Kink Stability of Isothermal Spherical Self-Similar Flow Revisited

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Anzhong Wang *

CASPER, Physics Department, Baylor University, 101 Bagby Avenue, Waco, TX76706

Yumei Wu †

Mathematics Department, Baylor University, Waco, TX76798

and

Institute of Mathematics, the Federal University of Rio de Janeiro, Caixa Postal 68530, CEP 21945-970, Rio de Janeiro, RJ, Brazil

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Abstract

The problem of kink stability of isothermal spherical self-similar flow in newtonian gravity is revisited. Using distribution theory we first develop a general formula of perturbations, linear or non-linear, which consists of three sets of differential equations, one in each side of the sonic line and the other along it. By solving the equations along the sonic line we find explicitly the spectrum, k, of the perturbations, whereby we obtain the stability criterion for the self-similar solutions. When the solutions are smoothly across the sonic line, our results reduce to those of Ori and Piran. To show such obtained perturbations can be matched to the ones in the regions outside the sonic line, we study the linear perturbations in the external region of the sonic line

*E-mail: Anzhong_Wang@baylor.edu

†E-mail: Yumei_Wu@baylor.edu; yumei@im.ufrj.br

(the ones in the internal region are identically zero), by taking the solutions obtained along the line as the boundary conditions. After properly imposing other boundary conditions at spatial infinity, we are able to show that linear perturbations, satisfying all the boundary conditions, exist and do not impose any additional conditions on k. As a result, the complete treatment of perturbations in the whole spacetime does not alter the spectrum obtained by considering the perturbations only along the sonic line.

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I. INTRODUCTION

In this paper we re-consider the problem of kink instability of isothermal spherical self-similar flow in newtonian gravity. This problem was first considered by Ori and Piran [1] and found that a large class of solutions was not stable against kink perturbations.

The kink modes result from the existence of sonic lines (points), along which discontinuities of (higher order) derivatives of some physical quantities can propagate. Such weak discontinuities cannot occur spontaneously. They are always the results of some singularities of the initial and/or boundary conditions. For example, they may occur due to the presence of angles on the surface of a body past which the flow takes place. In this case the first spatial derivatives of the velocity are discontinuous. They may also be formed when the curvature of the surface of the body is discontinuous (but without angles), in which the second spatial derivatives of the velocity are discontinuous. In addition, any singularity in the time variation of the flow also results in a non-steady weak discontinuity. The instability of such discontinuities is characterized by the divergence of them, and the blow-up may imply the formation of shock waves [2].

An example that discontinuities of derivatives can propagate along a sonic line is given by the linear perturbation, $\delta\varphi(\tau, x) = \varphi_1(x)e^{k\tau}$, of a massless scalar field in $2 + 1$ gravity, which satisfies the equation [3],

$$x\,(1-x)\,\varphi_1{}'' + \frac{1}{2}\left[(1+2k) - x(3+2k)\right]\varphi_1{}' - \frac{1}{2}k\varphi_1 = f(x), \tag{1.1}$$

where a prime denotes the ordinary differentiation with respect to the indicated argument, $f(x)$ is a smooth function of x, and $x = 1$ is the location of the sonic line [1]. From the above equation we can see that it is possible for φ_1 to have discontinuous derivatives across the sonic line $x = 1$. In fact, assume that φ_1 is continuous across $x = 1$ (but not its first-order derivative), we can write it in the form

$$\varphi_1(x) = \varphi_1^+(x)H(x-1) + \varphi_1^-(x)\left[1 - H(x-1)\right], \tag{1.2}$$

where $H(x-1)$ denotes the Heavside (step) function, defined as

$$H(x-1) = \begin{cases} 1, & x \geq 1, \\ 0, & x < 1. \end{cases} \tag{1.3}$$

Then, we find that

$$\begin{aligned} \varphi_1{}' &= \varphi_1^{+\prime} H(x-1) + \varphi_1^{-\prime}\left[1 - H(x-1)\right], \\ \varphi_1{}'' &= \varphi_1^{+\prime\prime} H(x-1) + \varphi_1^{-\prime\prime}\left[1 - H(z-1)\right] + \left[\varphi_1{}'\right]^- \delta(x-1), \end{aligned} \tag{1.4}$$

where $\delta(x-1)$ $[= dH(x-1)/dx]$ denotes the Dirac delta function, and

$$\left[\varphi_1{}'\right]^- \equiv \lim_{x\to 1^{+0}} \left(\frac{d\varphi_1^+(x)}{dx}\right) - \lim_{x\to 1^{-0}} \left(\frac{d\varphi_1^-(x)}{dx}\right). \tag{1.5}$$

Substituting Eq.(1.4) into Eq.(1.1) and considering the facts

$$\begin{aligned} &H^m(x) = H(x), \quad \left[1 - H(x)\right]^m = \left[1 - H(x)\right], \\ &\left[1 - H(x)\right] H(x) = 0, \;\; x\delta(x) = 0, \end{aligned} \tag{1.6}$$

