EFFECTS OF INHOMOGENEITIES ON COSMIC EXPANSION

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ABSTRACT

We evaluate the effect of inhomogeneity energy on the expansion rate of the universe. Our method is to expand to Newtonian order in potential and velocity but to take into account fully nonlinear density inhomogeneities. To linear order in density, kinetic and gravitational potential energy contribute to the total energy of the universe with the same scaling with expansion factor as spatial curvature. In the strongly nonlinear regime, growth saturates, and the net effect of the inhomogeneity energy on the expansion rate remains negligible at all times. In particular, inhomogeneity contributions never mimic the effects of dark energy or induce an accelerated expansion .

Subject headings: cosmological parameters — cosmology : theory — large-scale structure of universe

1. INTRODUCTION

Recent observations of type-Ia supernovae [\(Riess 2000](#page-3-0)) and the cosmic microwave background [\(Bennett et al.](#page-3-1) [2003\)](#page-3-1) in tandem suggest that the cosmological expansion is accelerating. Understanding the source of this accelerated expansion is one of the greatest current unsolved problems in cosmology [\(Peebles & Ratra 2003\)](#page-3-2). Acceleration seems to render inadequate a universe consisting entirely of matter, and appears to require an additional, unknown type of energy (dark energy, perhaps realized as a cosmological constant). An alternative to dark energy is that acceleration arises from a known component of the universe whose effects on the cosmic expansion have not been fully examined. One possibility currently being examined is that inhomogeneities in a matter dominated universe, on either sub-horizon [\(Rasanen 2005;](#page-3-3) [Notari](#page-3-4) [2005\)](#page-3-4) or super-horizon scales [\(Kolb et al. 2004](#page-3-5), [2005a](#page-3-6)[,b](#page-3-7); [Barausse et al. 2005\)](#page-3-8), may influence the expansion rate at late times. The central idea is that the energy induced by inhomogeneities leads to additional source terms in the Friedmann equations, with effects on the dynamics that leave no need for a separate dark energy component. In their entirety, these proposals present conflicting claims and a general state of much confusion: does the inhomogeneity energy produce an accelerated expansion, acting in effect as dark energy [\(Kolb et al. 2005b\)](#page-3-7), or does it behave as curvature [\(Geshnizjani et al. 2005\)](#page-3-9)? Is the magnitude of the effect small, large, or even divergent, on either large scales [\(Kolb et al. 2005b\)](#page-3-7), or on small scales at late times [\(Notari 2005\)](#page-3-4)?

Part of the confusion arises from the fully relativistic perturbation theory formulation of many of these calculations. Although this is undeniably a valid approach, the number of terms in a perturbation theory calculation can be large and can mask the underlying physics. In this Letter, taking advantage of phenomenological results that have been derived from a combination of quasilinear perturbation theory, nonlinear theory, and numerical simulations, we compute the potential and kinetic inhomogeneity energies within the horizon to Newtonian order in potential and velocity for fully nonlinear density contrasts. We find these energies to be small at present, and their projected values remain small, even far into the

future. Section 2 considers the effect of inhomogeneities, for weak gravity and slow motions but for arbitrary density perturbations, characterized in terms of the density power spectrum. Section 3 presents the results for the kinetic and potential energies in both the linear and the fully nonlinear regimes, as a function of the cosmological expansion factor. Finally, section 4 discusses the implications of these results.

2. EFFECTS OF INHOMOGENEITIES

The purpose of our work is to investigate whether inhomogeneity energy can mimic the effects of dark energy for a universe containing only matter. To this end, we work in an $\Omega_m = 1$ Einstein-de Sitter universe, with no curvature or cosmological constant, and compute the effects of inhomogeneities on the cosmic expansion rate. The dynamics of cosmological expansion are governed by the Friedmann equations,

$$
\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho, \qquad \left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi}{3}G(\rho + 3p). \tag{1}
$$

Any mass or energy density that makes up a significant fraction of the total can influence the evolution of the cosmological scale factor $a(t)$. A contribution to the energy density of the universe with equation of state $p_i = w \rho_i$ has $\rho_i \propto a^{-3(1+w)}$, or $\rho_i/\rho_m \propto a^{-3w}$; in particular, a component with $\rho \propto a^{-2}$ behaves as $w = -\frac{1}{3}$ or curvature, and a component with constant ρ behaves as a cosmological constant or dark energy.

