On the thermal conduction in tangled magnetic fields in clusters of galaxies

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ABSTRACT

Thermal conduction in tangled magnetic fields is reduced because heat conducting electrons must travel along the field lines longer distances between hot and cold regions of space than if there were no fields. We consider the case when the tangled magnetic field has a weak homogeneous component. We examine two simple models for temperature in clusters of galaxies: a time-independent model and a time-dependent one. We find that the actual value of the effective thermal conductivity in tangled magnetic fields depends on how it is defined for a particular astrophysical problem. Our final conclusion is that the heat conduction never totally suppressed but is usually important in the central regions of galaxy clusters, and therefore, it should not be neglected.

Subject headings: magnetic fields: conduction — cooling flows — magnetic fields: diffusion — methods: analytical

1. Introduction

In order to solve the problem of electron thermal conduction in a stochastic magnetic field, one should consider separately two effects that reduce the conduction (Pistinner & Shaviv 1996; Chandran & Cowley 1998). The first effect is that the heat conducting electrons have to travel along tangled magnetic field lines, and therefore, they have to travel longer distances between hot and cold regions of space (Tribble 1989, Tao 1995). The second effect is that electrons, while they are traveling along the field lines, become trapped and detrapped between magnetic mirrors, regions of strong magnetic field (Chandran, Cowley, & Ivanushkina 1999).

In our recent paper (Malyshkin & Kulsrud 2000) we studied the second effect, and we found the reduction of the thermal conductivity parallel to the magnetic field lines, $\kappa_\|$, relative to the Spitzer value for the thermal conductivity, $\kappa_{\rm S}$, caused by the presence of magnetic mirrors. In this paper we consider the first effect, and we calculate the further reduction of thermal conduction relative to $\kappa_\|$, caused by the tangled structure of the magnetic field lines. As a result, we obtain the total reduction caused by the both effects, and we calculate the effective thermal conductivity $\kappa_{\rm eff}$, which one has to apply for astrophysical problems. It turns out that there is no an universal definition of $\kappa_{\rm eff}$, and the result depends on the particular astrophysical problem under consideration.

The structure of this paper is the following. In the next section we consider the random walk of tangled magnetic field lines by making use of a diffusion approximation for this random walk. We find the expressions for the probability distributions of the field line lengths and for the Laplace transform, in the field line length, of these distributions. In Section 3 we use the Laplace transforms to calculate the effective thermal conductivity for our stationary model of a galaxy cluster. In Section 4 we consider a time-dependent model and study the time evolution of temperature in a galaxy cluster. Finally, in Section 5 we discuss our results, compare them with the results reported in previous papers, and consider the applications to cooling flow problem. For two opposite limiting cases our results for $\kappa_{\rm eff}$ are: one is similar to that of Tribble (1989), and the other coincides to that of Tao (1995).

2. Diffusion approximation for the random walk of magnetic field lines

In a particularly simple approach, the behavior of tangled magnetic field lines can be considered by using the one-dimensional random walk model suggested by Tribble (1989). In this model the lines are assumed to random walk between two infinite boundary plates, which are placed at $x = 0$ and $x = X_0$ perpendicular to the x-direction. In this section we consider this simple model in order to find the probability distributions of the lengths of the field lines in clusters of galaxies (we then use these distributions to find the effective thermal conductivity in the next sections). Contrary to the discrete calculations of Tribble, we use the continuum diffusion approximation for the random walk of the field lines. This allows us to include a weak homogeneous mean magnetic field component into our calculations.

2.1. Diffusion equation

We assume that the mean magnetic field component, $\langle\mathbf{B}\rangle$, is homogeneous, i.e. $\langle\mathbf{B}\rangle = \text{const}$. Let choose the coordinate system in a such way that $\langle\mathbf{B}\rangle$ lies in the x-z-plane: $\langle B\rangle_x = \langle B\rangle\cos\theta$, $\langle B\rangle_y = 0$, $\langle B\rangle_z = \langle B\rangle\sin\theta$. Here θ is the angle between the mean field component and the x-direction. Further, assume that the random component of the magnetic field, $\boldsymbol{\delta}\mathbf{B}$, is much stronger than the mean component, i.e. $\langle B\rangle/\delta B \ll 1$. Let express x-, y- and z-components of $\boldsymbol{\delta}\mathbf{B}$ in the spherical system of coordinates: $\delta B_x = \delta B\cos\varphi$, $\delta B_y = \delta B\sin\varphi\cos\psi$ and $\delta B_z = \delta B\sin\varphi\sin\psi$. The random component is isotropically distributed, so $\cos\varphi$ and ψ are uniformly distributed over $[-1, 1]$ and $[0, 2\pi)$ respectively. Under these assumptions, the cosine of the angle between the total magnetic field, $\mathbf{B} = \boldsymbol{\delta}\mathbf{B} + \langle\mathbf{B}\rangle$, and the x-direction is

$$\cos\alpha = \cos\varphi + [\langle B\rangle/\delta B][\cos\theta - \cos\theta\cos^2\varphi - \sin\theta\sin\varphi\cos\varphi\sin\psi], \tag{1}$$

where we keep terms only up to first order in $\langle B\rangle/\delta B \ll 1$.

Let the largest scale of the random component of the magnetic field be $l_0 \ll X_0$. The largest scale component has more magnetic energy in it than all smaller scale components have. Therefore, the decorrelation length of the total random component is l_0, and over each decorrelation length the random field changes into an entirely new direction. As a result, the mean step and the mean-squared step of the field line random walk in the x-direction are

$$\begin{aligned} \langle \Delta x \rangle &= l_0 \langle \cos\alpha \rangle_{\varphi,\psi,\delta B} &= (2/3) l_0 \epsilon \cos\theta, \\ \langle (\Delta x)^2 \rangle &= l_0^2 \langle \cos^2\alpha \rangle_{\varphi,\psi,\delta B} &= (1/3) l_0^2 \end{aligned} \tag{2}$$

over each decorrelation length. Here, to obtain these final expressions, we average $\cos\alpha$ and $\cos^2\alpha$ over direction and over absolute value of the field random component (i.e. over φ, ψ and δB). We also introduce the parameter $\epsilon \stackrel{\text{def}}{=} \langle B \rangle / \langle \delta B \rangle \ll 1$, which is the mean ratio of the magnetic field mean component to the random component.

There is the unique magnetic field line that goes through any given point of space between the two boundary plates. This field line leaves the point along two branches: a *positive branch* that starts at the initial point and goes always in the direction of the local field, and a *negative branch* that goes always opposite to the direction of the local field (see Figure 1). Let consider the following problem. Start at point $(x = X, y, z)$ and follow the positive branch of the magnetic field line going through this point, i.e. along the field line and always in the direction of the local field. The positive field line branch random walks in space according to equations (2). Let $P^+(s,x)\,ds\,dx$ be the probability that we are at x-position $x \in [x, x+dx)$ [and at any y-, z-positions] after we "have walked" along the line a distance $s \in [s, s+ds)$. The upper index "(+)" refers to the positive branch of the magnetic field line. Using equations (2), it is straightforward to write the diffusion equation for $P^+(s,x)$ as [1]

$$\frac{\partial P^+}{\partial s} = -\frac{\langle \Delta x \rangle}{l_0} \frac{\partial P^+}{\partial x} + \frac{1}{2} \frac{\langle (\Delta x)^2 \rangle}{l_0} \frac{\partial^2 P^+}{\partial x^2} = -\frac{2\epsilon\cos\theta}{3} \frac{\partial P^+}{\partial x} + \frac{l_0}{6} \frac{\partial^2 P^+}{\partial x^2}, \tag{3}$$

$$P^+(0,x) = \delta(x - X). \tag{4}$$

This is a Fokker-Planck equation. Here, $\delta(x - X)$ is the Dirac delta-function, and the formula on the second line is the initial condition on $P^+(s,x)$, which means we start walking along the field line at $x = X$. If we now consider the field line's negative branch, which goes always opposite to the direction of local field, then we need to replace ϵ by $-\epsilon$ in formula (3) in order to obtain the differential equation for the probability function $P^-(s,x)$ for this negative branch. (Note, that this replacement corresponds to the substitution $\pi - \theta$ for θ.)

