{\Omega e^{-\Lambda}\sin\theta}{r}\left(r^2B^{\hat r}\right)_{,r}+\Omega\left(\sin\theta B^{\hat\theta}\right)_{,\theta}\,. \qquad (141)
\end{aligned}
$$

A first important feature of equations (139)–(141) is that besides the relativistic corrections due the monopolar part of the gravitational field (proportional to M/R and already present in the non-rotating case), the rotation of spacetime introduces additional corrections related to the dipolar part of the gravitational field (and proportional to ω) to the decay of the magnetic field. A second relevant aspect of equations (139)–(141) is that they do not show the degeneracy encountered in the time evolution of the magnetic field in a non-rotating spacetime [cf. equation (146)]. In that case, in fact, the three induction equations for the components of the magnetic field reduce to a single evolution equation (Sengupta, 1997; Geppert et al. 2000). Here, instead, the three equations remain distinct and a particular field component might be favoured during the decay. Finally, equations (139)–(141) do not factor out the angular part as it is the case for a non-rotating, aligned dipole [cf. equation (146)] and the evolution of the magnetic field has therefore properties which depend on the angular position in the star. As a consequence of this, an initially dipolar magnetic field might not remain as such during its decay. This could be relevant for the evolution of the magnetic field of pulsars and more particularly of magnetars.

Using now the ansatz (60)–(62), we can write equations (139)–(141) in the more compact form

$$\frac{\partial F}{\partial t}\left(\cos\theta\cos\chi+\sin\theta\sin\chi\cos\lambda\right)\sin\theta=\frac{c^2e^{-\Lambda}}{4\pi\sigma r^2}\Bigg\{\left[e^{\Phi}r\left(G-H\right)\right]_{,r}\sin\chi\cos\lambda$$
$$-2\left[\left(e^{\Phi}rG\right)_{,r}+e^{\Phi+\Lambda}F\right]\sin\theta\left(\cos\theta\cos\chi+\sin\theta\sin\chi\cos\lambda\right)$$
$$-\frac{1}{4\pi\sigma}\sin\chi\sin\lambda\Bigg\{\left[\omega r(H-G)\right]_{,r}(1-2\sin^2\theta)+2\omega e^{-\Phi}\left[\left(e^{\Phi}rH\right)_{,r}+e^{\Phi+\Lambda}F\right]\sin^2\theta$$
$$-\Omega r\left(G-H\right)\Phi_{,r}\left(1-2\sin^2\theta\right)\Bigg\}\Bigg\}\,, \tag{142}$$

$$\frac{\partial G}{\partial t}\left(\sin\theta\cos\chi-\cos\theta\sin\chi\cos\lambda\right)=\frac{c^2}{4\pi\sigma r}\Bigg\{\frac{e^{\Phi}\left(G-H\right)}{r\sin^2\theta}\cos\theta\sin\chi\left[\cos\lambda+\frac{\omega e^{-\Phi}}{4\pi\sigma}\sin\lambda\right]$$
$$+e^{-\Lambda}\left[e^{-\Lambda}\left(e^{\Phi}rG\right)_{,r}+e^{\Phi}F\right]_{,r}\left(\sin\theta\cos\chi-\cos\theta\sin\chi\cos\lambda\right)$$
$$-\frac{e^{-\Lambda}}{4\pi\sigma}\cos\theta\sin\chi\sin\lambda\Bigg\{e^{-\Lambda}\left[\omega r\left(G-H\right)\right]_{,r}+\omega\left[F+e^{-(\Lambda+\Phi)}\left(re^{\Phi}H\right)_{,r}\right]+\Omega\left[\Phi_{,r}e^{-\Lambda}r\left(G-H\right)\right]\Bigg\}_{,r}\Bigg\}\,, \tag{143}$$

$$\frac{\partial H}{\partial t}\sin\lambda=\frac{c^2e^{-\Lambda}}{4\pi\sigma r}\left\{\left[e^{-\Lambda}\left(e^{\Phi}rH\right)_{,r}+e^{\Phi}F\right]\left[\sin\lambda-\frac{\omega e^{-\Phi}}{4\pi\sigma}\cos\lambda\right]\right\}_{,r}+\frac{c^2e^{\Phi}\left(G-H\right)}{4\pi\sigma r^2\sin^2\theta}\left[\sin\lambda-\frac{\omega e^{-\Phi}}{4\pi\sigma}\cos\lambda\right]\,, \tag{144}$$

where F, G, and H satisfy the constraint equation (30)

$$\left[\left(r^2F\right)_{,r}+2e^{\Lambda}rG\right]\sin\theta\left(\cos\chi\cos\theta+\sin\chi\sin\theta\cos\lambda\right)+e^{\Lambda}r\left(H-G\right)\sin\chi\cos\lambda=0\,, \tag{145}$$