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[1]Note the different notations used here and those in [3]. In particular, here we use x in the place of z of [3].

where m is an integer, one can see that Eq.(1.1) holds also on the sonic line $x = 1$ even when $[\varphi_1{}']^- \neq 0$. This is because $(1-x)\,[\varphi_1{}']^-\,\delta(x-1) = 0$ for any finite $[\varphi_1{}']^-$. Thus, the quantity

$$\left[\delta\varphi_{,x}\right]^- = [\varphi_1{}']^-\, e^{k\tau}, \tag{1.7}$$

has support only along the sonic line, where $(\ )_{,x} \equiv \partial(\ )/\partial x$. The above expression shows clearly how the discontinuity, $\left[\delta\varphi_{,x}\right]^-$, of the first derivative of the perturbation $\delta\varphi(\tau, x)$ propagates along the sonic line. When $Re(k) > 0$ the perturbation grows exponentially as $\tau \to \infty$, and is said unstable with respect to the kink perturbation. When $Re(k) = 0$, the perturbation oscillates in time τ, and is usually also considered as unstable. When $Re(k) < 0$ the perturbation decays exponentially and is said stable. Note that if the discontinuity happened on other places, say, $x = x_0 \neq 1$, clearly Eq.(1.1) would not hold on $x = x_0$, because now $(1-x)\,[\varphi_1{}']^-\,\delta(x - x_0) \neq 0$. This explains why the discontinuities are allowed only along sonic lines.

From the above example we can also see that in general one may divide the whole space into three different regions, $x > x_c$, $x = x_c$, and $x < x_c$, where $x = x_c$ is the location of the sonic line. In each of the three regions, the perturbations, collectively denoted by δy, satisfy a set of differential equations,

$$G^+\left(\delta y^{(m)}, \tau, x, k\right) = 0, \quad (x \geq x_c)\,, \tag{1.8}$$

$$G^-\left(\delta y^{(m)}, \tau, x, k\right) = 0, \quad (x \leq x_c)\,, \tag{1.9}$$

$$G^{(c)}\left(\left[\delta y^{(m)}\right]^-, \left[\delta y^{(m)}\right]^+, \tau, x_c, k\right) = 0, \quad (x = x_c)\,, \tag{1.10}$$

where $m = 0, 1, 2, ...$, and

$$\begin{aligned}
\delta y(\tau, x) &= \delta y^+(\tau, x) H\left(x - x_c\right) + \delta y^-(\tau, x)\left[1 - H(x - x_c)\right],\\
\delta y^{(m)}(\tau, x) &\equiv \frac{\partial^m \left(\delta y(\tau, x)\right)}{\partial x^m},\\
\left[\delta y^{(m)}\right]^{\pm} &\equiv \lim_{x\to x_c^+} \frac{\partial^m \left(\delta y^+(\tau, x)\right)}{\partial x^m} \pm \lim_{x\to x_c^-} \frac{\partial^m \left(\delta y^-(\tau, x)\right)}{\partial x^m}.
\end{aligned} \tag{1.11}$$

Therefore, in the presence of weak discontinuities the field equations actually have three different parts. Eq.(1.10) describes the evolution of the discontinuities, $\left[\delta y^{(m)}\right]^{\pm}$, along the sonic line, while Eqs.(1.8) and (1.9) describe the evolution of $\delta y^{\pm}$ in the regions $x > x_c$ and $x < x_c$. Although $\delta y^{\pm}$ and $\left[\delta y^{(m)}\right]^{\pm}$ satisfy different equations, *they have to match each other according to Eq.(1.11)*. In addition, to have the perturbations physically acceptable, $\delta y^{\pm}$ usually needs to satisfy some boundary conditions in the regions $x > x_c$ and $x < x_c$, such as, the regularity conditions on the symmetry axis, and those at spacelike infinities.

In [1] the field equation (1.10) along the sonic line was studied, by assuming that $\delta y^{-}(\tau, x) = 0$ in the region $|x| < |x_c|$. Hanawa and Matsumoto applied such a study to a fluid with a polytrope equation of state, $p = K\rho^{\gamma}$, where p and ρ denote, respectively, the pressure and mass density of the fluid, and K and γ are two constants [4]. Lately, Harada, and Harada and Maeda generalized such a study to the relativistic case [5,6]. Since $\delta y^{-}(\tau, x)$ vanishes identically, it is necessary that $\delta y^{+}(\tau, x)$ does not, in order to have non-vanishing $\left[\delta y^{(m)}\right]^{-}$. Then, a natural question is how the perturbations obtained along the sonic line match to the ones in $|x| > |x_c|$?