Hereafter, we use convenient dimensionless variables

$$\mathsf{s} = (3X_0^2/l_0)^{-1} s, \quad \mathsf{x} = x/X_0, \quad \mathsf{X} = X/X_0, \quad \mathsf{P}^\pm = X_0 P^\pm. \tag{5}$$

Note that the positions of the boundary plates are $\mathsf{x} = 0$ and $\mathsf{x} = 1$. In these variables

$$\frac{\partial \mathsf{P}^\pm}{\partial \mathsf{s}} = \mp\beta \frac{\partial \mathsf{P}^\pm}{\partial \mathsf{x}} + \frac{1}{2} \frac{\partial^2 \mathsf{P}^\pm}{\partial \mathsf{x}^2}, \qquad \mathsf{P}^\pm(0,\mathsf{x}) = \delta(\mathsf{x} - \mathsf{X}). \tag{6}$$

Here, we introduce an important parameter

$$\beta \stackrel{\text{def}}{=} 2 X_0 l_0^{-1} \epsilon \cos\theta. \tag{7}$$

[1] Note that the diffusion approximation is valid if $x \gg l_0$ and $X_0 - x \gg l_0$.

In the limit $\beta \ll 1$, the effect of the mean magnetic field component is negligible and the "diffusion term" $(1/2)\,(\partial^2 \mathsf{P}^\pm/\partial \mathsf{x}^2)$ in equation (6) is dominant. In this small β limit the random walk of field lines is controlled by their diffusion in space, and for a given $\mathsf{s} \sim 1$, the functions $\mathsf{P}^\pm(\mathsf{s},\mathsf{x})$ have considerable spreads in x. On the other hand, in the limit $\beta \gg 1$, the effect of the mean magnetic field component is large and the "flux term" $-\beta\,(\partial \mathsf{P}^\pm/\partial \mathsf{x})$ in equation (6) is dominant. In this large β limit the probability functions $\mathsf{P}^\pm$ stay very narrow in x for a given value of s, $\mathsf{P}^\pm(\mathsf{s},\mathsf{x}) \approx \delta(\mathsf{x} - \mathsf{X} \mp \beta \mathsf{s})$. When $\beta \sim 1$, both terms in the right-hand side of equation (6) are equally important. Thus, β indicates whether the influence of the mean field component on the field line random walk is negligible ($\beta \ll 1$), important ($\beta \sim 1$) or dominant ($\beta \gg 1$).

2.2. Boundary conditions and probability fluxes

Equations (6) need the boundary conditions on the probability functions $\mathsf{P}^\pm(\mathsf{s},\mathsf{x})$ at the two boundary plates. Let assume for definiteness, that the plate at $\mathsf{x} = 0$ represents the boundary of a galaxy cluster, the plate at $\mathsf{x} = 1$ represents the cluster core (see Figure 1), and the mean component of the magnetic field has a positive x-component, i.e. $0 < \theta < \pi/2$ (clearly the sign of $\cos\theta$ has no bearing on thermal diffusion problem). As we said above, there are two field line branches that we can follow starting at $\mathsf{x} = \mathsf{X}$, the positive branch and the negative branch. Each branch random walks in space, and can reach the cluster boundary at $\mathsf{x} = 0$ and the cluster core at $\mathsf{x} = 1$.

Let first consider the boundary conditions at the cluster core. The core is assumed to be much denser and much colder than the rest of the cluster, so we can neglect the temperature T in the core and set it to zero. As a result, whenever the positive or the negative branch reaches the core at $\mathsf{x} = 1$, it looses its thermal energy and cools down to zero temperature immediately. Therefore, there is no any point in following the field line branches after they first reach the core, and the corresponding boundary conditions on the probability functions $\mathsf{P}^\pm$ at the core are the absorption conditions: $\mathsf{P}^\pm(\mathsf{s},1) = 0$.

Let now consider the boundary conditions at the cluster boundary. We assume that the density drops significantly there. Therefore, the random component of the magnetic field also drops considerably at the cluster boundary, because this random component is believed to be created by MHD dynamo action inside the cluster. On the other hand, the mean component of the field hardly changes at the boundary, because it is associated with the open magnetic field lines that leave the cluster. As a result, the parameter β, given by equation (7), increases and becomes large at the cluster boundary (while the mean field component can still be less than the random component, so that our diffusion approximation is still valid). In other words, we consider β to be constant inside the cluster and to become large at the boundary. Therefore, at the cluster boundary $\mathsf{x} = 0$, the term $-\beta\,(\partial \mathsf{P}^\pm/\partial \mathsf{x})$ in equation (6) is always dominant. As a result, the positive field line branch is reflected at the cluster boundary, while the negative branch is absorbed and leaves the cluster.

To summarize, we write the boundary conditions for the positive and negative field line branches as

$$\begin{aligned} &\mathsf{P}^-(\mathsf{s},0) = 0, \quad \text{absorption of the negative branch at the boundary,} \\ &\mathsf{P}^-(\mathsf{s},1) = 0, \quad \text{absorption of the negative branch at the core;} \end{aligned} \tag{8}$$

$$\begin{aligned} \beta\, \mathsf{P}^{+}(\mathsf{s},0) - (1/2)(\partial \mathsf{P}^{+}/\partial \mathsf{x})|_{\mathsf{x}=0} = 0, &\quad \text{reflection of the positive branch at the boundary,} \\ \mathsf{P}^{+}(\mathsf{s},1) = 0, &\quad \text{absorption of the positive branch at the core,} \end{aligned} \tag{9}$$

see Figure 1.

Let for a moment consider the positive field line branch and integrate the diffusion equation (6) over $\mathsf{x} \in (0,1)$, using boundary conditions (9). As a result, we obtain the negative change of the total probability that we are still inside the cluster, $\mathsf{x} \in (0,1)$, after we walked a distance s along the line's positive branch:

$$-\frac{d}{d\mathsf{s}} \int_0^1 \mathsf{P}^{+}(\mathsf{s},\mathsf{x})\, d\mathsf{x} = -\frac{1}{2}\frac{\partial \mathsf{P}^{+}}{\partial \mathsf{x}}\bigg|_{\mathsf{x}=1} = \mathsf{F}_1^{+}(\mathsf{s}). \tag{10}$$

Here we introduce a "probability flux" into the cluster core at $\mathsf{x} = 1$:

$$\mathsf{F}_1^{+}(\mathsf{s}) = -(1/2)\,(\partial \mathsf{P}^{+}/\partial \mathsf{x})|_{\mathsf{x}=1} \tag{11}$$

(the upper index "+" refers to the field line's positive branch). Note, that the probability flux across the cluster boundary at $\mathsf{x} = 0$ is zero because of the reflection condition there [see eqs. (9)]. The probability, that we leave the cluster and enter the cluster core when $\mathsf{s} \in [\mathsf{s}, \mathsf{s} + d\mathsf{s})$, is equal to $\mathsf{F}_1^{+}(\mathsf{s})\, d\mathsf{s}$. Therefore, $\mathsf{F}_1^{+}(\mathsf{s})$ is the *probability distribution* of the lengths s of all positive field line branches that start at $\mathsf{x} = \mathsf{X}$, random walk in space and finally come to the cluster core at $\mathsf{x} = 1$.

Now consider the negative field line branch. In this case, we integrate the appropriate diffusion equation (6) over $\mathsf{x} \in (0,1)$ and use the boundary conditions (8). We find that the negative change of the total probability is now given by the sum of two probability fluxes $\mathsf{F}_1^{-}(\mathsf{s})$ and $\mathsf{F}_0^{-}(\mathsf{s})$ into the core (at $\mathsf{x} = 1$) and across the boundary (at $\mathsf{x} = 0$) respectively:

$$\mathsf{F}_1^{-}(\mathsf{s}) = -(1/2)\,(\partial \mathsf{P}^{-}/\partial \mathsf{x})|_{\mathsf{x}=1}, \qquad \mathsf{F}_0^{-}(\mathsf{s}) = (1/2)\,(\partial \mathsf{P}^{-}/\partial \mathsf{x})|_{\mathsf{x}=0}. \tag{12}$$

Similarly to the case of the positive branch, $\mathsf{F}_1^{-}(\mathsf{s})$ and $\mathsf{F}_0^{-}(\mathsf{s})$ are the probability distributions of the lengths of all negative field line branches that start at $\mathsf{x} = \mathsf{X}$, random walk in space and finally come to either the cluster boundary at $\mathsf{x} = 0$, or to the cluster core at $\mathsf{x} = 1$.