The set of equations (142)–(144) is too complicated to be solved analytically even when analytic expressions are available for the metric functions (e.g. for a constant density stellar model). The numerical solution of (142)–(144) for a number of equations of state together with a self-consistent evolution of the star's angular velocity and electrical conductivity will be presented in a separate paper (Rezzolla et al. 2000). Note that equations (142)–(144) could also be derived through a vector potential A_μ defined so that the electromagnetic tensor $F_{\mu\nu}=A_{\nu,\mu}-A_{\mu,\nu}$. Details of this derivation can be found in Appendix D.

An interesting limit of the induction equations (142), (143) is the one for a non-rotating dipole in a spherically symmetric spacetime. In this case: $\Omega=0=\omega$, $\chi=0$, and $H, H/\partial t$ are not determined [cf. equations (62) and (65)]. As mentioned before, the induction equations are degenerate in this case and the unique evolution equation is then

$$\frac{\partial F}{\partial t}=\frac{c^2e^{-\Lambda}}{4\pi\sigma r^2}\left\{\left[e^{\Phi-\Lambda}\left(r^2F\right)_{,r}\right]_{,r}-2e^{\Phi+\Lambda}F\right\}\,, \tag{146}$$

corresponding to the solution found by Geppert et al. (2000). When the metric functions Φ and Λ refer only to the vacuum region of spacetime external to the star, equation (146) further simplifies to

$$\frac{\partial F}{\partial t}=\frac{c^2}{4\pi\sigma r^2}\sqrt{\frac{r-2M}{r}}\left\{\left[\left(1-\frac{2M}{r}\right)\left(r^2F\right)_{,r}\right]_{,r}-2F\right\}\,, \tag{147}$$

and which now corresponds to the solution found by Sengupta (1997).

5 CONCLUSION

We have presented analytic general relativistic expressions for the electromagnetic fields internal and external to a slowly-rotating magnetized neutron star. The star is considered isolated and in vacuum, but no special assumption is made on the orientation of the dipolar magnetic field with respect to the rotation axis. The solutions to Maxwell equations have been considered both for an infinite and for a finite electrical conductivity.

In the first case, corresponding to stationary magnetic fields, we have shown that the general relativistic corrections due to the dragging of reference frames are not present in the form of the magnetic fields but emerge only in the form of the electric fields. In particular, we have shown that the frame-dragging provides an additional induced electric field which is analogous to the one introduced by the rotation of the star in the flat spacetime limit. In the case of finite electrical conductivity, on the other hand, corresponding to decaying magnetic fields, we have shown that corrections due both to the spacetime curvature and to the dragging of reference frames can be found in the induction equation. An interesting result obtained in this regime is that the rotation of the star eliminates the degeneracy in the components of the induction equation which remain therefore distinct. Furthermore, rotation and dipole misalignment do not eliminate the angular dependence in the induction equation and, as a result, an initially dipolar magnetic field might evolve towards a different configuration during its decay.

Because of their complexity, the evolution equations found for the magnetic field require a numerical integration which will discuss in detail in a forthcoming work (Rezzolla et al. 2000). There, we will also present direct comparisons between the flat and the curved spacetime solutions and quantify more precisely the importance of the general relativistic corrections.

One of the relevant aspects of the solutions presented in this paper is that they provide a lowest order analytic form for the electromagnetic field in the spacetime of a slowly rotating misaligned dipole subject to assumptions which, while giving simplifications, allow the major features of a realistic solution to be seen. In this sense, they reflect a rather general physical configuration and could therefore be used in a variety of astrophysical situations.

