

# Scintillations and Lévy flights through the interstellar medium

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## ABSTRACT

Temporal broadening of pulsar signals results from electron density fluctuations in the interstellar medium that cause the radiation to travel along paths of different lengths. The Gaussian theory of fluctuations predicts that the pulse temporal broadening should scale with the wavelength as  $\lambda^4$ , and with the dispersion measure (corresponding to distance to the pulsar) as  $DM^2$ . For large dispersion measure,  $DM > 20 \text{ pc/cm}^3$ , the observed scaling is  $\lambda^4 DM^4$ , contradicting the conventional theory. Although the problem has existed for 30 years, there has been no resolution to this paradox.

We suggest that scintillations for distant pulsars are caused by non-Gaussian, spatially intermittent density fluctuations with a power-like probability distribution. This probability distribution does not have a second moment in a large range of density fluctuations, and therefore the previously applied conventional Fokker-Planck theory does not hold. Instead, we propose to apply the theory of Lévy distributions (so-called Lévy flights). Using the scaling analysis (confirmed by numerical simulations of ray propagation) we show that the observed scaling is recovered for large  $DM$ , if the density differences,  $\delta N$ , have Lévy distribution decaying as  $|\delta N|^{-5/3}$ .

*Subject headings:* Turbulence — ISM: kinematics and dynamics

## 1. Introduction

Intensity fluctuations of pulsars radiation are caused by scattering of radio waves by electron density inhomogeneities in the interstellar medium. These fluctuations are a signature

of turbulent, non-equilibrium motion in the ISM, and as the phenomenon of turbulence itself, they have withstood full theoretical understanding for decades, see e.g. reviews by Sutton (1971) and Rickett (1977, 1990). Observationally, the presence of electron density fluctuations leads, among other effects, to temporal and angular broadening of the pulsar image. These two effects are naturally related – due to fluctuations of the refraction index, different rays from a pulsar travel along paths of different shapes and the stronger the deviation of the path from the straight line, the broader the pulsar image and the larger the time broadening of the arriving signal. Denoting the angular width of the image as  $\Delta\theta$ , and using simple geometrical consideration, one estimates the arrival time broadening as  $\tau_d \approx (\Delta\theta)^2 d/c$ , where  $d$  is the distance to the pulsar, and  $c$  is the speed of light, see more detailed discussion in (Blanford & Narayan 1985; Gwinn, Bartel, & Cordes 1993).

A ray propagating through the interstellar medium encounters many randomly distributed small “prisms” on its way, that make the scattering angle wander randomly. At each scattering event, the angle deflection is proportional to  $\lambda^2$  [see below], where  $\lambda$  is the wavelength of the scattered radiation. Taking into account that the scattering angle is small and exhibits the standard Gaussian random walk, we estimate  $(\Delta\theta)^2 \sim \lambda^4 d$ , and the time delay scales as  $\tau_d \sim \lambda^4 d^2$ , where  $d$  plays the role of time in this random walk. The distance to the pulsar is proportional to the dispersion measure,  $DM$ , and therefore this relation can be checked experimentally. As has been consistently noted for more than 30 years, observed scaling of scintillations of distant pulsars,  $DM > 20 \text{ pc/cm}^3$ , is far from this simple theoretical prediction, instead, it is well described by  $\tau_d \sim \lambda^4 DM^4$ , see e.g., (Sutton 1971; Rickett 1977). Sutton proposed that scaling for longer lines of sight arose from dramatically increased probability of intersection with strongly scattering HII regions. In this sense, he proposed that rare, large events dominated the line-of-sight averages.