Because the probability fluxes are the probability distributions of the lengths of field line branches, hereafter, we refer to them as to the probability distributions.

2.3. Laplace transform solutions

Although equations (6) are linear, they are still difficult to solve analytically for both sets of boundary conditions given by equations (8) and (9). We solve equations (6) numerically in Section 4. However, in some cases we do not need the solution of equations (6) in order to find the effective thermal conductivity in tangled magnetic fields. In the next section we will find this conductivity using a simple stationary model and the Laplace images of the probability distributions $\mathsf{F}_1^{+}(\mathsf{s})$, $\mathsf{F}_1^{-}(\mathsf{s})$ and $\mathsf{F}_0^{-}(\mathsf{s})$ that we obtain in this section.

To calculate these Laplace images, we first take the Laplace transforms $\mathsf{s} \to \tilde{\mathsf{s}}$, $\mathsf{P}^{\pm}(\mathsf{s},\mathsf{x}) \to \tilde{\mathsf{P}}^{\pm}(\tilde{\mathsf{s}},\mathsf{x})$ of equations (6). We have

$$\tilde{\mathsf{s}}\tilde{\mathsf{P}}^{\pm} - \delta(\mathsf{x} - \mathsf{X}) \;=\; \mp\beta\frac{\partial \tilde{\mathsf{P}}^{\pm}}{\partial \mathsf{x}} + \frac{1}{2}\frac{\partial^2 \tilde{\mathsf{P}}^{\pm}}{\partial \mathsf{x}^2}. \tag{13}$$

The Laplace images $\tilde{\mathsf{P}}^{\pm}(\tilde{\mathsf{s}},\mathsf{x})$ must be continuous functions of x. Thus, integrating equations (13) across $\mathsf{x}=\mathsf{X}$, i.e. over an infinitesimal interval $\mathsf{x}\in(\mathsf{X}-0,\mathsf{X}+0)$, we obtain an additional jump conditions together with the continuity conditions

$$[\partial\tilde{\mathsf{P}}^{\pm}/\partial\mathsf{x}]_{\mathsf{X}\pm 0}=-2, \qquad [\tilde{\mathsf{P}}^{\pm}]_{\mathsf{X}\pm 0}=0. \tag{14}$$

Here, we denote the jump at $\mathsf{x}=\mathsf{X}$ as $[...]_{\mathsf{X}\pm 0}$.

The boundary conditions on the images $\tilde{\mathsf{P}}^{\pm}(\tilde{\mathsf{s}},\mathsf{x})$ are the same as those on the original functions $\mathsf{P}^{\pm}(\mathsf{s},\mathsf{x})$. In the case of the negative branch they are given by equations (8), in the case of the positive branch they are given by equations (9). Equation (13) is a simple linear homogeneous differential equation provided $\mathsf{x}\neq\mathsf{X}$. There are two independent solutions of this equation expressed in terms of hyperbolic functions. We express the two general solutions of this equation in the regions $0\le\mathsf{x}<\mathsf{X}$ and $\mathsf{X}<\mathsf{x}\le 1$ as two linear combinations of these hyperbolic solutions. Then, we use the appropriate boundary conditions and jump conditions (14) to find the unique continuous solutions of equations (13) for $\mathsf{P}^{\pm}(\mathsf{s},\mathsf{x})$ in the whole interval of x, $0\le\mathsf{x}\le 1$. We have

$$\tilde{\mathsf{P}}^{+}(\tilde{\mathsf{s}},\mathsf{x})=\frac{2}{\xi_0}\frac{\exp\left[\beta(\mathsf{x}-\mathsf{X})\right]}{\xi_0\cosh\xi_0+\beta\sinh\xi_0}\times\begin{cases}[\xi_0\cosh(\xi_0\mathsf{X})+\beta\sinh(\xi_0\mathsf{X})]\sinh[\xi_0(1-\mathsf{x})], & \mathsf{X}\le\mathsf{x};\\ [\xi_0\cosh(\xi_0\mathsf{x})+\beta\sinh(\xi_0\mathsf{x})]\sinh[\xi_0(1-\mathsf{X})], & \mathsf{x}\le\mathsf{X}\end{cases} \tag{15}$$

for the positive field line branch, and

$$\tilde{\mathsf{P}}^{-}(\tilde{\mathsf{s}},\mathsf{x})=\frac{2}{\xi_0}\frac{\exp\left[-\beta(\mathsf{x}-\mathsf{X})\right]}{\sinh\xi_0}\times\begin{cases}\sinh(\xi_0\mathsf{X})\sinh[\xi_0(1-\mathsf{x})], & \mathsf{X}\le\mathsf{x};\\ \sinh(\xi_0\mathsf{x})\sinh[\xi_0(1-\mathsf{X})], & \mathsf{x}\le\mathsf{X}\end{cases} \tag{16}$$

for the negative branch. Here, we introduce

$$\xi_0(\tilde{\mathsf{s}})\stackrel{\mathrm{def}}{=}\sqrt{\beta^2+2\tilde{\mathsf{s}}}\,. \tag{17}$$

Now, we substitute these formulas into Laplace transformed equations (11) and (12) to find the Laplace images $\tilde{\mathsf{F}}_1^{+}(\tilde{\mathsf{s}})$, $\tilde{\mathsf{F}}_1^{-}(\tilde{\mathsf{s}})$ and $\tilde{\mathsf{F}}_0^{-}(\tilde{\mathsf{s}})$ of the probability distributions $\mathsf{F}_1^{+}(\mathsf{s})$, $\mathsf{F}_1^{-}(\mathsf{s})$ and $\mathsf{F}_0^{-}(\mathsf{s})$. We have

$$\tilde{\mathsf{F}}_1^{+}(\tilde{\mathsf{s}})=\int_0^\infty e^{-\tilde{\mathsf{s}}\mathsf{s}}\,\mathsf{F}_1^{+}(\mathsf{s})\,d\mathsf{s} \;=\; -\frac{1}{2}\frac{\partial\tilde{\mathsf{P}}^{+}}{\partial\mathsf{x}}\Big|_{\mathsf{x}=1}=\frac{\xi_0\cosh(\xi_0\mathsf{X})+\beta\sinh(\xi_0\mathsf{X})}{\xi_0\cosh\xi_0+\beta\sinh\xi_0}\,e^{\beta(1-\mathsf{X})}, \tag{18}$$

$$\tilde{\mathsf{F}}_1^{-}(\tilde{\mathsf{s}})=\int_0^\infty e^{-\tilde{\mathsf{s}}\mathsf{s}}\,\mathsf{F}_1^{-}(\mathsf{s})\,d\mathsf{s} \;=\; -\frac{1}{2}\frac{\partial\tilde{\mathsf{P}}^{-}}{\partial\mathsf{x}}\Big|_{\mathsf{x}=1}=\frac{\sinh(\xi_0\mathsf{X})}{\sinh\xi_0}\,e^{-\beta(1-\mathsf{X})}, \tag{19}$$

$$\tilde{\mathsf{F}}_0^{-}(\tilde{\mathsf{s}})=\int_0^\infty e^{-\tilde{\mathsf{s}}\mathsf{s}}\,\mathsf{F}_0^{-}(\mathsf{s})\,d\mathsf{s} \;=\; +\frac{1}{2}\frac{\partial\tilde{\mathsf{P}}^{-}}{\partial\mathsf{x}}\Big|_{\mathsf{x}=0}=\frac{\sinh[\xi_0(1-\mathsf{X})]}{\sinh\xi_0}\,e^{\beta\mathsf{X}}. \tag{20}$$

Here, the integrals are the definition of the Laplace images, and ξ_0 is given by formula (17).