We introduce the effects of inhomogeneities following the formulation of [Seljak & Hui \(1996](#page-3-10)). In the conformal Newtonian gauge, with metric

$$
ds^{2} = a^{2}(\tau)[-(1+2\psi)d\tau^{2} + (1-2\phi)dx^{2}], \qquad (2)
$$

the time-time Einstein equation (G_0^0) yields

$$
3\left(\frac{\dot{a}}{a}\right)^2 (1 - 2\psi) + (2 + 6\phi) \frac{1}{a^2} \nabla^2 \phi + \frac{1}{a^2} (\nabla \phi)^2
$$

= $8\pi G \bar{\rho} (1 + \delta) (1 + v^2),$ (3)

Where $\phi \simeq \psi$ from the space-space components of G^{μ}_{ν} . (Our numerical factors are slightly different from those of Seljak & Hui; they make little difference in the results.) The source on the right-hand side includes a density perturbation $\delta = \delta \rho / \bar{\rho}$ in the material rest frame, with the

transformation to the cosmological frame expanded to leading order for small v^2 . Ignoring $\phi \nabla^2 \phi$, $(\nabla \phi)^2$, and v^2 , the homogeneous part of this equation reproduces the usual Friedmann equation. The inhomogeneous part reveals that ϕ obeys the Poisson equation with source $4\pi G\bar{\rho}a^2\delta$. The volume average of the entire equation then leads to

$$
\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\bar{\rho}\left(1 - 5W + 2K\right),\tag{4}
$$

where W and K are the Newtonian potential and kinetic energy per unit mass,

$$
W = \frac{1}{2} \langle (1+\delta)\phi \rangle, \qquad K = \frac{1}{2} \langle (1+\delta)v^2 \rangle. \tag{5}
$$

These expressions are correct to first order in ϕ and v^2 , but neither an assumption nor an approximation in δ . We assume that $\langle \nabla^2 \phi \rangle = 0$; in all other places the Poisson equation is adequate to determine ϕ .

The Newtonian potential and kinetic energies thus can influence cosmological expansion. We can compute both W and also K completely and exactly from knowledge only of the density power spectrum. The potential is related to the density inhomogeneity by the Poisson equation, $\nabla^2 \phi = 4\pi G \bar{\rho} a^2 \delta$, an expression which holds even for nonlinear inhomogeneities. From this, we obtain

$$
W = -\frac{1}{2} 4\pi G \bar{\rho} a^2 \int \frac{d^3 k}{(2\pi)^3} \frac{P(k)}{k^2} = -\int \frac{dk}{k} \Delta_W^2(k),\tag{6}
$$

an expression correct in both linear and nonlinear regimes if $P(k)$ is the appropriate linear or nonlinear power spectrum. The last equality defines the dimensionless spectral density $\Delta_W^2(k)$.

In linear perturbation theory, valid for small inhomogeneities, the density contrast grows as $\delta = \delta_0(x)D(t)$, where in a matter dominated universe $D(t) \propto a(t) \propto t^{2/3}$ [\(Peebles 1980](#page-3-11)). The kinetic energy follows from the linearized equation of continuity, $\dot{\delta} + \nabla \cdot v/a = 0$ [\(Peebles](#page-3-11) [1980\)](#page-3-11),

$$
K_{\rm lin} = \frac{1}{2} \dot{a}^2 \int \frac{d^3k}{(2\pi)^3} \frac{P(k)}{k^2}
$$
 (7)