The problem of scintillations was addressed by many authors who developed thorough analytical models, see the discussion in (Tatarskii & Zavorotnyi 1980; Rumsey 1975; Gochelashvily & Shishov 1975; Lee & Jokipii 1975a,b; Goodman & Narayan 1985; Blanford & Narayan 1985; Lithwick & Goldreich 2001). These models account for both smooth and non-smooth density fluctuations, the latter can arise from turbulent cascades. The main object of the theories is the so-called projected correlator of density fluctuations. Denote as  $N(\mathbf{r}, t)$  the electron density, and  $\tilde{N}(\mathbf{x}, t) = \int^d dz N(\mathbf{r}, t)$  its projection perpendicular to the distance  $d$ . Here  $\mathbf{x}$  is a two dimensional vector in the plane perpendicular to the line of sight, and  $z$  is a coordinate along the line of sight, i.e.  $\mathbf{r} = (\mathbf{x}, z)$ . Note that these theories all assume that the distribution of projected density fluctuations is Gaussian. The density

and projected density correlators are related as:

$$\langle \tilde{N}(\mathbf{x}_1)\tilde{N}(\mathbf{x}_2) \rangle = \int_0^d \int_0^d dz dz' \langle N(\mathbf{r}_1)N(\mathbf{r}_2) \rangle, \quad (1)$$

where both fields inside the brackets are taken at the same time. Due to space homogeneity, these correlators depend only on the difference of the coordinates, e.g.,  $\langle N(\mathbf{r}_1)N(\mathbf{r}_2) \rangle = \kappa(\mathbf{r}_1 - \mathbf{r}_2)$ . Assuming that the density fluctuations have finite correlation length  $l$ , i.e. the  $\kappa$  function decays fast for  $|\mathbf{r}_1 - \mathbf{r}_2| > l$ , we obtain

$$\langle \tilde{N}(\mathbf{x}_1)\tilde{N}(\mathbf{x}_2) \rangle = d \int_0^\infty dz \kappa(\mathbf{x}_1 - \mathbf{x}_2, z) \equiv d \tilde{\kappa}(\mathbf{x}_1 - \mathbf{x}_2). \quad (2)$$

It is easy to show that if in the inertial range of turbulent fluctuations,  $|r| \ll l$ , the  $\kappa$  function behaves as  $\kappa(\mathbf{r}) \approx N_0^2[1 - B(r/l)^\alpha]$ , then the projected function is expanded as  $\tilde{\kappa}(x, t) \approx \tilde{N}_0^2[1 - \tilde{B}(x/l)^{1+\alpha}]$ . As an estimate, one has  $\langle N^2 \rangle = N_0^2 \sim \tilde{N}_0^2/l$ , and  $B$  and  $\tilde{B}$  are of the order 1. The analytical case corresponds to  $\alpha = 1$ , and in this case  $\tau_d \sim \lambda^4 d^2$ . In a general case, the density field should not be analytic, and  $\alpha \neq 1$ . For example, Kolmogorov turbulence would imply  $\alpha = 2/3$ . Such different possibilities have been exhaustively analyzed in the literature, see e.g. (Lee & Jokipii 1975a,b; Goodman & Narayan 1985; Lambert & Rickett 2000). Rigorous consideration shows that in the non-analytic case, the scaling of the broadening time changes. For  $\alpha \leq 1$ , one obtains

$$\tau_d \sim \lambda^{2(\alpha+3)/(\alpha+1)} d^{(\alpha+3)/(\alpha+1)}, \quad (3)$$

while for a more exotic case,  $\alpha > 1$ , one gets

$$\tau_d \sim \lambda^{8/(3-\alpha)} d^{(3+\alpha)/(3-\alpha)}. \quad (4)$$

In section 2 we present a simple derivation of these results. Since most observational data indicate that  $\lambda$ -scaling is close to  $\lambda^4$ , neither possibility provides enough freedom for changing the  $d$ -scaling from  $d^2$  to  $d^4$ .

In the present paper we propose a new model, that fully exploits the turbulent origin of the density fluctuations. We assume that the statistics of the density fluctuations is not Gaussian, but highly intermittent, and that the probability density function (PDF) of density differences has power-law decay,  $P(\delta N) \sim |\delta N|^{-1-\beta}$ . If this power-law distribution does not have a second moment ( $\beta < 2$ ), the Gaussian random walk approach does not work. Instead, we suggest to use the theory of Lévy distributions, see (Shlesinger, Zaslavsky, & Frisch 1995). Physically, the possibility of power-law density distribution seems rather natural for