As is well known from the theory of the Laplace transform [and can easily be checked by differentiation of equations (18)–(20) with respect to the Laplace variable $\tilde{\mathsf{s}}$ and setting $\tilde{\mathsf{s}}$ to zero], the following formulas stand for the integral moments of the probability distributions $\mathsf{F}_1^{+}(\mathsf{s})$, $\mathsf{F}_1^{-}(\mathsf{s})$ and $\mathsf{F}_0^{-}(\mathsf{s})$:

$$\int_0^\infty \mathsf{s}^n\mathsf{F}_1^{\pm}(\mathsf{s})\,d\mathsf{s}=(-1)^n\,\frac{d^n\tilde{\mathsf{F}}_1^{\pm}}{d\tilde{\mathsf{s}}^n}\Big|_{\tilde{\mathsf{s}}=0}, \qquad \int_0^\infty \mathsf{s}^n\mathsf{F}_0^{-}(\mathsf{s})\,d\mathsf{s}=(-1)^n\,\frac{d^n\tilde{\mathsf{F}}_0^{-}}{d\tilde{\mathsf{s}}^n}\Big|_{\tilde{\mathsf{s}}=0}. \tag{21}$$

For example, the zeroth integral moment of $\mathsf{F}_1^+(\mathsf{s})$ is

$$\int_0^\infty \mathsf{F}_1^+(\mathsf{s})\, d\mathsf{s} = \tilde{\mathsf{F}}_1^+(0) = 1, \tag{22}$$

which means that the total probability for the positive branch to ultimately reach the cluster core is unity, as it should be because positive branches always end up in the core. The zeroth integral moments of $\mathsf{F}_1^-(\mathsf{s})$ and $\mathsf{F}_0^-(\mathsf{s})$ are

$$\int\limits_0^\infty \mathsf{F}_1^-(\mathsf{s})\, d\mathsf{s} = \tilde{\mathsf{F}}_1^-(0) = \frac{\exp(2\beta\mathsf{X}) - 1}{\exp(2\beta) - 1}, \quad \int\limits_0^\infty \mathsf{F}_0^-(\mathsf{s})\, d\mathsf{s} = \tilde{\mathsf{F}}_0^-(0) = \frac{\exp(2\beta) - \exp(2\beta\mathsf{X})}{\exp(2\beta) - 1}, \tag{23}$$

which means that $\tilde{\mathsf{F}}_1^-(0)$ and $\tilde{\mathsf{F}}_0^-(0)$ are the probabilities for the negative branch to reach the cluster core (at $\mathsf{x} = 1$) and the cluster boundary (at $\mathsf{x} = 0$) respectively. Note, that the total probability for the negative branch to reach either the core or the boundary, is equal to one, $\tilde{\mathsf{F}}_1^-(0) + \tilde{\mathsf{F}}_0^-(0) = 1$, as it should be.

It follows from equations (23) that $\mathsf{F}_1^-(\mathsf{s})/\tilde{\mathsf{F}}_1^-(0)$ and $\mathsf{F}_0^-(\mathsf{s})/\tilde{\mathsf{F}}_0^-(0)$ are the *normalized* (to one) probability distributions of the lengths of only those negative field line branches that reach the cluster core and only those negative branches that reach the cluster boundary respectively (see Section 2.2). The probability distributions of the positive branches, $\mathsf{F}_1^+(\mathsf{s})$, does not need to be normalized, because all positive branches always end up in the cluster core.

3. Stationary model

In this section we consider a stationary one-dimensional temperature distribution in a cluster of galaxies. We assume that there is a stationary homogeneous source of heat q between the two Tribble boundary plates, which represent the cluster core and the cluster boundary. The heat is transported by electrons along the tangled magnetic field lines. We further assume that the thermal conductivity parallel to the field lines, $\kappa_\parallel$, is constant. We keep our assumption made in Section 2.2 that the cluster core is cold, i.e. $T = 0$ at $\mathsf{x} = 1$. On the other hand, the density drops significantly at the cluster boundary, so we use the heat reflection condition there, $\partial T/\partial s = 0$ at $\mathsf{x} = 0$ (here, s is the distance coordinate along a field line).

Let consider a point $(\mathsf{x} = \mathsf{X}, y, z)$ inside the cluster. The stationary temperature at this point depends on the lengths of the positive and the negative branches of the magnetic field line passing through this point. The positive branch always reaches the cluster core at $\mathsf{x} = 1$, where it is cooled to zero temperature. Let the positive branch have dimensionless length s_1^+. As for the negative branch, there are two possibilities. First, with probability $\tilde{\mathsf{F}}_1^-(0)$ it reaches the core at $\mathsf{x} = 1$ and cools down to $T = 0$. In this case we denote the dimensionless length of the negative branch by s_1^-. The second possibility is that the negative branch with probability $\tilde{\mathsf{F}}_0^-(0)$ reaches the cluster boundary at $\mathsf{x} = 0$, where $\partial T/\partial \mathsf{s} = 0$. In this case we denote the dimensionless length of the negative branch by s_0^-. In any case, the differential equation for the temperature distribution along the field line is

$$\frac{\partial^2 T}{\partial \mathsf{s}^2} + \frac{9X_0^4}{l_0^2}\frac{q}{\kappa_\parallel} = 0, \tag{24}$$

where s is the dimensionless coordinate along the field line, given by equations (5). [2]

Now, let us consider a plane $\mathsf{x} = \mathsf{X}$, which is perpendicular to the x-direction, and let us find the temperature averaged over points of this plane. We make this averaging by randomly choosing points of the plane, and then, by averaging temperature over the chosen points. [3] The lengths s_1^+, s_1^- and s_0^- of positive and negative branches, going through different points are uncorrelated, and these lengths have the probability distributions $\mathsf{F}_1^+(\mathsf{s})$, $\mathsf{F}_1^-(\mathsf{s})$ and $\mathsf{F}_0^-(\mathsf{s})$ respectively. Therefore, the averaged temperature and the averaged temperature squared are

$$\langle T \rangle = \frac{q}{2\kappa_\parallel}\frac{9X_0^4}{l_0^2}\left\{\tilde{\mathsf{F}}_1^-(0)\langle \mathsf{s}_1^+\rangle\langle \mathsf{s}_1^-\rangle + \tilde{\mathsf{F}}_0^-(0)\left[\langle(\mathsf{s}_1^+)^2\rangle + 2\langle \mathsf{s}_1^+\rangle\langle \mathsf{s}_0^-\rangle\right]\right\}, \tag{25}$$

$$\langle T^2 \rangle = \left[\frac{q}{2\kappa_\parallel}\frac{9X_0^4}{l_0^2}\right]^2\left\{\tilde{\mathsf{F}}_1^-(0)\langle(\mathsf{s}_1^+)^2\rangle\langle(\mathsf{s}_1^-)^2\rangle + \tilde{\mathsf{F}}_0^-(0)\left[\langle(\mathsf{s}_1^+)^4\rangle + 4\langle(\mathsf{s}_1^+)^3\rangle\langle \mathsf{s}_0^-\rangle + 4\langle(\mathsf{s}_1^+)^2\rangle\langle(\mathsf{s}_0^-)^2\rangle\right]\right\} \tag{26}$$

Here, the factors multiplying by $\tilde{\mathsf{F}}_1^-(0)$ represent the averaged temperature obtained by solving equation (24) in the case when both the positive and the negative branches reach the cluster core (where $T = 0$), as shown by the lower field line in Figure 1. The factors multiplying by $\tilde{\mathsf{F}}_0^-(0)$ represent the averaged temperature obtained by solving equation (24) when the positive branch reaches the cluster core and the negative branch reaches the cluster boundary (where $\partial T/\partial \mathsf{s} = 0$), as shown by the upper field line in Figure 1. Remember that $\tilde{\mathsf{F}}_1^-(0)$ and $\tilde{\mathsf{F}}_0^-(0)$ are the probabilities for the negative branch to reach the core and the boundary respectively, while the positive branch always reaches the core. All brackets $\langle ... \rangle$ mean averaging over appropriate probability distributions of lengths of branches. Note, that the expressions inside the brackets $\{...\}$ in the equations above are dimensionless, while q, $\kappa_\parallel$, X_0 and l_0 are not.