To answer the above question, in this paper we first solve the equations along the sonic line for $\delta y_c(\tau) \equiv \delta y(\tau, x_c)$, and find explicitly the spectrum, k, of the perturbations. In [1] only the case where the self-similar solutions are smooth across the sonic line was studied. Here we generalize such a study to include the case where the self-similar solutions can have discontinuous derivatives. Once $\delta y_c(\tau)$ is found, we then solve the equations for the perturbations $\delta y^{+}(\tau, x)$, by taking $\delta y_c(\tau)$ and its first derivatives as the boundary conditions [cf. Eqs.(5.6) and (5.7)],

$$\delta y^{+}(\tau, x)\Big|_{x=x_c} = \delta y_c(\tau), \tag{1.12}$$

$$\delta y^{+}_{,x}(\tau, x)\Big|_{x=x_c} = u(\tau). \tag{1.13}$$

In addition to these, we also impose conditions at spatial infinity $x = \infty$ [cf. Eq.(5.8)],

Perturbations grow slower than the background solutions. (1.14)

In general, conditions (1.12) are already sufficient to determine all the integration constants. Thus, conditions (1.13) and (1.14) represent additional restrictions. By carefully analyzing the perturbations, we are able to show that *there exist solutions to the perturbation equations that satisfy all the boundary conditions (1.12) - (1.14), and do not alter the spectrum k, obtained by considering the perturbations only along the sonic line.*

Specifically, the paper is organized as follows: in Sec. II we review some main properties of the self-similar solutions, and pay particular attention on their asymptotic behavior near the sonic line and at the spatial infinity. In particular, we work out the equations that the discontinuities of the first-order derivatives of the self-similar solutions have to satisfy across the sonic line. In Sec. III, by using distribution theory we first develop a general formula of perturbations, which are valid for both linear and non-linear perturbations. It consists of three different sets, one in each side of the sonic line, and the other along it. Then, in Sec. IV we restrict ourselves to the perturbations along the sonic line and find explicitly the spectrum, k, of the perturbations. When the solutions are analytical across the line, we re-discover the results of Ori and Piran. When they are not, we find explicitly the stability criterion for those solutions in terms of x_c. In Sec. V, we consider the linear perturbations $\delta y^{+}(\tau, x)$ in the region $|x| > |x_c|$, by taking the solutions $\delta y_c(\tau)$ and its derivatives obtained along the sonic line in Sec. IV as the boundary conditions. The paper is ended with Sec. VI, where our main conclusions are presented.

II. FIELD EQUATIONS AND SELF-SIMILAR SOLUTIONS

The spherical flow of self-gravitating isothermal perfect fluid is described by the hydrodynamic equations,

$$v_{,t} + v v_{,r} + \frac{c_s^2 \rho_{,r}}{\rho} + \frac{Gm}{r^2} = 0, \tag{2.1}$$

$$r^2 \rho_{,t} + \left(r^2 v \rho\right)_{,r} = 0, \tag{2.2}$$

$$m_{,t} = -4\pi r^2 \rho v, \tag{2.3}$$

$$m_{,r} = 4\pi r^2 \rho, \tag{2.4}$$

where $v_{,t} \equiv \partial v/\partial t$, and r, t, c_s, G, $v(t,r)$, $\rho(t,r)$ and $m(t,r)$ denote, respectively, the radial coordinate, the time, the speed of sound, the gravitational constant, the velocity, the mass density, and the total mass enclosed within the radius r of the perfect fluid at the moment t, with the equation of state

$$p = c_s^2 \rho, \tag{2.5}$$

where p denotes the pressure of the fluid. The boundary conditions at $r = 0$ are

$$v(t,0) = 0, \quad \rho(t,0) = 0. \tag{2.6}$$

Note that Eqs.(2.1) - (2.4) are unchanged under the coordinate transformations,

$$t = t' + \epsilon, \quad r = r', \tag{2.7}$$

where ϵ is a constant. We call such a gauge as the *Galilean gauge*, and shall be back to it when we consider the linear perturbations in Sec. V.

Following Ori and Piran, we first introduce the dimensionless quantities,

$$\begin{aligned} x &= \frac{r}{c_s t}, \quad \tau = -\ln\left(\frac{-t}{t_0}\right), \\ V &= \frac{v}{c_s}, \quad D = \frac{4\pi G\rho}{c_s^2} r^2, \quad M = \frac{Gm}{c_s^2 r}, \end{aligned} \tag{2.8}$$

where t_0 is a dimensional constant with the dimension of length, which, without loss of generality, shall be set to one. Then, it can be shown that Eqs.(2.1)-(2.4) take the form

$$(V - x)\,V_{,x} + E_{,x} - V_{,\tau} + \frac{1}{x}(M-2) = 0, \tag{2.9}$$

$$(V - x)\,E_{,x} + V_{,x} - E_{,\tau} = 0, \tag{2.10}$$

$$xM_{,x} + M - e^E = 0, \tag{2.11}$$

$$xM_{,x} + M_{,\tau} - \frac{1}{x} V e^E = 0, \tag{2.12}$$

where $E \equiv \ln(D)$. Since in this paper we are mainly concerned with the gravitational collapse of the fluid, from now on we shall restrict ourselves only to the region where $t \leq 0$, in which a singularity is usually developed at the moment $t = 0$. From Eq.(2.8) we can see that this also means $x \leq 0$.