strong turbulent fluctuations. Indeed, the ISM turbulence can be near-sonic, i.e. velocity and density fields can develop shock discontinuities. From the theory of shock turbulence (Burgers turbulence) one knows that shocks or large negative velocity gradients have a power-law distribution, (Polyakov 1995; E *et al* 1997; Boldyrev 1998). Jump conditions on a shock then show that the velocity and density discontinuities are proportional to each other, therefore density jumps may also have power-law distribution. Taking the Lévy distribution of the density fluctuations as a working conjecture, we demonstrate that the scaling of the broadening time with respect to  $d$  is sensitive to the exponent of the distribution,  $\beta$ , and the scaling  $\tau_d \sim \lambda^4 d^4$  is reproduced for  $\beta = 2/3$ .

In the next section we review the ray-tracing model of pulse propagation, considered before by Williamson (1972, 1973); Blanford & Narayan (1985). In particular, we re-derive the results cited above for the Gaussian density fluctuations in a general, non-analytic case. In Section 3 we apply the model to the non-Gaussian, Lévy distributed density fluctuations. We then numerically calculate the distribution of pulse-arrival times in the case of a smooth density field, and demonstrate that if  $P(\delta N) \sim |\delta N|^{-5/3}$ , the width of this distribution changes with the distance to the pulsar as  $\lambda^4 d^4$ , in agreement with our scaling arguments. Conclusions and future research are outlined in Section 4.

## 2. Ray-tracing method

This method is applicable in the limit of geometrical optics, i.e. when the wave length is much smaller than the characteristic size of density inhomogeneities (Lifshitz, Landau, & Pitaevsky 1995). This rather effective method was applied to the problem of scintillations by Williamson (1972, 1973); Blanford & Narayan (1985); we present it here in the form that allows us to apply it in the next section to Lévy walks. In the limit considered, signal propagation can be characterized by rays,  $\mathbf{r}(t)$ , along which wave packets travel similar to particles obeying the following system of Hamilton equations:

$$\begin{aligned}\dot{\mathbf{r}} &= \partial\omega(k, r)/\partial\mathbf{k}, \\ \dot{\mathbf{k}} &= -\partial\omega(k, r)/\partial\mathbf{r}.\end{aligned}\tag{5}$$

In this representation,  $\omega$  plays the role of Hamiltonian,  $\omega^2 = \omega_{pe}^2(r) + k^2 c^2$ , where  $\omega_{pe}^2(r) = 4\pi N(r)e^2/m_e$  is the electron plasma frequency and  $\mathbf{k}$  is a wave vector. Differentiating the first equation in (5) with respect to  $t$  and using the second equation one obtains:

$$\ddot{\mathbf{r}} = -2\pi c^2 \lambda^2 r_0 \partial N(r)/\partial\mathbf{r},\tag{6}$$

where  $r_0 = e^2/m_e c^2$  is the classical radius of electron. Taking into account that the ray propagates at small angles to the line of sight, chosen as the  $z$ -axis, we are interested in ray

displacement in the perpendicular,  $\mathbf{x}$  direction, and instead of time we will use  $z$  variable,  $z = ct$ . Consider now two rays, separated by a vector  $\Delta\mathbf{x}$  in the direction perpendicular to the  $z$ -axis. As follows from (6), this vector obeys the following equation

$$\begin{aligned}\frac{d(\Delta\mathbf{x})}{dz} &= \Delta\mathbf{v}, \\ \frac{d(\Delta\mathbf{v})}{dz} &= A\Delta\frac{\partial N(x, z)}{\partial\mathbf{x}},\end{aligned}\tag{7}$$

where  $A = -2\pi\lambda^2 r_0$ , and  $\Delta\mathbf{v}$  is an auxiliary variable having the meaning of velocity of beam spreading in the  $\mathbf{x}$  direction, clearly  $\Delta\theta \sim |\Delta\mathbf{v}|$ . Let us now assume that the electron density is a Gaussian random function with the correlation length  $l$ . Then,  $\Delta\mathbf{v}(z)$  is a Gaussian random walk, whose elementary time step has the length  $l$ . Since we are interested in very large propagation distances,  $z \gg l$ , and the scattering angles are very small, one can effectively assume that the random density is short-time correlated, i.e., the characteristic “ $z$ -time” of change of vectors  $\Delta\mathbf{v}$  and  $\Delta\mathbf{x}$  is much larger than  $l$ .