Now we use equations (21)–(23) to express averaged powers of the branch lengths in the formulas (25) and (26) in terms of the Laplace images of the probability distributions. We obtain

$$\langle T \rangle = \frac{q}{2\kappa_\parallel}\frac{9X_0^4}{l_0^2}\left\{\tilde{\mathsf{F}}_0^-\frac{d^2\tilde{\mathsf{F}}_1^+}{d\tilde{\mathsf{s}}^2} + \frac{d\tilde{\mathsf{F}}_1^-}{d\tilde{\mathsf{s}}}\frac{d\tilde{\mathsf{F}}_1^+}{d\tilde{\mathsf{s}}} + 2\frac{d\tilde{\mathsf{F}}_0^-}{d\tilde{\mathsf{s}}}\frac{d\tilde{\mathsf{F}}_1^+}{d\tilde{\mathsf{s}}}\right\}_{\tilde{\mathsf{s}}=0}, \tag{27}$$

$$\langle T^2 \rangle = \left[\frac{q}{2\kappa_\parallel}\frac{9X_0^4}{l_0^2}\right]^2\left\{\tilde{\mathsf{F}}_0^-\frac{d^4\tilde{\mathsf{F}}_1^+}{d\tilde{\mathsf{s}}^4} + 4\frac{d\tilde{\mathsf{F}}_0^-}{d\tilde{\mathsf{s}}}\frac{d^3\tilde{\mathsf{F}}_1^+}{d^3\tilde{\mathsf{s}}^3} + \frac{d^2\tilde{\mathsf{F}}_1^-}{d\tilde{\mathsf{s}}^2}\frac{d^2\tilde{\mathsf{F}}_1^+}{d\tilde{\mathsf{s}}^2} + 4\frac{d^2\tilde{\mathsf{F}}_0^-}{d\tilde{\mathsf{s}}^2}\frac{d^2\tilde{\mathsf{F}}_1^+}{d\tilde{\mathsf{s}}^2}\right\}_{\tilde{\mathsf{s}}=0}. \tag{28}$$

We can substitute the Laplace images of the three probability distributions given by formulas (18)–(20) into these equations and obtain analytical formulas for $\langle T \rangle$ and $\langle T^2 \rangle$. However, the resulting expressions are too complicated to be usefully interpreted. Therefore, we calculated all derivatives in the equations above numerically. Figure 2 shows the relative dispersion of the temperature, $(\langle T^2 \rangle - \langle T \rangle^2)/\langle T \rangle^2$, as a function of parameter β for two choices of x-position inside the cluster, $X = 0.5X_0$ and $X = 0.1X_0$. We see that the dispersion is high for small values of β, while it is $\propto 1/\beta \ll 1$ when $\beta \gg 1$, according to the discussion in the last paragraph of Section 2.1.

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[2] Our description for the temperature distribution along the field lines really refers to a space average of temperature over a length interval long compared to the field decorrelation length, l_0. The effect of regions of weak and strong magnetic field (magnetic mirrors) is taken into account by introducing $\kappa_\parallel$, which is reduced relative to the Spitzer conductivity, see Malyshkin and Kulsrud 2000.

[3] Formally, this plane is infinite, so are the number of points and distances between the points.

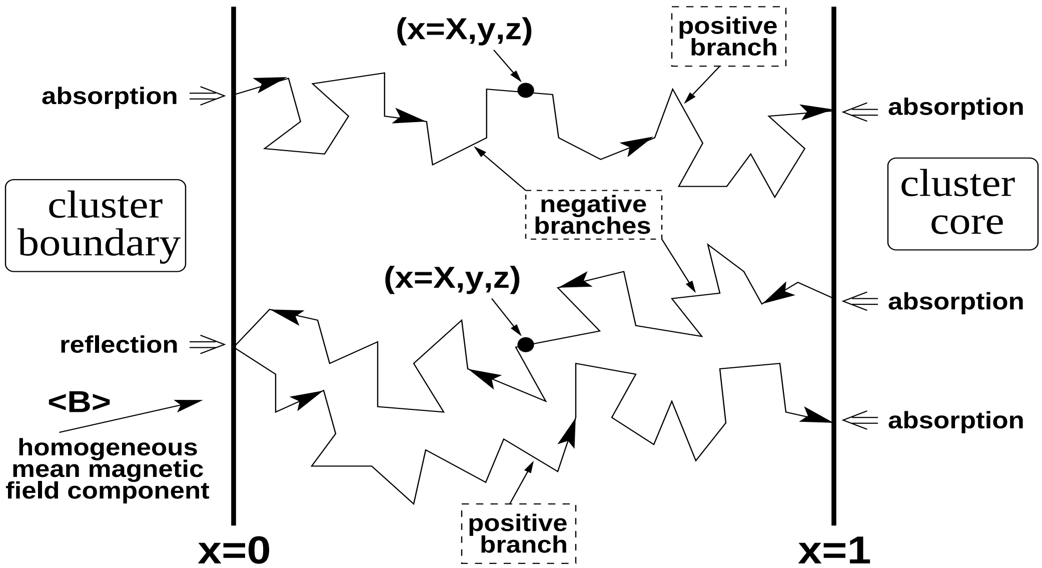

Fig. 1.— There is the unique magnetic field line that goes through any given point $(\mathsf{x}=\mathsf{X}, y, z)$ of space, this field line leaves the point along the positive branch and along the negative branch. All positive branches are reflected at the cluster boundary at $\mathsf{x}=0$, and are absorbed at the cluster core at $\mathsf{x}=1$. All negative branches are absorbed both at the boundary and at the core. Thus, there are two types of magnetic field lines: first, there are lines that go from the cluster boundary to the cluster core (see the upper line in this figure), and second, there are lines that leave the core and finally come back to the core (see the lower line in this figure).

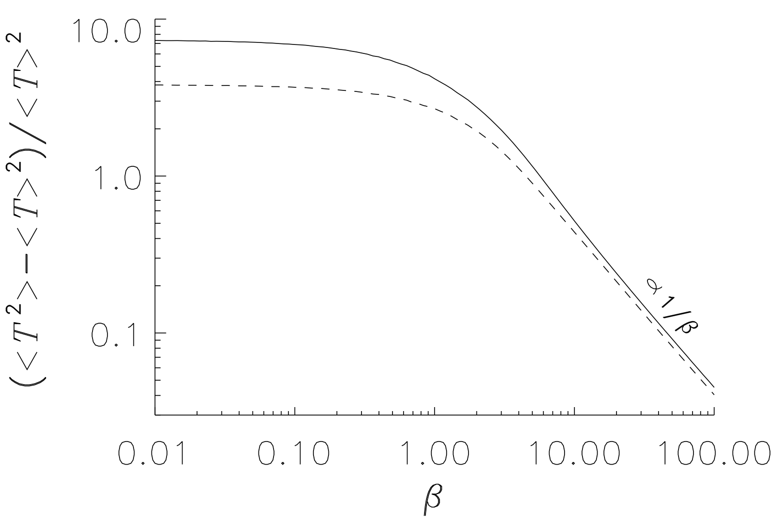

Fig. 2.— The relative temperature dispersion, $(\langle T^2\rangle - \langle T\rangle^2)/\langle T\rangle^2$, versus β. The solid and the dashed lines represent $X = 0.5X_0$ and $X = 0.1X_0$ cases respectively.

If there were no magnetic field at all, the temperature dispersion would be zero and the stationary temperature at $\mathsf{x} = \mathsf{X}$ plane would be

$$T_{\rm S} = \frac{q}{2\kappa_{\rm eff}}(X_0^2 - X^2), \tag{29}$$

where the effective thermal conductivity $\kappa_{\rm eff}$ would be equal to the Spitzer thermal conductivity, $\kappa_{\rm eff} = \kappa_{\rm S}$. Because there is tangled magnetic field, the actual stationary temperature is given by formula (27) and it is higher. However, instead of using formula (27) we can use the familiar formula (29) for the temperature if we choose the appropriate reduced value of the effective thermal conductivity. Equating formulas (27) and (29), we define the effective thermal conductivity for our stationary model as

$$\kappa_{\rm eff} = \kappa_\parallel \frac{l_0^2}{X_0^2}\,\frac{X_0^2 - X^2}{9X_0^2}\left\{\tilde{\mathsf{F}}_0^- \frac{d^2\tilde{\mathsf{F}}_1^+}{d\tilde{\mathsf{s}}^2} + \frac{d\tilde{\mathsf{F}}_1^-}{d\tilde{\mathsf{s}}}\frac{d\tilde{\mathsf{F}}_1^+}{d\tilde{\mathsf{s}}} + 2\frac{d\tilde{\mathsf{F}}_0^-}{d\tilde{\mathsf{s}}}\frac{d\tilde{\mathsf{F}}_1^+}{d\tilde{\mathsf{s}}}\right\}_{\tilde{\mathsf{s}}=0}^{-1}. \tag{30}$$

The dashed lines in Figures 3(a) and (b) show this effective conductivity normalized to $\kappa_\parallel l_0^2/X_0^2$ as a function of β for the two choices of position inside the cluster, $X = 0.1X_0$ and $X = 0.5X_0$. To obtain the plots we again calculated all derivatives of the Laplace images in equation (30) numerically. Note, that $\kappa_\parallel$ is the parallel thermal conductivity reduced by random magnetic mirrors (Malyshkin and Kulsrud 2000).