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APPENDIX A: THE ELECTROMAGNETIC TENSOR

For completeness, we provide below the explicit expressions for the components of the electromagnetic tensor used throughout the paper. In a coordinate basis, and at first order in Ω, these components are given by

$$F_{\alpha\beta} = \begin{pmatrix} 0 & -e^{\Phi}E_r - \omega e^{\Lambda} r^2 \sin\theta B^{\theta} & -e^{\Phi}E_{\theta} + \omega e^{\Lambda} r^2 \sin\theta B^{r} & -e^{\Phi}E_{\phi} \\ e^{\Phi}E_r + \omega e^{\Lambda} r^2 \sin\theta B^{\theta} & 0 & e^{\Lambda} r^2 \sin\theta B^{\phi} & -e^{\Lambda} r^2 \sin\theta B^{\theta} \\ e^{\Phi}E_{\theta} - \omega e^{\Lambda} r^2 \sin\theta B^{r} & -e^{\Lambda} r^2 \sin\theta B^{\phi} & 0 & e^{\Lambda} r^2 \sin\theta B^{r} \\ e^{\Phi}E_{\phi} & e^{\Lambda} r^2 \sin\theta B^{\theta} & -e^{\Lambda} r^2 \sin\theta B^{r} & 0 \end{pmatrix} . \tag{A1}$$

The matrix (A1) can also be expressed in terms of the electromagnetic field measured by the ZAMO observers, in which case it takes the form

$$F_{\alpha\beta} = \begin{pmatrix} 0 & -e^{\Phi+\Lambda}E^{\hat{r}} - \omega e^{\Lambda} r \sin\theta B^{\hat{\theta}} & -e^{\Phi} r E^{\hat{\theta}} + \omega r^2 \sin\theta B^{\hat{r}} & -e^{\Phi} r \sin\theta E^{\hat{\phi}} \\ e^{\Phi+\Lambda}E^{\hat{r}} + \omega e^{\Lambda} r \sin\theta B^{\hat{\theta}} & 0 & e^{\Lambda} r B^{\hat{\phi}} & -e^{\Lambda} r \sin\theta B^{\hat{\theta}} \\ e^{\Phi} r E^{\hat{\theta}} - \omega r^2 \sin\theta B^{\hat{r}} & -e^{\Lambda} r B^{\hat{\phi}} & 0 & r^2 \sin\theta B^{\hat{r}} \\ e^{\Phi} r \sin\theta E^{\hat{\phi}} & e^{\Lambda} r \sin\theta B^{\hat{\theta}} & -r^2 \sin\theta B^{\hat{r}} & 0 \end{pmatrix} . \tag{A2}$$

Finally, we note that the components of the electromagnetic tensor in the ZAMO frame can be derived from (A1) with the transformation

$$F_{\hat{\alpha}\hat{\beta}} = \boldsymbol{e}^{\mu}_{\hat{\alpha}} \boldsymbol{e}^{\nu}_{\hat{\beta}} F_{\mu\nu} \ , \tag{A3}$$

to obtain

$$F_{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} 0 & -cE^{\hat{r}} & -cE^{\hat{\theta}} & -cE^{\hat{\phi}} \\ cE^{\hat{r}} & 0 & B^{\hat{\phi}} & -B^{\hat{\theta}} \\ cE^{\hat{\theta}} & -B^{\hat{\phi}} & 0 & B^{\hat{r}} \\ cE^{\hat{\phi}} & B^{\hat{\theta}} & -B^{\hat{r}} & 0 \end{pmatrix} . \tag{A4}$$

APPENDIX B: RADIAL EIGENFUNCTIONS

In this Appendix we briefly sketch the procedure for the calculation of the radial eigenfunctions and that have lead to the solutions (90)–(92), (118)–(119), (122)–(123). In general, we look for a solution of the equation

$$\frac{d}{dx}\left\{\left(\frac{1+x}{1-x}\right)\frac{d}{dx}\left[(1-x)^2 q_{\ell}\right]\right\} + \ell\,(\ell+1)\,q_{\ell} = 0 \ , \tag{B1}$$

for the function q_{ℓ}. Equation (B1) can also be written as

$$\left(1-x^2\right) q''_{\ell} - 2(1+2x)q'_{\ell} + \left[\ell\,(\ell+1) - 2\right] q_{\ell} = 0 \ , \tag{B2}$$

with the dash representing a total derivative with respect to x. The solution of (B1) has then form