The diffusion coefficient for this random walk is

$$D = -A^2 \frac{d^2 \tilde{\kappa}(\Delta x)}{d(\Delta x)^2} \sim \lambda^4 r_0^2 N_0^2 \left(\frac{\Delta x}{l}\right)^{\alpha-1} \frac{1}{l},\tag{8}$$

and the diffusion is described by  $(\Delta\theta)^2 \sim Dz$ . We however observe that the diffusion coefficient depends of the distance  $\Delta x$ , and its behavior differs qualitatively for  $\alpha < 1$  and  $\alpha > 1$ . In the first case,  $\alpha < 1$ , diffusion is larger for smaller distances, therefore two rays are effectively attracting each other in the course of propagation. This means that at some point the geometrical ray picture will break down, and one needs to consider the effects of interference (interaction) of different rays. This happens when the beam is compressed to the size limited by the uncertainty condition in the perpendicular direction,  $k\Delta\theta\Delta x \sim 1$ . Upon substituting  $\Delta\theta \sim D^{1/2}z^{1/2}$ , and using the expression for the diffusion coefficient (8), we can obtain the minimal size of contraction, and, equivalently, the diffraction angle corresponding to the aperture of this size. Assuming that the contraction happens at about half the distance between the pulsar and the Earth,  $z \sim d/2$ , we find:

$$(\Delta\theta)^2 \sim [N_0^4 r_0^4 l^{-2\alpha} \lambda^{2(\alpha+3)} d^2]^{1/(\alpha+1)}, \quad \alpha < 1.\tag{9}$$

Recalling now that  $\tau_d \sim (\Delta\theta)^2 d$ , we recover the result (3). In the second case,  $\alpha > 1$ , the rays effectively repel, so geometrical optics does not break down. In this case  $\Delta x \sim \Delta v z \sim D^{1/2}z^{3/2}$ . This equation gives

$$(\Delta\theta)^2 \sim [N_0^4 r_0^4 l^{-2\alpha} \lambda^8 d^{2\alpha}]^{1/(3-\alpha)}, \quad 1 \leq \alpha < 3,\tag{10}$$

which agrees with the result (4). Both expressions give the same result for the analytic case,  $\alpha = 1$ . The above standard results have been obtained by many authors and by a variety of different methods, see e.g., (Williamson 1972; Lee & Jokipii 1975a,b; Goodman & Narayan 1985; Blanford & Narayan 1985). As we mentioned in the introduction, neither one of the expressions (9) or (10) allows us to recover the observed scaling  $\tau_d \sim \lambda^4 d^4$ . In the next section we address the problem, assuming that the density-difference distribution has a slowly decaying power-law tail, such that the second moment of the distribution does not exist. In this case the diffusion approximation does not hold, and one needs to work directly with Eq. (7) to establish the scaling of the probability of pulse arrival times.

### 3. Lévy model for scintillations

In previous section we implicitly used the central limit theorem, which states that the sum of many independent random variables has Gaussian distribution if second moments of these variables exist. More precisely, a convolution of many distribution functions that have second moments, converges to an appropriately rescaled Gaussian distribution. Therefore, the convolution of two Gaussian functions is a Gaussian function again. One can generalize this question for distribution functions without finite second moments: if their convolution converges, what is going to be the limit? The answer is the so-called Lévy distribution (Shlesinger, Zaslavsky, & Frisch 1995). As is the Gaussian distribution, the Lévy distribution is stable: convolution of this distribution with itself gives the same distribution after proper rescaling. In other words, if two independent random variables are drawn from a Lévy distribution, their sum has the same distribution, appropriately rescaled. Analogously to a Gaussian random walk, a sum of independent, Lévy distributed random variables is called a Lévy walk or Lévy flight. The latter name reflects the highly intermittent behavior of a typical Lévy trajectory: it has sudden large jumps or “flights,” see Fig. 1. Lévy flights are common in completely different random systems and often replace diffusion in turbulent systems. For example, a particle exhibiting a Brownian random motion in an equilibrium fluid, exhibits a Lévy walk in a turbulent fluid. For a variety of further illustrations see (Shlesinger, Zaslavsky, & Frisch 1995).