Note, that for large values of β the effective conductivity is independent of X and is simply $\kappa_{\rm eff} = \kappa_\parallel \beta^2 l_0^2/9X_0^2 = \kappa_\parallel (\Delta x/l_0)^2$.[4] This result exactly coincides with that given by the equation (4) of Tao (1995), who gives this result as a lower limit on $\kappa_{\rm eff}$. On the other hand, for small and moderate values of β the effective conductivity depends on X as on a parameter. This dependence results from our definition of $\kappa_{\rm eff}$ given above. In other words, the actual value of $\kappa_{\rm eff}$ depends on how one defines it for a particular problem under consideration. The calculations in the next section further support this important statement.

4. Time evolution of temperature in clusters of galaxies

In the previous section we considered a stationary one-dimensional temperature distribution in a cluster of galaxies, assuming a constant heat source inside the cluster. In this section we solve a time-dependent problem and find the evolution of temperature of the cluster in time. Let assume, that when the cluster was formed at zero time, $t = 0$, the temperature was homogeneous inside the cluster, i.e. $T = T_0$, $0 \le \mathsf{x} \le 1$ (here x is the dimensionless x-coordinate inside the cluster; $\mathsf{x} = 0$ and $\mathsf{x} = 1$ correspond to the boundary and the core of the cluster respectively, see Figure 1). We

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[4]This result can be found by the following calculations. The solutions of equations (6) in the limit $\beta \gg 1$ are $\mathsf{P}^\pm(\mathsf{s},\mathsf{x}) \approx \delta(\mathsf{x} - \mathsf{X} \mp \beta\mathsf{s})$. Thus, all positive field line branches reach the core and have dimensionless lengths $\mathsf{s}_1^+ \approx (1-\mathsf{X})/\beta$, while all negative branches reach the boundary and have lengths $\mathsf{s}_0^- \approx \mathsf{X}/\beta$. Substituting these expressions and $\tilde{\mathsf{F}}_1^-(0) \approx 0$, $\tilde{\mathsf{F}}_0^-(0) \approx 1$ into formula (25), and equating the result with formula (29), we obtain $\kappa_{\rm eff} = \kappa_\parallel \beta^2 l_0^2/9X_0^2 = \kappa_\parallel(\Delta x/l_0)^2$, see also eqs. (7) and (2). This result is valid only if $\beta^2 l_0^2/X_0^2 \ll 1$, which is equivalent to the assumed condition $\epsilon = \langle B\rangle/\langle \delta B\rangle \ll 1$, see Sec. 2.1.

assume that the cluster cools down in time by the heat conduction into the dense cluster core. As in the previous section, we again assume that the parallel thermal conductivity $\kappa_\parallel$ is constant, that the core is cold, $T = 0$ at $\mathsf{x} = 1$, and that there is the heat reflection condition along magnetic field lines at the cluster boundary, $\partial T/\partial \mathsf{s} = 0$ at $\mathsf{x} = 0$ (here s is the coordinate along field lines).

Because electrons travel along magnetic field lines, each field line cools down in time individually. Let us consider a temperature evolution at a point $(\mathsf{x} = \mathsf{X}, y, z)$ inside the cluster. There is a single field line going through this point. Initially, at $t = 0$, the temperature at this point is T_0. Then, as the field line cools down, the temperature drops in time. It is convenient to introduce a dimensionless time variable

$$\tau = \frac{l_0^2}{X_0^4} \frac{\kappa_\parallel}{\rho C_{\rm H}}\, t, \tag{31}$$

where ρ and $C_{\rm H}$ are the mass density and the heat capacity (per unit mass) of gas inside the cluster. For simplicity, we assume that the product $\rho C_{\rm H}$ is constant. Then, the time evolution of the temperature at the point (X, y, z) is given by the following simple equations:

$$\frac{\partial T(\tau, \mathsf{s})}{\partial \tau} = \frac{1}{9} \frac{\partial^2 T}{\partial^2 \mathsf{s}}, \qquad T(0, \mathsf{s}) = T_0, \tag{32}$$

where s is the dimensionless distance counted along the field line starting at the point (X, y, z), see equations (5).

There are two possibilities for the field line going through the point (X, y, z). First, both the line's positive branch of length s_1^+ and the line's negative branch of length s_1^- reach the cold cluster core (see the lower line in Figure 1). In this case the boundary conditions on the temperature distribution along the line are

$$T(\tau, -\mathsf{s}_1^-) = 0, \qquad T(\tau, \mathsf{s}_1^+) = 0.$$

With these boundary conditions the solution of equations (32) for the temperature at the point (X, y, z), where $\mathsf{s} = 0$, is

$$T_{11}(\tau, \mathsf{s}_1^-, \mathsf{s}_1^+) = T_0 \frac{4}{\pi} \sum_{n=0}^{\infty} (1+2n)^{-1} \exp\left[-\frac{\pi^2 \tau}{9(\mathsf{s}_1^- + \mathsf{s}_1^+)^2} (1+2n)^2 \right] \sin\left[\frac{\pi \mathsf{s}_1^+}{\mathsf{s}_1^- + \mathsf{s}_1^+} (1+2n) \right]. \tag{33}$$

Here, the index "11" indicates that both branches reach the core at $\mathsf{x} = 1$. The second possibility is that the line's negative branch of length s_0^- reaches the cluster boundary, while the line's positive branch of length s_1^+ ends up in the cluster core (see the upper line in Figure 1). In this case we have

$$(\partial T/\partial \mathsf{s})\big|_{\mathsf{s} = -\mathsf{s}_0^-} = 0, \qquad T(\tau, \mathsf{s}_1^+) = 0,$$

and the solution of equations (32) for the temperature at the point (X, y, z), where $\mathsf{s} = 0$, is

$$T_{01}(\tau, \mathsf{s}_0^-, \mathsf{s}_1^+) = T_0 \frac{4}{\pi} \sum_{n=0}^{\infty} (1+2n)^{-1} \exp\left[-\frac{\pi^2 \tau}{36(\mathsf{s}_0^- + \mathsf{s}_1^+)^2} (1+2n)^2 \right] \sin\left[\frac{\pi \mathsf{s}_1^+}{2(\mathsf{s}_0^- + \mathsf{s}_1^+)} (1+2n) \right]. \tag{34}$$

Here, the index "01" indicates that the negative branch goes to the boundary at $\mathsf{x} = 0$, while the positive branch goes to the core at $\mathsf{x} = 1$.

Now, let find the temperature averaged over all points in the plane $\mathsf{x} = \mathsf{X}$ in the same way as we did in the previous section. We have

$$\langle T \rangle(\tau) = \int_0^\infty \int_0^\infty T_{11}(\tau, \mathsf{s}_1^-, \mathsf{s}_1^+) \mathsf{F}_1^-(\mathsf{s}_1^-) \mathsf{F}_1^+(\mathsf{s}_1^+)\, d\mathsf{s}_1^- d\mathsf{s}_1^+ + \int_0^\infty \int_0^\infty T_{01}(\tau, \mathsf{s}_0^-, \mathsf{s}_1^+) \mathsf{F}_0^-(\mathsf{s}_0^-) \mathsf{F}_1^+(\mathsf{s}_1^+)\, d\mathsf{s}_0^- d\mathsf{s}_1^+. \quad (35)$$

Here, we have averaged over the probability distributions of s_1^+, s_1^- and s_0^-, which are given by equations (11) and (12). The two integrals in formula (35) correspond to the two possibilities for the field line branches, considered above. We have carried out these integrals numerically. Figures 4(a) and (b) show contour plots of $\langle T \rangle(\tau)/T_0$ in the τ–β parameter space for the two choices of X, $X = 0.5X_0$ and $X = 0.1X_0$. To obtain these plots, first, we numerically solved equations (6) with the boundary conditions (8) and (9), by making use of the standard implicit algorithm for partial differential equations. [5] Then we calculated the probability distributions (11) and (12) for different values of β. Finally, we substituted these distributions and formulas (33), (34) into equation (35), and we calculated the temperature for different values of τ and β. (The Laplace transform method was not useful for this problem because we need more than simple moments of the s_1^+ etc.)