$$q_{\ell} = \frac{d}{dx}\left[(1+x)\,\frac{d}{dx}Q_{\ell}\right] \ , \tag{B3}$$

where Q_{ℓ} are Legendre functions of second kind (Jeffrey 1995). A proof of this comes from substituting (B3) into (B2) to obtain

$$(1+x)\left[(1-x^2)q''''_{\ell} + (1-7x)q'''_{\ell} + \left[\ell\,(\ell+1)-2\right] q''_{\ell}\right] - 4(1+2x)q''_{\ell} + \left[\ell\,(\ell+1)-2\right] q'_{\ell} = 0 \ , \tag{B4}$$

and which can be rewritten as

$$(1+x)\left[(1-x^2)q''''_{\ell} - 6xq'''_{\ell} + \left[\ell\,(\ell+1)-6\right] q''_{\ell}\right] + (1-x^2)q'''_{\ell} - 4xq''_{\ell} + \left[\ell\,(\ell+1)-2\right] q'_{\ell} = 0 \ . \tag{B5}$$

It is now easy to realize that (B5) is identically satisfied since the content of the square brackets is, in fact, the second derivative of Legendre's equation

$$(1-x^2)q''_\ell - 2xq'_\ell + \ell\,(\ell+1)\,q_\ell = 0\,, \tag{B6}$$

while all the remainder of (B5) is the first derivative of Legendre's equation (B6).

APPENDIX C: SURFACE CHARGES AND CURRENTS

We here give explicit expressions for the surface charge distribution σ_s and surface currents $i^{\hat{k}}$ resulting from the discontinuities across the star's surface of the $r-$component of the electric field and of $\theta,\ \phi-$components of the magnetic field.

Defining now $[A]^{\pm} \equiv [A(r=R)]_{EXT} - [A(r=R)]_{IN}$, the surface charge density σ_s can be found as

$$\sigma_s = \frac{1}{4\pi}[E^{\hat{r}}]^{\pm}\,, \tag{C1}$$

and is given by

$$\begin{aligned}
\sigma_s \;=\; & \frac{1}{4\pi}\left\{\frac{15\omega_R R^3}{8M^5 c}\left\{C_3\left[\left(3-\frac{2R}{M}\right)\ln N_R^2 + \frac{2M^2}{3R^2} + \frac{2M}{R} - 4\right] + \frac{2M^2}{5R^2}\ln N_R^2 + \frac{4M^3}{5R^3}\right\}\right.\\
& \left. + \frac{\Omega}{3cR^2}C_1 C_2\left[\left(3-\frac{2R}{M}\right)\ln N_R^2 + \frac{2M^2}{3R^2} + \frac{2M}{R} - 4\right]\right\}(\cos\chi)\mu\\
& -\frac{1}{4\pi}\left\{\frac{45\omega_R R^3}{16M^5 c}\left\{C_3\left[\left(3-\frac{2R}{M}\right)\ln N_R^2 + \frac{2M^2}{3R^2} + \frac{2M}{R} - 4\right] + \frac{2M^2}{5R^2}\ln N_R^2 + \frac{4M^3}{5R^3}\right\}\right.\\
& \left. + \frac{\Omega}{2cR^2}C_1 C_2\left[\left(3-\frac{2R}{M}\right)\ln N_R^2 + \frac{2M^2}{3R^2} + \frac{2M}{R} - 4\right] - \frac{\bar{\omega}_R}{cR^2}C_1\right\}\sin\theta(\cos\chi\sin\theta - \sin\chi\cos\lambda\cos\theta)\mu\,,
\end{aligned} \tag{C2}$$

where $\omega_R \equiv \omega(r=R)$ and $\bar{\omega}_R \equiv \bar{\omega}(r=R)$

In a similar way, imposing that

$$i^{\hat{\phi}} = \frac{c}{4\pi}[B^{\hat{\theta}}]^{\pm}\,, \qquad\qquad i^{\hat{\theta}} = \frac{c}{4\pi}[B^{\hat{\phi}}]^{\pm}\,, \tag{C3}$$

we obtain

$$i^{\hat{\theta}} = \frac{3c}{16\pi}\frac{N_R}{M^2 R}\left[\frac{R}{M}\ln N_R^2 + \frac{1}{N_R^2} + 1 + \frac{4M^2}{3R^2}C_1\right](\sin\chi\sin\lambda)\,\mu\,, \tag{C4}$$

$$i^{\hat{\phi}} = \frac{3c}{16\pi}\frac{N_R}{M^2 R}\left[\frac{R}{M}\ln N_R^2 + \frac{1}{N_R^2} + 1 + \frac{4M^2}{3R^2}C_1\right](\cos\chi\sin\theta - \sin\chi\cos\theta\cos\lambda)\,\mu\,. \tag{C5}$$