If a random variable  $y$  has a Lévy probability density,  $P(y)$ , then the Fourier transform of this distribution (the characteristic function) has the form:

$$\Phi(\mu) = \int_{-\infty}^{\infty} dy P(y) \exp(i\mu y) = \exp(-C|\mu|^\beta), \quad (11)$$

where  $0 < \beta < 2$ , and  $C$  is some positive constant. For  $\beta = 2$  we recover a Gaussian

distribution. This formula can be taken as the definition of a symmetric Lévy walk. One can verify that  $P(y) \sim |y|^{-1-\beta}$  as  $|y| \rightarrow \infty$ . Of course, a distribution of a physical quantity usually has a second moment. This does not contradict our case, since the far tails of the PDF, which are not described by the Lévy formula, make the dominant contribution to the second moment. However, if we are interested in effects caused by small fluctuations,  $y \ll y_{rms}$ , it is the central part of the PDF that is important.

The characteristic function of a convolution of  $n$  Lévy distributions is just a product of  $n$  characteristic functions (11). We therefore conclude that the sum of  $n$  Lévy distributed random variables has the distribution

$$P_n(y) = P(yn^{-1/\beta})n^{-1/\beta}. \quad (12)$$

This is the demonstration of the convolution *stability* of the Lévy distribution. Formula (12) teaches us that the displacement  $y$  of the Lévy random walk scales with the number of steps as  $y \sim n^{1/\beta}$ . In the Gaussian case,  $\beta = 2$ , we recover the well known diffusion result.

We now would like to apply this result to our scintillation problem. Let us *assume* that the dimensionless density difference  $\Delta N(x)/N_0$  has a Lévy distribution with parameter  $\beta$ . We then obtain from (5),  $\Delta \mathbf{v} \sim -A\Delta N$ , and [compare this result to (8)!]:

$$(\Delta\theta)^2 \sim \lambda^4 r_0^2 N_0^2 \left(\frac{\Delta x}{l}\right)^{\alpha-1} \left(\frac{z}{l}\right)^{2/\beta}. \quad (13)$$

In this formula,  $\alpha$  describes the scaling of the density fluctuations with distance, while  $\beta$  is the exponent of the power-law decay of the density-difference probability distribution function. The scaling in Eq. (13) is understood not in the sense of averaging (the moments of  $\Delta\theta$  of the order higher than  $\beta$  do not satisfy this scaling), but in the sense of scaling of the central part of the distribution  $P_z(\Delta\theta)$ . We now proceed exactly as we did in the derivation of formulae (9) and (10), and obtain for  $\alpha < 1$ :

$$(\Delta\theta)^2 \sim [N_0^4 r_0^4 l^{2-2\alpha-4/\beta} \lambda^{2(\alpha+3)} d^{4/\beta}]^{1/(\alpha+1)}, \quad (14)$$

and for  $1 \leq \alpha < 3$ :

$$(\Delta\theta)^2 \sim [N_0^4 r_0^4 l^{2-2\alpha-4/\beta} \lambda^8 d^{2\alpha-2+4/\beta}]^{1/(3-\alpha)}. \quad (15)$$

In the smooth (analytic) case,  $\alpha = 1$ , the scaling of the time broadening is

$$\tau_d \sim (N_0^2 r_0^2 l^{-2/\beta} / c) \lambda^4 d^{(2+\beta)/\beta}. \quad (16)$$

We see that this scaling is sensitive to the exponent of the power distribution of the density fluctuations. This result was obtained by rather general arguments, and describes the scaling

of the arrival time distribution, rather than the moments of this distribution. Observations measure precisely the time width of the arriving signal, not its moments, i.e. they infer exactly the quantity corresponding to scaling (16).