If there were no magnetic field, the time evolution of the temperature would be given by equations

$$\frac{\partial T}{\partial t} = \frac{\kappa_{\rm eff}}{\rho C_{\rm H}} \frac{\partial^2 T}{\partial^2 x}, \quad T(0,x) = T_0, \quad (\partial T/\partial x)|_{x=0} = 0, \quad T(t, X_0) = 0, \quad (36)$$

resulting in the following expression for the temperature similar to equation (34):

$$\begin{aligned} T_{\rm S}(\tau) &= T_0 \frac{4}{\pi} \sum_{n=0}^{\infty} (1+2n)^{-1} \exp\left[\frac{\pi^2}{4} \frac{\kappa_{\rm eff} t}{X_0^2 \rho C_{\rm H}} (1+2n)^2\right] \sin\left[\frac{\pi(X_0 - X)}{2X_0}(1+2n)\right] \\ &= T_0 \frac{4}{\pi} \sum_{n=0}^{\infty} (1+2n)^{-1} \exp\left[\frac{\pi^2 \tau}{4} \frac{X_0^2 \kappa_{\rm eff}}{l_0^2 \kappa_\parallel} (1+2n)^2\right] \sin\left[\frac{\pi(X_0 - X)}{2X_0}(1+2n)\right], \quad (37) \end{aligned}$$

where we use definition (31). The effective thermal conductivity $\kappa_{\rm eff}$ would be equal to the Spitzer thermal conductivity, $\kappa_{\rm eff} = \kappa_{\rm S}$, if there were no magnetic field.

For each set of values T, τ, β and X there is a value of $\kappa_{\rm eff}$, which makes equations (35) and (37) agree. This value can be taken as the effective thermal conductivity, but it does depend

[5] We used a grid with equal steps $\Delta\mathsf{s}$ and $\Delta\mathsf{x}$ in s- and x-coordinates. The goal was to find the tabulated solution of equation (6) at all grid points. The solution at $s = 0$ is a delta-function. To find the solution at $s > 0$, we used a method similar to mathematical induction. This method allowed us, after we obtained the tabulated solution at s, to find the unknown tabulated solution at $\mathsf{s} + \Delta\mathsf{s}$ by the following algorithm. First, we wrote equation (6) as a finite difference operators, taking all x derivatives at $\mathsf{s} + \Delta\mathsf{s}$ (the implicit algorithm). As a result, we obtained a system of linear equations, which was a tridiagonal matrix. Second, we solved this system by the method of the Gaussian decomposition with backsubstitution, and found the solution at $\mathsf{s} + \Delta\mathsf{s}$.

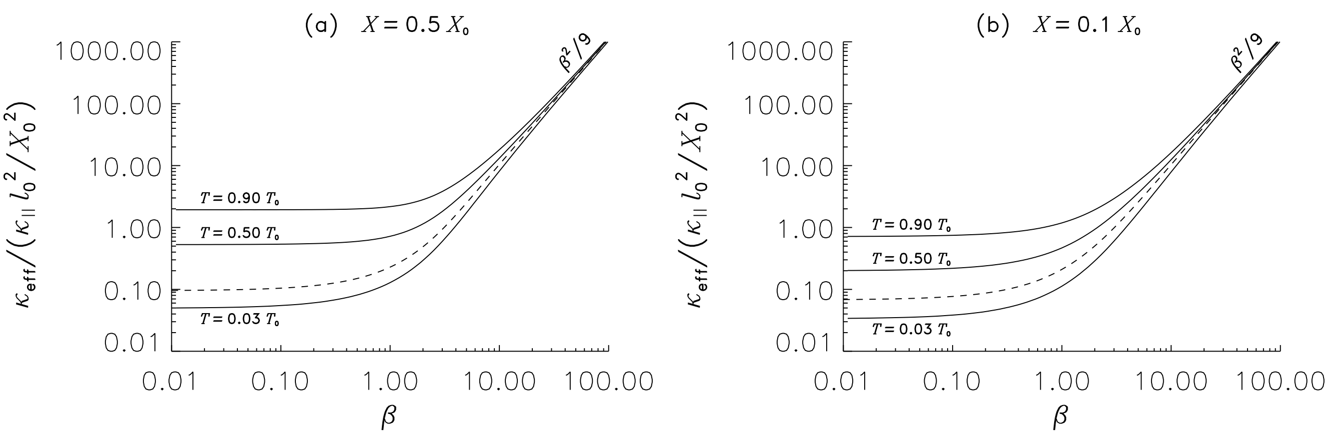

Fig. 3.— The effective thermal conductivity $\kappa_{\rm eff}$ normalized to $\kappa_{\|}l_0^2/X_0^2$ is plotted as function of parameter β. The function $\kappa_{\rm eff}(\beta)$ is not universal and it depends on how $\kappa_{\rm eff}$ is defined.

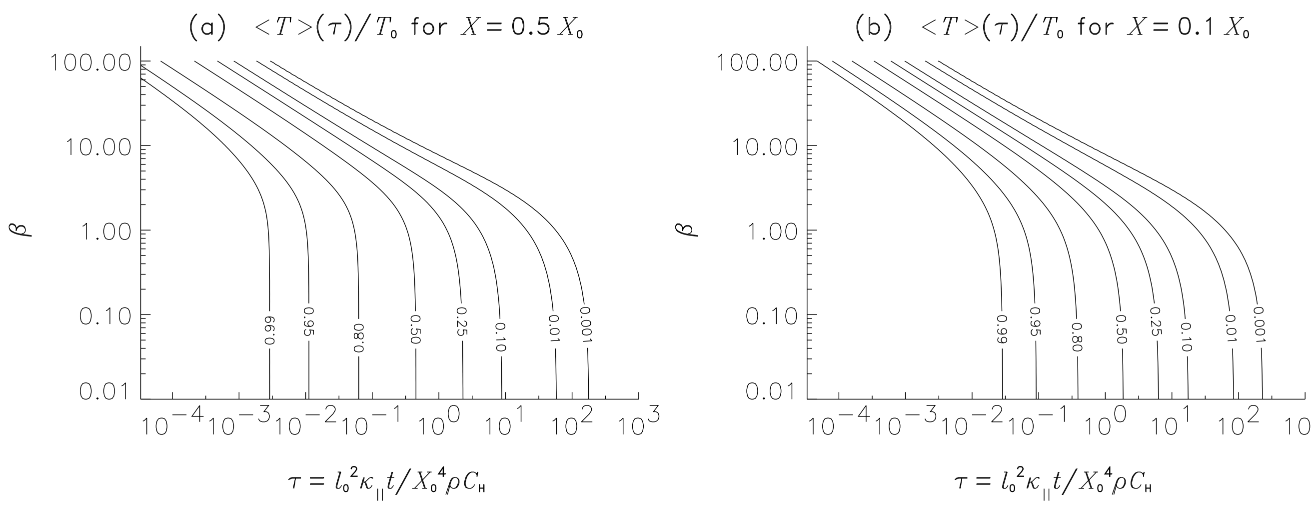

Fig. 4.— Contour plots of the average temperature normalized to the initial temperature, $\langle T\rangle(\tau)/T_0$, for two choices of X: (a) $X = 0.5X_0$ and (b) $X = 0.1X_0$.

on conditions. In the case of the presence of the tangled magnetic field, instead of using the complicated formula (35) we can use the simple formula (37) with this $\kappa_{\rm eff}$ reduced appropriately from the Spitzer value. To find the reduced effective conductivity, we first choose fixed values of position X and temperature T. Then, we find $\kappa_{\rm eff}$ as a function of β in a such way, that both formula (35) and formula (37) give the same chosen value of temperature, T, at the same time τ. The calculation of $\kappa_{\rm eff}(\beta)$ was done numerically for two choices of X: $X = 0.5X_0$ and $X = 0.1X_0$; and for three choices of T: $T = 0.03\,T_0$, $T = 0.50\,T_0$ and $T = 0.90\,T_0$. [In other words, we numerically found such functions $\kappa_{\rm eff}(\beta)$, that the temperature contours $T/T_0 = 0.03$, $T/T_0 = 0.50$ and $T/T_0 = 0.90$ appeared the same on Figures 4 when we used formulas (35) and (37) for the temperature.] The resulting functions $\kappa_{\rm eff}(\beta)$ normalized to $\kappa_\parallel l_0^2/X_0^2$ are given by the solid lines in Figures 3. We again see, that $\kappa_{\rm eff}(\beta)$ is not an universal function, and that it varies for different choices of T and X, except when the β is large. For smaller values of temperature the effective conductivity becomes less because very long field lines, which keep the initial temperature, become more important.