Self-similar solutions are given by

$$F(\tau, x) = F_{ss}(x), \tag{2.13}$$

where $F \equiv \{E,\ M,\ V\}$. Setting all the terms that are the derivatives of τ be zero in Eqs.(2.9)-(2.12), we find that

$$(V_{ss} - x)\, {V_{ss}}' + {E_{ss}}' + \frac{1}{x}\,(M_{ss} - 2) = 0, \tag{2.14}$$

$$(V_{ss} - x)\, {E_{ss}}' + {V_{ss}}' = 0, \tag{2.15}$$

$$x{M_{ss}}' + M_{ss} - e^{E_{ss}} = 0, \tag{2.16}$$

$$x{M_{ss}}' - \frac{1}{x} e^{E_{ss}} V_{ss} = 0. \tag{2.17}$$

The boundary conditions of Eq.(2.6) become

$$V_{ss}(0) = 0 = D_{ss}(0). \tag{2.18}$$

From Eqs.(2.16) and (2.17) we find that

$$M_{ss} = e^{E_{ss}} \left(1 - \frac{V_{ss}}{x}\right). \tag{2.19}$$

The solutions of these equations have been studied extensively [7–10]. In particular, it was shown that they must pass a sonic line at $x = x_c$, defined by [2]

$$V_{ss}(x_c) - x_c = 1, \tag{2.20}$$

which divides the (t,r)-plane into two regions: the interior, $|x| < |x_c|$, where $V_{ss} - x - 1 < 0$, and the exterior, $|x| > |x_c|$, where $V_{ss} - x - 1 > 0$, as shown in Fig. 1. The motion of inwardingly moving perturbations is given by $dr/dt = v(t, r) - c_s$, or

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[2]The line $V_{ss}(x_c) - x_c = -1$ is also a sonic line, with the segment $t \leq 0$ being saddle. However, this part is physically irrelevant, so in this paper we shall not consider it.

$$\frac{dx}{dt} = -\frac{1}{|t|}\left(V_{ss} - x - 1\right) = \begin{cases} < 0, & |x| > |x_c|, \\ = 0, & |x| = |x_c|, \\ > 0, & |x| < |x_c|, \end{cases} \tag{2.21}$$

where $t \leq 0$. Thus, in the exterior all the perturbations, including those that move inward with respect to the fluid, are dragged outward in terms of x, while in the interior perturbations can move in both directions. As a results, perturbations cannot penetrate from the exterior into interior [1]. this observation is important when we consider the kink perturbations.

FIGURES

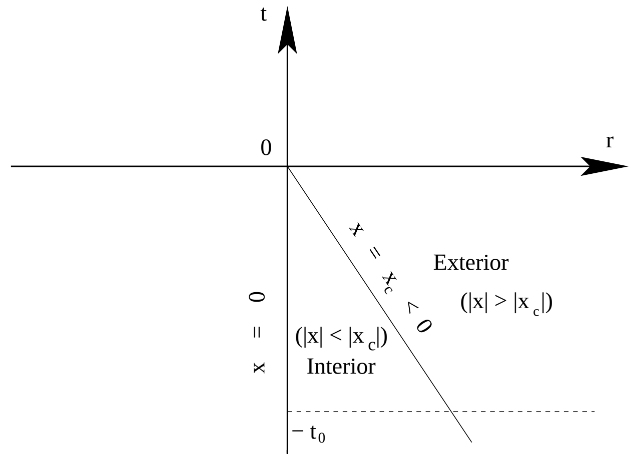

FIG. 1. The (t, r)-plane. The line $x = x_c < 0$ represents the sonic line. The whole plane is divided into three regions: $\Omega^+ \equiv \{t, r : |x| \geq |x_c|\}$; $\Sigma \equiv \{t, r : x = x_c\}$; and $\Omega^- \equiv \{t, r : |x| \leq |x_c|\}$.