In the rest of this section we would like to verify the scaling (16), by numerical simulation of Eq. (7). Our simulations also provide the time-shape of the arriving signal. Let us assume that the distance to the pulsar,  $d$ , is much larger than the scale of an elementary scatter,  $l$ , i.e.,  $n = d/l \gg 1$ , where  $n$  is the number of scattering events. At each scattering event, the angle of the ray changes by  $\delta\theta \ll 1$ , where  $\delta\theta$  is a Lévy-distributed random variable. [We denote  $\delta N$  and  $\delta\theta$  the characteristic changes of the density field and of the angle of propagation on one scattering segment of length  $l$  along the line of sight. This should not be confused with the changes of these variables between two *different* rays in a perpendicular plane  $\mathbf{x}$ , denoted by  $\Delta$ 's.] The time delay (compared to the straight propagation) introduced by each scattering segment is  $\delta\tau_d \sim l\theta^2/c$ . We need to find the probability distribution of the total travel time delay:

$$\tau_d = \frac{l}{c} \sum_{m=1}^n \theta_m^2 = \frac{l}{c} \sum_{m=1}^n \left[ \sum_{s=1}^m \delta\theta_s \right]^2, \quad (17)$$

assuming that each  $\delta\theta_s$  [where  $\delta\theta_s \approx -A\delta N$  due to (7)] is distributed identically, independently, and according to the Lévy law (11) with  $\beta = 2/3$ . This random variable can be generated in the following manner, see (Klafter, Zumofen, & Shlesinger 1995). Choose two positive numbers  $a > b > 1$ . Let  $\delta\theta = \theta_0 b^i$  with probability  $P(i) = (a-1)/(2a^{i+1})$ , and  $\delta\theta = -\theta_0 b^i$  with the same probability, where  $i = 0, 1, 2, \dots$ . This is the so-called Weierstrass self-similar random walk, that can be considered as a discrete analog of the Lévy walk with  $\beta = \ln(a)/\ln(b)$ .

In Fig. 2 we plot intensity of the arriving signal (the number of arriving rays) as a function of time. The arrival time was calculated with the aid of (17), where in the distribution of  $\delta\theta$  we have chosen  $\beta = 2/3$ ,  $b = 4$ , and  $\theta_0 = 0.0002$ . We considered the number of scattering events (the distance to the pulsar) to be  $n_1 = 100$ ,  $n_2 = 100 \times 2^{1/4} \approx 119$ , and  $n_3 = 100 \times 2^{-1/4} \approx 84$ . From Fig. 2 one can see that the widths of the curves (estimated at the half of their maximum values) indeed differ by a factor of 2, as the scaling  $\tau_d \sim \lambda^4 d^4$  would predict for these distances.

#### 4. Conclusions

In conclusion, we suggest a novel explanation for the observed scaling of time broadening of pulsar signals for large distances (large dispersion measures,  $DM > 20 \text{ pc/cm}^3$ ),  $\tau_d \sim$



$\lambda^4 d^4$ . The central concept is that the density fluctuations in the interstellar medium have a Lévy probability distribution function that has power-law decay and does not have a second moment. The angle of pulse propagation, deviated by these density fluctuations, exhibits not a conventional Brownian motion, but rather a Lévy flight. The exponent  $\beta$  is the parameter of the probability distribution of density differences, and the pulse broadening time is rather sensitive to it, as is described by our main formulae (14) and (15). The scaling  $\tau_d \sim \lambda^4 d^4$  is recovered for  $\beta = 2/3$ , i.e. for the  $|\delta N|^{-5/3}$  decay of the distribution function of density differences. This tail of the PDF appears as a result of turbulent density fragmentation, and it would be highly desirable to develop an analytical explanation for it. This is a concrete prediction of our model for the turbulence in the ISM, that can in principle be checked numerically.