5. Discussion

We would like to start the discussion with by stressing one of the main results we found: the actual value of this effective conductivity in the tangled magnetic fields depends on how this effective conductivity is defined. In our paper we used two "natural" definitions of $\kappa_{\rm eff}$ for two very simple models of a cluster of galaxies: a time-independent model in Section 3 and a time-dependent model in Section 4. The results for $\kappa_{\rm eff}$, reported in figures 3(a) and 3(b), are the same only provided $\beta \gg 1$, and they are significantly different (by up to a factor of ten) when $\beta \lesssim 1$. We conclude that there does not exist an universal result for the effective conductivity in the tangled magnetic fields. One has to define $\kappa_{\rm eff}$ and to calculate it for a particular astrophysical problem that he/she considers. We believe that such calculation is possible for many problems, including numerical simulations of galaxy cluster formation and of cooling flows, by making use of our diffusion approximation method for the random walk of tangled magnetic field lines.

It is useful to compare our results for the effective conductivity in the tangled magnetic fields with those reported in previous papers. As we said in the last paragraph of Section 3, in the limit $\beta \gg 1$ our result for $\kappa_{\rm eff}$ coincides with the result obtained by Tao (1995). This simple formula (for β large) corresponds to lines being in order X_0/ϵ in length, a result sometimes believed (remember, that $\epsilon \ll 1$ is the ration of the field mean component to the field random component). But in fact, it is only valid as a lower limit on $\kappa_{\rm eff}$, except for large β. However, we can speculate that it revolves around the very definition of ϵ. We give ϵ first and then impose the statistics of the field. If the statistics is given first and the mean field is defined afterwards, as the rms value of the mean field on the scales $\sim X_0$, then $\kappa_{\rm eff}$ would be closer to our results.

In the limit $\beta \ll 1$, our result, $\kappa_{\rm eff} \sim 0.1\kappa_\parallel l_0^2/X_0^2$, is consistent with the result of Tribble (1989). [6] To the best of our knowledge, there have no results for $\kappa_{\rm eff}$ when $\beta \sim 1$ obtained before.

[6] Tribble has $\kappa_{\rm eff} = 4\kappa_\parallel l_0^2/X_0^2$, see his equation (11). The difference arises because: first, our boundary conditions (8) and (9) for the random walk of field lines are different from those of Tribble; second, the magnetic field lines

Recently Chandran and Cowley (1998) have suggested that the diffusion of the heat conducting electrons perpendicular to the magnetic field lines is crucially important for the heat transport. In their model the statistically independent random step by the electrons is equal to $\sqrt{2L_{\rm RR}l_0}$, where $L_{\rm RR} \sim l_0 \ln{(l_0/\rho_e)}$ is the Rechester-Rosenbluth length. This length is the distance along magnetic field lines over which a separation between two initially neighboring lines grows exponentially from the electron gyro-radius ρ_e up to the field decorrelation length l_0 (because of the Kolmogorov-Lyapunov exponential divergence of field lines). Chandran and Cowley suggested that the perpendicular diffusion of electrons and this exponential divergence of the magnetic field lines enable the electrons to conduct heat between different field lines, which may enhance the effective thermal conductivity considerably. However, we believe that the picture considered by Chandran and Cowley is rigorously valid only if electrons can move a distance $L_{\rm RR}$ along field lines without collisions. In fact, there are collisions, and if the electron mean free path λ is less than the Rechester-Rosenbluth length $L_{\rm RR}$, the exponential divergence of field lines does not help electrons to diffuse perpendicular field lines. In clusters of galaxies, the ratio of $L_{\rm RR}$ to λ is $L_{\rm RR}/\lambda \approx 5000\,(l_0/10\,{\rm kpc})\,(T/10^7{\rm K})^{-2}\,(n/10^{-3}{\rm cm}^{-3})$. As a result, the perpendicular heat transport can generally be neglected (we accept 3–10 kpc as a typical scale for magnetic fields in clusters of galaxies, see Kronberg 1994, Eilek 1999). The Chandran and Cowley's model can be very important for some other astrophysical problems, but the consideration of their model is beyond of the scope of this paper.

Finally, let estimate the importance of the heat conduction for the formation and evolution of galaxy clusters, using figures 4. The parameter β and the dimensional time τ, given by equations (7) and (31), can be estimated as

$$\beta = 20\,\frac{X_0}{1\,{\rm Mpc}}\,\frac{10\,{\rm kpc}}{l_0}\,\frac{\epsilon\cos\theta}{0.1}, \quad \tau = 5\times10^{-6}\,\frac{\kappa_\|}{\kappa_{\rm S}}\,\frac{t}{10^{10}{\rm yrs}}\,\frac{10^{-3}{\rm cm}^3}{n}\left[\frac{T}{10^7K}\right]^{5/2}\left[\frac{l_0}{10\,{\rm kpc}}\right]^2\left[\frac{1\,{\rm Mpc}}{X_0}\right]^4.$$

The reduction of the parallel thermal conductivity relative to the Spitzer value, caused by magnetic mirrors, depends on the ratio $l_0/\lambda = 160\,(l_0/10\,{\rm kpc})\,(T/10^7{\rm K})^{-2}\,(n/10^{-3}{\rm cm}^{-3})$ [Malyshkin & Kulsrud 2000]. The typical values are $\kappa_\|/\kappa_{\rm S} \sim 1/10$ for the hot and low-density halo of a galaxy cluster, and $\kappa_\|/\kappa_{\rm S} \sim 1$ for the cluster central region. We see that for the halo, where $X_0 \sim 1\,{\rm Mpc}$, $l_0 \sim 10\,{\rm kpc}$, $T \sim 5\times10^7\,{\rm K}$ and $n \sim 2\times10^{-4}{\rm cm}^{-3}$, we have $\beta \sim 20$ and $\tau \sim 10^{-4}$. Thus, according to figures 4, the heat conduction is unimportant there, while according to figures 3, the effective thermal conductivity is reduced by a factor of ~ 2000. At the same time, for the cluster central region we have $X_0 \sim 0.1\,{\rm Mpc}$, $l_0 \sim 3\,{\rm kpc}$, $T \sim 2\times10^7\,{\rm K}$, $n \sim 10^{-3}\,{\rm cm}^{-3}$, $\beta \sim 7$ and $\tau \sim 0.03$, so the heat conduction is important there, and the effective thermal conductivity is reduced by a factor of ~ 200. This conclusion agrees with those of Tau (1995) and of Rosner and Tucker (1989). Thus, with certain parameters, such as these, thermal conduction can be important even with such a large reduction factor. Figures 4(a) and 4(b) give the direct temperature evolution without reference to reduction factors, and perhaps because of the sensitivity of the reduction factor to its definition, they are probably the more useful expressions of thermal evolution.

Suginohara and Ostriker in their hydrodynamic simulations of galaxy cluster formation en-

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in our model are allowed to random walk in three dimensions, while Tribble considered a restricted one-dimensional random walk.

countered a "cooling catastrophe", which appears as a steep non-realistic rise in the density profile of the relaxed core of a galaxy cluster because of the fast cooling in the core (Suginohara & Ostriker 1998). The heat conduction into the core was among solutions suggested by them. However, they believed that the conduction can be neglected if it is reduced by a factor of 30 or more. In this paper we find that the thermal conduction is very important in the cluster central region, despite the fact that in our models it is reduced by a factor of ~ 200. Thus, the conduction should be included in hydrodynamic simulations of galaxy clusters, even when the reduction factor is large.

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