Assuming that $F_{ss}(x)$ is at least C^2 in $\Omega^{\pm}$, it can be shown that the solutions near the sonic line $x = x_c$ are given by [8,10]

$$\begin{aligned} V_{ss}(x) &= (1 + x_c) + \frac{1 + x_c}{x_c}(x - x_c) - \frac{1 + x_c}{2x_c^2}(x - x_c)^2 + R_V(x, x_c), \\ E_{ss}(x) &= \ln(-2x_c) - \frac{1 + x_c}{x_c}(x - x_c) + \frac{1 + x_c}{x_c^2}(x - x_c)^2 + R_E(x, x_c), \\ M_{ss}(x) &= 2 - \frac{2(1 + x_c)}{x_c}(x - x_c) + \frac{(1 + x_c)(2 + x_c)}{x_c^2}(x - x_c)^2 + R_M(x, x_c), \end{aligned} \tag{2.22}$$

for type 1, and

$$\begin{aligned} V_{ss}(x) &= (1 + x_c) - \frac{1}{x_c}(x - x_c) - \frac{x_c^2 + 5x_c + 5}{2x_c^2(2x_c + 3)}(x - x_c)^2 + R_E(x, x_c), \\ E_{ss}(x) &= \ln(-2x_c) + \frac{1}{x_c}(x - x_c) + \frac{x_c^2 - 2}{2x_c^2(2x_c + 3)}(x - x_c)^2 + R_E(x, x_c), \\ M_{ss}(x) &= 2 - \frac{2(1 + x_c)}{x_c}(x - x_c) + \frac{2 + x_c}{x_c^2}(x - x_c)^2 + R_M(x, x_c), \end{aligned} \tag{2.23}$$

for type 2, where $R_F(x, x_c)$ denotes the errors of the corresponding expansion. Note that when $x_c = -2$ the two types coincide. For our present purpose, we can consider this degenerate case as a particular one of any of the two types.

The qualitative behavior of the solutions near the critical point depends on the values of x_c: (a) When $|x_c| < 1$, the critical point is a *saddle*, and only two solutions pass through it, one for each type. (b) When $|x_c| > 1$, the critical point is a *node* [9]. In this case, the directions that the solutions pass the critical point are further classified into *primary* and *secondary* directions. In particular, one of the directions, type 1 solutions for $1 < |x_c| < 2$ and type 2 solutions for $|x_c| > 2$, is called the secondary direction, while the other the primary direction. Only one solution can pass a node in the secondary direction, but an infinite number of solutions pass it along the primary direction [10]. Physically relevant solutions are these across the sonic line through nodes [cf. Fig. 2]. Thus, in the following we shall consider only the case where $x_c \leq -1$.

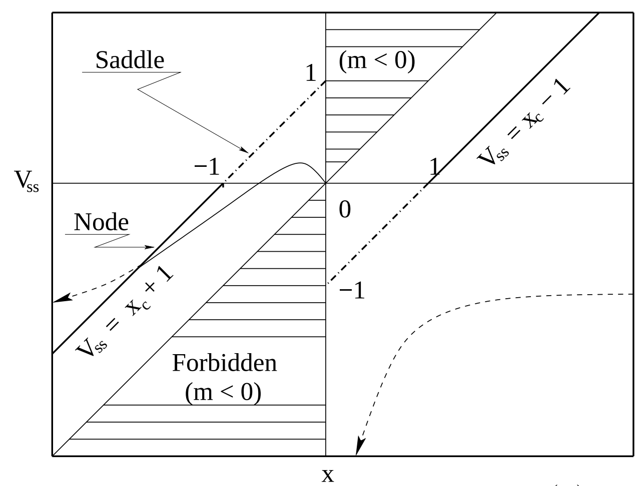

FIG. 2. The (V_{ss}, x)-plane. The region bound by the lines $V_{ss}(x) = x$ and $x = 0$ is excluded, because in this region the mass of the fluid is negative, $m < 0$. The lines with arrows represent a possible trajectory of the fluid.

It is possible to match solutions with different types across the critical point. But in this case the matching is only C^0, that is, $F_{ss}(x)$ is only continuous, and its first-order derivatives normal to the surface $x = x_c$ are usually not [9]. Ori and Piran considered only the case where the matching is smooth, that is, solutions that belong to the same type in both sides of the sonic line. In this paper, we shall consider all the possibilities.

Dividing the whole plane into three regions, $\Omega^{\pm}$ and Σ, where $\Omega^{+} \equiv \{t, r : |x| \geq |x_c|\}$, $\Sigma \equiv \{t, r : x = x_c\}$, and $\Omega^{-} \equiv \{t, r : |x| \leq |x_c|\}$, we can write $F_{ss}(x)$ as,

$$F_{ss}(x) = \begin{cases} F_{ss}^{+}(x), & x \in \Omega^{+}, \\ F_c, & x \in \Sigma, \\ F_{ss}^{-}(x), & x \in \Omega^{-}. \end{cases} \tag{2.24}$$

Since F_{ss} is C^0 across the surface $x = x_c$, we must have

$$\begin{aligned} E_{ss}^{+}(x_c) &= E_{ss}^{-}(x_c) = \ln(-2x_c), \\ M_{ss}^{+}(x_c) &= M_{ss}^{-}(x_c) = 2, \\ V_{ss}^{+}(x_c) &= V_{ss}^{-}(x_c) = 1 + x_c, \end{aligned} \tag{2.25}$$

as we can see from Eqs.(2.22) and (2.23), this is indeed the case. On the other hand, from Eqs. (2.19), (2.22) and (2.23) we find that $M_{ss}{}'$ is also continuous and given by

$$M_{ss}^{+\prime}(x_c) = M_{ss}^{-\prime}(x_c) = -\frac{2(1 + x_c)}{x_c}. \tag{2.26}$$