(the usual factor $f(\Omega) \simeq \Omega^{0.6} = 1$ for $\Omega_m = 1$). The kinetic energy scales with $a(t)$ as \dot{a}^2D^2 , while the potential energy scales as $\bar{\rho}a^2D^2$; and so both W and K grow as $D^2/a \propto a(t)$, or $\rho_U = \bar{\rho}(W+K) \propto a^{-2}$. As was noted by [Geshnizjani et al. \(2005\)](#page-3-9) for super-horizon inhomogeneities, in perturbation theory inhomogeneity energy has the same effect on the expansion rate as spatial curvature. We note that $K_{\text{lin}}/|W_{\text{lin}}| = H^2/4\pi\bar{G}\bar{\rho} = \frac{2}{3}$, a fixed ratio in the linear regime. The full kinetic energy in principle involves higher order correlation functions and is not a simple integral over the power spectrum. Nonetheless, the full kinetic energy can be obtained simply from the potential energy through the cosmic energy equation of [Irvine \(1961\)](#page-3-12) and [Layzer \(1963\)](#page-3-13),

$$
\left(\frac{d}{dt} + \frac{2\dot{a}}{a}\right)K = -\left(\frac{d}{dt} + \frac{\dot{a}}{a}\right)W,\tag{8}
$$

with initial conditions set in the linear regime, $K_{lin} =$ $\frac{2}{3}|W_{lin}|$. Equations [\(6\)](#page-1-0) and [\(8\)](#page-1-1) provide us with expressions sufficient to calculate nonperturbative contributions to the expansion rate for both the gravitational potential perturbation and kinetic energy components. The results of these calculations are given in the next section.

3. RESULTS

Equations [\(6\)](#page-1-0) and [\(8\)](#page-1-1) determine the inhomogeneity energy of the universe as a function of epoch, which we characterize by the expansion factor $a/a₀$. For the primordial power spectrum, we use the CDM power spectrum as given by [Bardeen et al. \(1986\)](#page-3-14), with spectral index $n = 1$, $\Omega_m = 1$, and COBE normalized amplitude $\delta_H = 1.9 \times 10^{-5}$ [\(Bunn & White 1997\)](#page-3-15). To obtain the nonlinear power spectrum we use the linear-nonlinear mapping of [Peacock & Dodds \(1994,](#page-3-16) [1996](#page-3-17)). The results of these calculations are shown in Figures 1 and 2.

Fig. 1.— Spectral density of gravitational potential energy $\Delta^2_W(k)$ [the integrand of eq. [\(6\)](#page-1-0)], evaluated at the present, plotted as a function of wavenumber k. The dashed line shows Δ_W^2 in linear perturbation theory; the solid line shows the fully nonlinear form.

Figure [1](#page-1-2) shows the dimensionless spectral density of gravitational potential energy $\Delta_W^2(k)$ defined in eq. [\(6\)](#page-1-0), evaluated at the present, plotted as a function of wavenumber k . The dashed curve shows the density in linear perturbation theory, and the solid curve shows its fully nonlinear form.

Fig. 2.— Fractional contributions of gravitational potential energy W (long-dashed line) and kinetic energy K (solid line) to the total energy density of the universe, plotted as a function of past and future expansion factor for an $\Omega_m = 1$ universe. The short-dashed line is the sum of contributions from inhomogeneities. The dotted lines show results from linear perturbation theory.

Figure [2](#page-1-3) shows the contributions of potential energy and kinetic energy to the energy density of the universe, for past and future expansion factors in an $\Omega_m = 1$ universe. At early times, perturbation theory gives an accurate result, but at $a/a_0 \approx 0.05$ (redshift $z \approx 20$) the behavior starts to change, for an interval growing faster than a^1 with the fastest growth as $a^{1.2}$, and then saturating and growing significantly more slowly, eventually as ln a.