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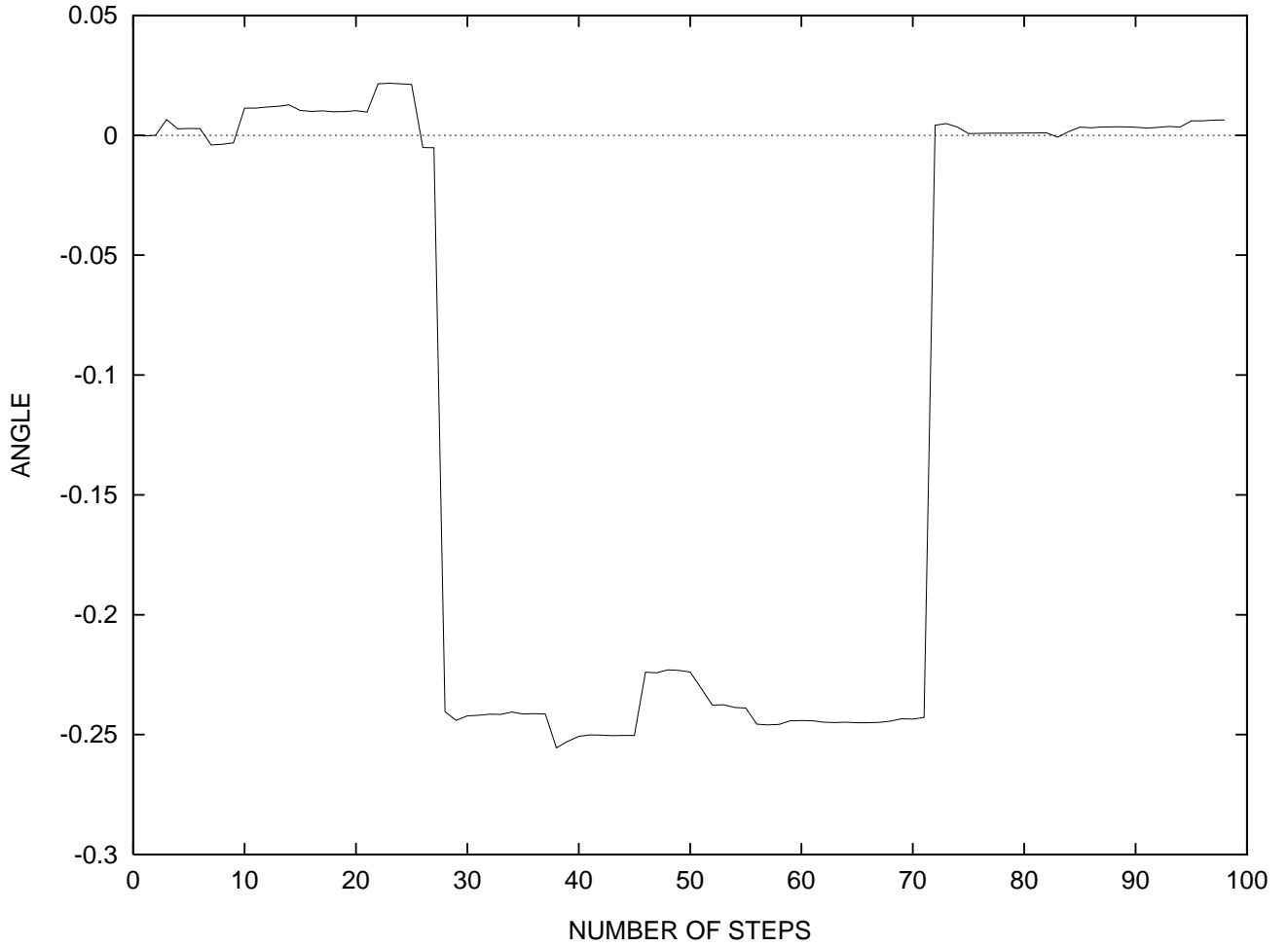


Fig. 1.— A typical realization of a Lévy random walk. The trajectory exhibits sudden large deviations, “flights.” In the case of ray propagation through the ISM, the ray angle performs a Lévy walk. Large angular deviations occur when the ray encounters regions of large electron density inhomogeneities, such as shocks or HII regions. (Angular scale is arbitrary.)