Taking the limits $x \to x_c^{\pm 0}$ in Eqs.(2.14) - Eqs.(2.17) and then subtracting them, we find that

$$[V_{ss}{}']^{-} + [E_{ss}{}']^{-} = 0, \quad (x = x_c). \tag{2.27}$$

Note that in writing Eq.(2.27) we had used Eqs.(2.20) and (2.25).

As assumed previously, $F_{ss}(x)$ is at least C^2 in $\Omega^{\pm}$, we can take derivative of Eqs.(2.14) - (2.17) with respect to x, and obtain

$$(V_{ss} - x)\, V_{ss}{}'' + E_{ss}{}'' + (V_{ss}{}' - 1)\, V_{ss}{}' + \frac{1}{x} M_{ss}{}' - \frac{1}{x^2}(M_{ss} - 2) = 0, \tag{2.28}$$

$$(V_{ss} - x)\, E_{ss}{}'' + V_{ss}{}'' + (V_{ss}{}' - 1)\, E_{ss}{}' = 0, \tag{2.29}$$

$$x M_{ss}{}'' + 2 M_{ss}{}' - e^{E_{ss}} E_{ss}{}' = 0, \tag{2.30}$$

$$x M_{ss}{}'' + M_{ss}{}' - \frac{1}{x} e^{E_{ss}} (V_{ss}{}' + V_{ss} E_{ss}{}') + \frac{1}{x^2} e^{E_{ss}} V_{ss} = 0, \tag{2.31}$$

where the quantities $F_{ss}(x)$ in the above equations should be understood as $F_{ss}^{+}(x)$ in Ω^{+} and $F_{ss}^{-}(x)$ in Ω^{-}. The subtraction of the limits $x \to x_c^{\pm 0}$ of Eqs.(2.28) - Eqs.(2.31) yields

$$[V_{ss}{}'']^{-} + [E_{ss}{}'']^{-} - [V_{ss}{}']^{-} \left(1 - [V_{ss}{}']^{+}\right) = 0, \tag{2.32}$$

$$[V_{ss}{}'']^{-} + [E_{ss}{}'']^{-} - [E_{ss}{}']^{-} + \frac{1}{2}\left([E_{ss}{}']^{+} [V_{ss}{}']^{-} + [E_{ss}{}']^{-} [V_{ss}{}']^{+}\right) = 0, \tag{2.33}$$

$$[M_{ss}{}'']^{-} + 2\, [E_{ss}{}']^{-} = 0, \quad (x = x_c), \tag{2.34}$$

where

$$[F_{ss}{}']^{+} \equiv \lim_{x \to x_c^{+0}} \left(\frac{dF^{+}_{ss}(x)}{dx} \right) + \lim_{x \to x_c^{-0}} \left(\frac{dF^{-}_{ss}(x)}{dx} \right). \tag{2.35}$$

Note that in writing the above equations we had used Eqs.(2.25) and (2.27), so that Eqs.(2.30) and (2.31) yield the same equation, (2.34). From Eqs.(2.32) and (2.33), on the other hand, we find that

$$[V_{ss}{}']^{-} \left([E_{ss}{}']^{+} - 3\,[V_{ss}{}']^{+} + 4 \right) = 0, \quad (x = x_c), \tag{2.36}$$

which has the solutions,

$$(a) \quad [V_{ss}{}']^{-} = 0, \tag{2.37}$$

$$(b) \quad [E_{ss}{}']^{+} - 3\,[V_{ss}{}']^{+} + 4 = 0. \tag{2.38}$$

Clearly, Case (a) requires that the solutions inside and outside the surface $x = x_c$ belong to the same type, and in this case from Eqs.(2.22), (2.23) and (2.27) we find that

$$[E_{ss}{}']^{+} - 3\,[V_{ss}{}']^{+} + 4 = \frac{4\,(2 - |x_c|)}{|x_c|} \times \begin{cases} 1, & \text{Type 1}, \\ -1, & \text{Type 2}, \end{cases} \quad \left([V_{ss}{}']^{-} = 0 \right). \tag{2.39}$$

III. GENERAL FORMULA OF PERTURBATIONS

In this section using distribution theory, we shall develop formulas that are valid for any kind of perturbations, linear or non-linear, and write down the equations of perturbations in $\Omega^{\pm}$ and Σ separately.