APPENDIX D: AN ALTERNATIVE DERIVATION OF THE INDUCTION EQUATION

To confirm the results presented in Section 4 and to compare with the results presented in the literature in the case of Schwarzschild background spacetime (Sengupta, 1997) we here present a derivation of equations (142), (143), (144) in terms of a vector potential A_α defined as

$$F_{\alpha\beta} \equiv A_{\beta,\alpha} - A_{\alpha,\beta}\,. \tag{D1}$$

The use of a vector potential is sometimes looked at with skepticism (Geppert, Page and Zannias, 2000) in view of the non-commutativity of the covariant derivative, which could lead to ambiguities if Maxwell equations are expressed through double covariant derivatives of a vector potential. All of these ambiguities, however, are easily removed if the vector potential is introduced only when Maxwell equations (15) are recast in a form not involving covariant derivatives

$$\frac{1}{\sqrt{-g}}\left(\sqrt{-g}F^{\alpha\beta}\right)_{,\beta} = 4\pi J^\alpha \tag{D2}$$

In our derivation we start by using equations (7), (15), (17), neglecting the displacement current and taking the four-velocity of the conductor u^α in the form (38) [i.e. neglecting terms proportional to $g^{a0}F_{0b}u^b \approx \mathcal{O}(\omega^2)$] we get

$$\frac{1}{4\pi\sigma\sqrt{-g}u^0}\left(\sqrt{-g}F^{ab}\right)_{,b} = \rho_e u^a + \sigma g^{ab}\left(F_{bc}u^c + F_{b0}u^0\right)\,, \tag{D3}$$

which can be written as

$$F_{a0} = \frac{g_{ab}}{4\pi\sigma\sqrt{-g}u^0}\left(\sqrt{-g}F^{bc}\right)_{,c} - \frac{F_{ab}u^b}{u^0} - \frac{\rho_e g_{ab}u^b}{\sigma u^0}\,. \tag{D4}$$

Using now (D2) we can write the general expression for the evolution of the vector potential A_i as

$$A_{i,0} = -F_{i0} = -\frac{g_{ij}}{4\pi\sigma\sqrt{-g}u^0}\left(\sqrt{-g}F^{jk}\right)_{,k} + \frac{F_{ij}u^j}{u^0} + \frac{\rho_e g_{ij}u^j}{\sigma u^0} \ . \tag{D5}$$

Using (D5), the induction equation for the evolution of the ϕ component of vector potential can be written as

$$\begin{aligned}\frac{\partial A_\phi}{\partial t} &= -\frac{c^2 e^{-\Lambda}}{4\pi\sigma}\sin\theta\Bigg\{\frac{1}{\sin\theta}\left(e^{\Phi-\Lambda}F_{\phi r}\right)_{,r} + \left(\frac{e^{\Phi+\Lambda}F_{\phi\theta}}{r^2\sin\theta}\right)_{,\theta} + \frac{\omega e^{-\Phi}}{4\pi\sigma}\left[\sin\theta\left(e^{\Phi-\Lambda}F_{\theta r}\right)_{,r} + \frac{e^{\Phi+\Lambda}}{r^2\sin\theta}F_{\theta\phi,\phi}\right]_{,\theta} \\ &\quad + \frac{1}{4\pi\sigma}\left\{\omega e^{-\Lambda}\left[\left(\sin\theta F_{r\theta}\right)_{,\theta} + \frac{1}{\sin\theta}F_{r\phi,\phi}\right]\right\}_{,r}\Bigg\} + \frac{c^2\Omega e^{-2\Lambda}}{16\pi^2\sigma^2}\Phi_{,r}\left[\frac{1}{\sin\theta}F_{\phi r,\phi} + \left(\sin\theta F_{\theta r}\right)_{,\theta}\right]\sin\theta \ ,\end{aligned} \tag{D6}$$