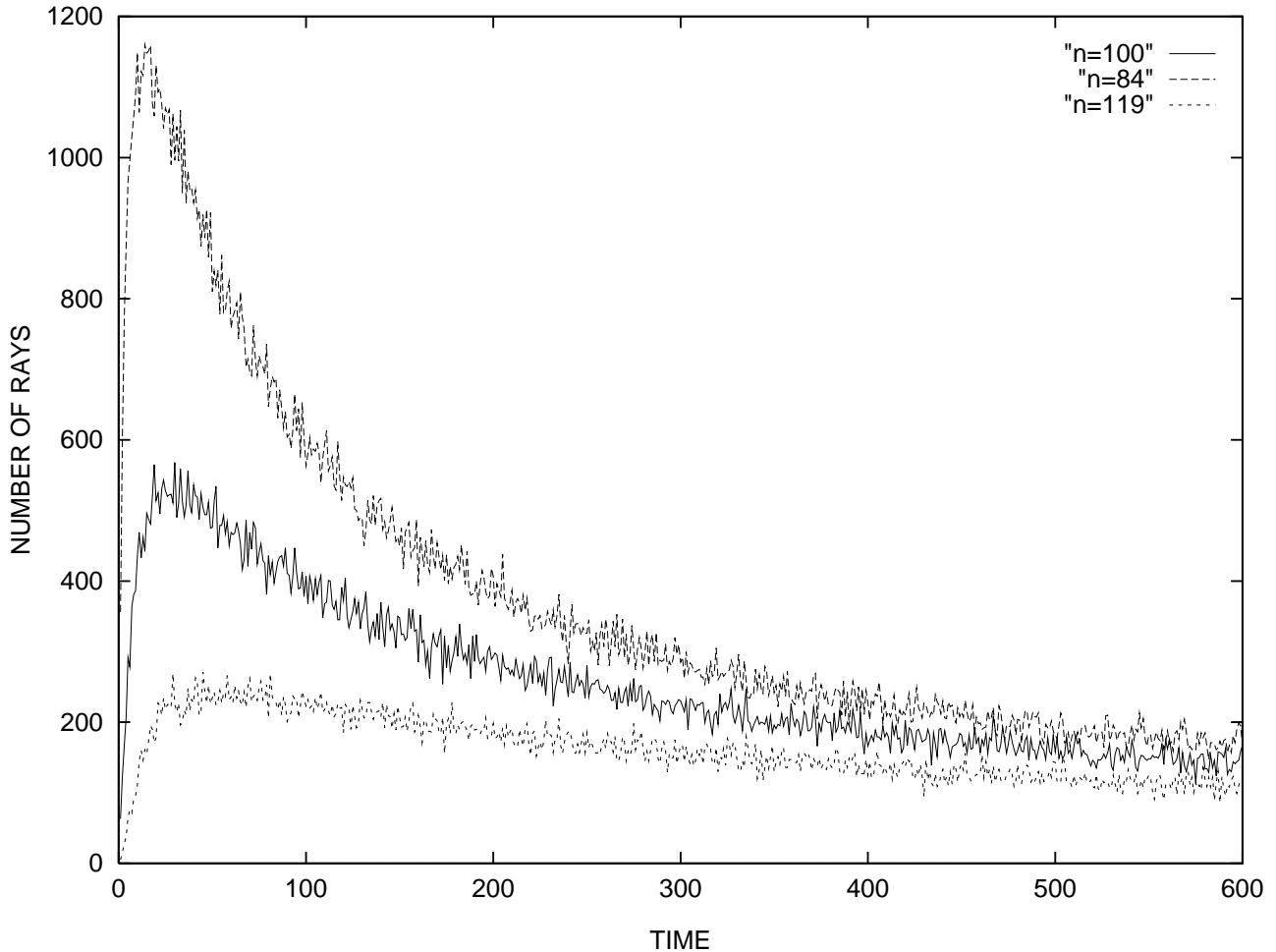


Fig. 2.— Numerical calculation of the number of arriving rays vs time (time units are arbitrary). We used formula (17), and the Lévy distributed density fluctuations with  $\beta = 2/3$ . We calculated arrival times of  $10^6$  rays for three different distances to the source,  $n_1 = 100$ ,  $n_2 = 84 \approx 100 \times 2^{-1/4}$ , and  $n_3 = 119 \approx 100 \times 2^{1/4}$ . One observes that the width of the plot “n=84” is twice as small, and the width of the plot “n=119” is twice as large, as the width of the plot “n=100.” This corresponds to the scaling  $\tau_d \sim d^4$ .