4. DISCUSSION

In this Letter we have evaluated the size and the time evolution of the contribution of inhomogeneities to the expansion dynamics of a matter-dominated universe, including the effects of fully nonlinear density inhomogeneities. When density fluctuations are in the linear regime, the ratio of the inhomogeneity contribution to the matter density grows linearly with expansion factor, as does curvature in an open universe, making only a very small contribution to the expansion rate. As density fluctuations begin to go nonlinear, the inhomogeneity energy grows at a slightly faster rate, at most as $a^{1.2} \propto a^{-3w}$, or $w = -0.4$. This by itself, even if the dominant energy component, would be only temporarily and only very slightly accelerating, with deceleration parameter $q_0 = \frac{1}{2}(1+3w) = -0.1$. Since, at this time, the total fraction of inhomogeneity energy is $\Omega_U \approx 10^{-5} \ll 1$, this has a negligible effect on cosmological expansion dynamics.

As the universe further evolves, so that the main contributions to W and K come from deeply nonlinear scales, we compute the potential energy from integration of the nonlinear power spectrum, and obtain kinetic energy from the cosmic energy equation (eq. [\[8\]](#page-1-1)). In a scaleinvariant model with power spectrum $P \sim k^n$ as $k \to 0$, the kinetic and potential energies K and W scale with the expansion factor as $a^{(1-n)/(3+n)}$ [\(Davis & Peebles 1977](#page-3-18)) (logarithmically in a as $n \to 1$), with ratio

$$
\frac{K}{|W|} = \frac{4}{7+n}.\tag{9}
$$

Numerical simulations show that this continues to hold for the CDM spectrum with effective index $n =$ $d \log P/d \log k$ at an appropriate scale, the basis of the linear-nonlinear mapping [\(Peacock & Dodds 1994,](#page-3-16) [1996\)](#page-3-17). For the CDM spectrum, with $n \to 1$ on large scales, this means that growth stops, and the ratio tends to the virial value $K/|W| \rightarrow \frac{1}{2}$ at late times. We note that aside from the integration of the Layzer-Irvine equation, many of these results were obtained by [Seljak & Hui](#page-3-10) [\(1996\)](#page-3-10).

Our results show that the contributions of the potential and kinetic energies of inhomogeneities has never been strong enough to dominate the expansion dynamics of the universe. For a universe with $\Omega_m = 1$ today, normalized to the large scale fluctuations in the microwave background, the net effect of inhomogeneities today is that of a slightly open universe, with $\Omega_k \approx 10^{-4}$ in curvature. The maximum contribution comes from scales of order 1 Mpc, falling off rapidly for smaller and larger k, as illustrated in Figure [1.](#page-1-2) The behavior on asymptotically small scales $(k \gg 10^6 h \text{ Mpc}^{-1})$ depends on an

extrapolation that ignores such details as star formation, but [Fukugita & Peebles \(2004\)](#page-3-19) estimate that the net contribution of dissipative gravitational settling from baryon-dominated parts of galaxies, including main sequence stars and substellar objects, white dwarfs, neutron stars, stellar mass black holes, and galactic nuclei, is in total $10^{-4.9}$ of the critical energy density.

The suggestion that nonlinear effects for large inhomogeneities may mimic the effect of dark energy is not the case for the fully nonlinear theory. It is true that higher order terms in perturbation theory grow faster; the general *n*-th order term grows as $D^n(t)$. There indeed comes a scale in space or an evolution in time where the behavior of higher order terms appears to diverge. Nevertheless, the fully nonlinear result is well behaved. It is only the perturbation expansion that breaks down, and the actual energy saturates and grows more and more slowly at late times. As illustrated in Figure [2,](#page-1-3) the nonlinear potential and kinetic energies remain small compared to the total matter density at all times, even an expansion factor of 10^3 into the future. Inhomogeneity effects do not substantially affect the expansion rate at any epoch.

Fig. 3.— The expected fluctuation in the potential energy per unit mass $\langle (\Delta W)^2 \rangle^{1/2}$ evaluated at the present as a function of in-
frared cutoff k_{min} for $n = 0.95$, $n = 1$, and $n = 1.05$ (solid lines, top to bottom). Dashed lines are analytic approximations that asymptotically become $k^{-0.025}$, $(\log k)^{1/2}$, or constant, respectively. The dotted line shows the result for a rolling spectral index that has $n = 0.95$ on the horizon today but approaches $n = 1$ as $k \to 0$, as predicted by most models of slow-roll inflation. The mean value $\langle W \rangle = 3.1 \times 10^{-5}$ is shown as the horizontal dashed line.