For any given function $F(x,\tau)$ that is C^2 in $\Omega^{\pm}$ and C^0 across the surface $x = x_c$ can be written as[3],

$$F(x,\tau) = F^{+}(x,\tau)\left[1 - H(x - x_c)\right] + F^{-}(x,\tau) H(x - x_c)\,, \tag{3.1}$$

where F^{+} (F^{-}) is defined in Ω^{+} (Ω^{-}), and

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[3]Note that in Ω^{+} (Ω^{-}) we now have $x < x_c$ ($x > x_c$).

$$\lim_{x \to x_s^{+0}} F^+(x,\tau) = \lim_{x \to x_s^{-0}} F^-(x,\tau) \equiv F_c(\tau). \tag{3.2}$$

In the present case, it is clear that

$$F^{\pm}(x,\tau) = F^{\pm}_{ss}(x) + \delta F^{\pm}(x,\tau), \tag{3.3}$$

where $F^{\pm}_{ss}(x)$ denotes the self-similar solutions of Eqs.(2.14) - (2.17) with the boundary conditions of Eqs. (2.18), (2.25) and (2.26). The perturbations $\delta F^{\pm}(x,\tau)$ satisfy the conditions,

$$\lim_{x \to x_s^{+0}} \delta F^+(x,\tau) = \lim_{x \to x_s^{-0}} \delta F^-(x,\tau) \equiv \delta F_c(\tau). \tag{3.4}$$

Then, using Eqs. (1.6), (3.1), (3.2), and the fact that

$$H\left(x - x_c\right)\delta\left(x - x_c\right) = \frac{1}{2}\delta\left(x - x_c\right), \quad \left[1 - H\left(x - x_c\right)\right]\delta\left(x - x_c\right) = \frac{1}{2}\delta\left(x - x_c\right), \tag{3.5}$$

we find

$$\begin{aligned} F_{,x} &= F^+{}_{,x}\left[1 - H\left(x - x_c\right)\right] + F^-{}_{,x}H\left(x - x_c\right), \\ F_{,xx} &= F^+{}_{,xx}\left[1 - H\left(x - x_c\right)\right] + F^-{}_{,xx}H\left(x - x_c\right) - \left[F_{,x}\right]^- \delta\left(x - x_c\right), \\ F^{(n)} &= F^{+(n)}\left[1 - H\left(x - x_c\right)\right] + F^{-(n)}H\left(x - x_c\right) - \sum_{i=0}^{n-2}\left[F^{(n-i-1)}\right]^- \delta^{(i)}\left(x - x_c\right), \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} \left[F^{(n)}\right]^- &\equiv \lim_{x \to x_c^{+0}}\left(\frac{\partial^n F^+\left(x,\tau\right)}{\partial x^n}\right) - \lim_{x \to x_c^{-0}}\left(\frac{\partial^n F^-\left(x,\tau\right)}{\partial x^n}\right), \\ \delta^{(n)}\left(x - x_c\right) &\equiv \frac{d^n\delta\left(x - x_c\right)}{dx^n}, \\ \int_{-\infty}^{\infty} F(x,\tau)\delta^{(k)}\left(x - x_c\right)dx &= (-1)^k F^{(k)}\left(x_c,\tau\right). \end{aligned} \tag{3.7}$$

Inserting Eqs.(3.1)-(3.6) into Eqs.(2.9) - (2.12), and considering Eq.(3.5), we find that

$$\left(V_{ss} - x\right)\delta V_{,x} + V_{ss,x}\delta V + \delta V \delta V_{,x} - \delta V_{,\tau} + \delta E_{,x} + \frac{1}{x}\delta M = 0, \tag{3.8}$$

$$\left(V_{ss} - x\right)\delta E_{,x} + E_{ss,x}\delta V + \delta V \delta E_{,x} - \delta E_{,\tau} + \delta V_{,x} = 0, \tag{3.9}$$

$$x\delta M_{,x} + \delta M - e^{E_{ss}}\left(e^{\delta E} - 1\right) = 0, \tag{3.10}$$

$$x\delta M_{,x} + \delta M_{,\tau} + \frac{1}{x}e^{E_{ss}}\left\{V_{ss}\left(1 - e^{\delta E}\right) - e^{\delta E}\delta V\right\} = 0. \tag{3.11}$$

Again, the quantities F, δF in the above equations should be understood as F^{+}, δF^{+} (F^{-}, δF^{-}) in Ω^{+} (Ω^{-}). The boundary conditions (2.6), together with those of Eq.(2.18) require

$$\delta V^{-}(t,0) = 0 = \delta D^{-}(t,0). \tag{3.12}$$

Subtracting the limits $x \to x_c^{\pm 0}$ of Eqs.(3.8) - (3.11), on the other hand, we find,

$$(1+\delta V_c)\, W_c(\tau) + U_c(\tau) + [V_{ss}{}']^{-}\, \delta V_c = 0, \tag{3.13}$$

$$(1+\delta V_c)\, U_c(\tau) + W_c(\tau) + [E_{ss}{}']^{-}\, \delta V_c = 0, \tag{3.14}$$