which coincides with equation (11) of Sengupta's 1998 paper when $\chi = 0 = \omega$ and $e^{2\Phi} = N^2 = 1 - 2M/r$. Similarly, the evolution equations for the other components of the vector potential are given by

$$\frac{\partial A_\theta}{\partial t} = -\frac{c^2 e^{-\Lambda}}{4\pi\sigma\sin\theta}\left\{\sin\theta\left(e^{\Phi-\Lambda}F_{\theta r}\right)_{,r} + \frac{e^{\Phi+\Lambda}}{r^2\sin\theta}F_{\theta\phi,\phi} - \frac{\omega e^{-\Phi}}{4\pi\sigma}\left[\sin\theta\left(e^{\Phi-\Lambda}F_{\theta r}\right)_{,r\phi} + \frac{e^{\Phi+\Lambda}}{r^2\sin\theta}F_{\theta\phi,\phi\phi}\right]\right\} + \Omega F_{\theta\phi} \ , \tag{D7}$$

$$\frac{\partial A_r}{\partial t} = -\frac{c^2 e^{\Lambda}}{4\pi\sigma r^2\sin\theta}\left\{e^{\Phi-\Lambda}\left(\sin\theta F_{r\theta}\right)_{,\theta} + \frac{e^{\Phi-\Lambda}}{\sin\theta}F_{r\phi,\phi} - \frac{\omega e^{-\Lambda}}{4\pi\sigma}\left[\left(\sin\theta F_{r\theta}\right)_{,\theta\phi} + \frac{1}{\sin\theta}F_{r\phi,\phi\phi}\right]\right\} + \Omega F_{r\phi} \ . \tag{D8}$$

In the case of a misaligned rotator, the explicit expressions for the "magnetic" components of the electromagnetic tensor are

$$F_{r\theta} = e^{\Lambda} r H \sin\chi\sin\lambda \ , \tag{D9}$$

$$F_{\phi r} = G e^{\Lambda} r\sin\theta\left(\sin\theta\cos\chi - \sin\chi\cos\lambda\cos\theta\right) \ , \tag{D10}$$

$$F_{\theta\phi} = F r^2\sin\theta\left(\cos\theta\cos\chi + \sin\chi\cos\lambda\sin\theta\right) \ , \tag{D11}$$

using which the induction equations (D6)–(D8) assume the form

$$\begin{aligned}\frac{\partial A_\phi}{\partial t} &= \frac{c^2 e^{-\Lambda}\sin\theta}{4\pi\sigma}\Bigg\{-\left[\left(e^{\Phi} r G\right)_{,r} + e^{\Phi+\Lambda}F\right]\left(\sin\theta\cos\chi - \sin\chi\cos\lambda\cos\theta\right) \\ &\quad + \frac{1}{4\pi\sigma}\left\{\left[\left(e^{\Phi} r H\right)_{,r} + e^{\Phi+\Lambda}F\right]\omega e^{-\Phi} + \left[\omega r(G-H)\right]_{,r} + \Omega r\left(G-H\right)\Phi_{,r}\right\}\cos\theta\sin\chi\sin\lambda\Bigg\} \ ,\end{aligned} \tag{D12}$$

$$\frac{\partial A_\theta}{\partial t} = \frac{c^2}{4\pi\sigma}\left[e^{-\Lambda}\left(e^{\Phi} r H\right)_{,r} + e^{\Phi}F\right]\left[\sin\chi\sin\lambda - \frac{\omega e^{-\Phi}}{4\pi\sigma}\sin\chi\cos\lambda\right] + \Omega F r^2\sin\theta\left(\cos\theta\cos\chi + \sin\chi\cos\lambda\sin\theta\right) \ , \tag{D13}$$

$$\frac{\partial A_r}{\partial t} = \frac{c^2 e^{\Phi+\Lambda}\left(G-H\right)}{4\pi\sigma r\sin\theta}\sin\chi\cos\theta\left[\sin\lambda - \frac{\omega e^{-\Phi}}{4\pi\sigma}\cos\lambda\right] - \Omega G e^{\Lambda} r\sin\theta\left(\sin\theta\cos\chi - \sin\chi\cos\lambda\cos\theta\right) \ . \tag{D14}$$

Using now the definition (D1) it is possible to show that equations (D12), (D13), (D14) are equivalent to equations (142), (143) and (144) derived in the main text.