It has been pointed out that although the average inhomogeneity energy is small, its variance has a logarithmically divergent contribution from the variance of the potential on super-horizon scales [\(Kolb et al. 2005a\)](#page-3-6),

$$
\langle (\Delta W)^2 \rangle = \frac{1}{V^2} \int d^3x \, d^3x' \, \frac{1}{4} \, \bar{\rho} \langle \phi(x) \, \phi(x') \rangle
$$

$$
= (2\pi G \bar{\rho} a^2)^2 \int \frac{d^3k}{(2\pi)^3} \, \frac{P(k)}{k^4} \, W^2(kR), \, (10)
$$

windowed over the horizon volume (for calculational convenience we use a Gaussian rolloff rather than a sharp radial edge). For $n \to 1$ as $k \to 0$, this is indeed logarithmically dependent on the low-k cutoff (and if $n < 1$) the divergence is worse), but the rest of the integral is finite for the CDM spectrum. The fluctuation in potential

energy, $\langle (\Delta W)^2 \rangle^{1/2}$, is shown in Figure [3](#page-2-0) as a function of the infrared cutoff k_{min} . The integral is dominated by the smallest values of k , where perturbations are deep in the linear regime. For $n = 1$ the result is very accurately $\Delta W = 1.45 \times 10^{-5} |\ln k_{\min} R_H|^{1/2}$. (We note that for $n \to 1$ the units of k_{min} are unimportant.) The fluctuation is comparable to the mean $\langle W \rangle = 3.1 \times 10^{-5}$ when the cutoff is near the scale of the horizon $k = H_0/c$, and does not become of order 1 until $k_{\text{min}} \sim 10^{-170}$ (for $n = 0.95$), or $k_{\text{min}} \sim 10^{-10^9}$ (for $n \to 1$), or ever (for $n > 1$). While such an exponentially vast range of scales may not be beyond the range of possibility in an inflationary universe, it requires a fearless extrapolation well beyond what is known directly from observation. The fluctuation ΔW is dominated by contributions from modes that are deep in the linear perturbation regime, and scales with expansion factor as $\Delta W \propto \bar{\rho} a^2 D$, constant in time. This contribution to the energy will appear dynamically in the Friedmann equation as another matter component. Furthermore, in the presence of a true dark energy component, any effects on cosmological expansion arising from inhomogeneities quickly becomes unimportant once dark energy becomes dominant [\(Seljak & Hui 1996](#page-3-10)).

The fact that fluctuations in the potential diverge remains troublesome. It has been recognized for some time that potential fluctuations in the standard model with $n \to 1$ are logarithmically divergent, but since for most purposes the value of the potential is unimportant, this has not been perceived as a significant problem. The ef-

fect of potential on the expansion dynamics is real, but the weak logarithmic divergence and the fact that it is a feedback of a gravitational energy on gravitational dynamics may lead one to hope that this divergence is alleviated in a renormalized quantum theory of gravity.

We have found that, to leading order in ϕ and v^2 but with fully nonlinear density fluctuations, inhomogeneities on sub-horizon scales have only a minimal effect on the cosmological expansion dynamics, even far into the future, and in particular never result in an accelerated expansion. Other authors have also shown that recent attempts to explain an accelerated expansion through super-horizon perturbations face significant difficulties [\(Flanagan 2005;](#page-3-20) [Geshnizjani et al. 2005](#page-3-9); [Hirata & Seljak 2005](#page-3-21)). The possibility that a known component of the universe may be responsible for the accelerated expansion remains intriguing. However, we conclude that sub-horizon perturbations are not a viable candidate for explaining the accelerated expansion of the universe.

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