$$[\delta M_{,x}]^{-} = 0, \quad (x = x_c), \tag{3.15}$$

where

$$U_c(\tau) \equiv [\delta E_{,x}]^{-}\,, \quad W_c(\tau) \equiv [\delta V_{,x}]^{-}\,, \tag{3.16}$$

and $[\delta F_{,x}]^{-}$ is defined as

$$[\delta F_{,x}]^{\pm} \equiv \lim_{x\to x_c^{+0}} \left(\frac{\partial\left(\delta F^{+}(x,\tau)\right)}{\partial x}\right) \pm \lim_{x\to x_c^{-0}} \left(\frac{\partial\left(\delta F^{-}(x,\tau)\right)}{\partial x}\right). \tag{3.17}$$

Note that in writing Eqs.(3.13) -(3.15), we used the fact that

$$\left.\frac{\partial}{\partial \tau}\delta F^{+}(x,\tau)\right|_{x=x_c^{+0}} = \left.\frac{\partial}{\partial \tau}\delta F^{-}(x,\tau)\right|_{x=x_c^{-0}}. \tag{3.18}$$

In addition, Eqs.(3.10) and (3.11) give the same equation, (3.15). Combining Eq.(2.27) with Eqs.(3.13)-(3.15) we find the following solutions,

$$(i)\;\; \delta V_c = 0, \quad U_c(\tau) = -W_c(\tau), \quad [\delta M_{,x}]^{-} = 0, \tag{3.19}$$

$$(ii)\;\; \delta V_c = -2, \quad U_c(\tau) = W_c(\tau) + 2\,[V_{ss}{}']^{-}\,, \quad [\delta M_{,x}]^{-} = 0, \tag{3.20}$$

$$(iii)\;\; U_c(\tau) = -W_c(\tau) = [V_{ss}{}']^{-}\,, \quad [\delta M_{,x}]^{-} = 0, \quad (x = x_c). \tag{3.21}$$

On the other hand, taking the derivatives of Eqs.(2.9) - (2.12) with respect to x, we find that

$$(V-x)\,V_{,xx}+E_{,xx}-V_{,x\tau}+(V_{,x}-1)\,V_{,x}+\frac{1}{x}M_{,x}-\frac{1}{x^2}\,(M-2)=0, \tag{3.22}$$

$$(V-x)\,E_{,xx}+V_{,xx}-E_{,x\tau}+(V_{,x}-1)\,E_{,x}=0, \tag{3.23}$$

$$xM_{,xx}+2M_{,x}-e^{E}E_{,x}=0, \tag{3.24}$$

$$xM_{,xx}+M_{,\tau x}+M_{,x}-\frac{1}{x}e^{E}\,(V_{,x}+VE_{,x})+\frac{1}{x^2}Ve^{E}=0. \tag{3.25}$$

Substituting Eqs.(3.3) - (3.6) into the these equations and using Eq.(3.5), we find that each of these equations can be written in the form,

$$G^{+}(\tau,x)\,[1-H\,(x-x_c)]+G^{-}(\tau,x)H\,(x-x_c)+G_c\,(\tau)\,\delta\,(x-x_c)=0, \tag{3.26}$$

which is equivalent to

$$G^{+}(\tau,x)=0,\quad \left(x\in\Omega^{+}\right) \tag{3.27}$$

$$G^{-}(\tau,x)=0,\quad \left(x\in\Omega^{-}\right) \tag{3.28}$$

$$G_c\,(\tau)=0,\quad (x\in\Sigma)\,. \tag{3.29}$$

It can be shown that Eqs.(3.27) and (3.28) take the same forms as these given by Eqs.(3.22) - (3.25) after replying F by $F^{\pm}$. But, on the surface $x=x_c$ Eq.(3.29) gives

$$[V_{,x}]^{-}+[E_{,x}]^{-}=0, \tag{3.30}$$

$$[M_{,x}]^{-}=0. \tag{3.31}$$

Eq.(3.31) is consistent with Eqs.(2.26) and (3.15), while Eq.(3.30) together with Eq.(2.27) yields

$$U_c(\tau)+W_c(\tau)=0. \tag{3.32}$$

On the other hand, subtracting the limits $x\rightarrow x_c^{\pm 0}$ of Eqs.(3.22)-(3.25), we find that

$$\begin{aligned}&\frac{dW_c(\tau)}{d\tau}-(1+\delta V_c)\,[\delta V_{,xx}]^{-}-[\delta E_{,xx}]^{-}-\delta V_c\,[V_{ss}'']^{-}\\&\quad+\left(1-[V_{ss}']^{+}-[\delta V_{,x}]^{+}\right)W_c(\tau)-[V_{ss}']^{-}\,[\delta V_{,x}]^{+}=0,\end{aligned} \tag{3.33}$$

$$\frac{dU_c(\tau)}{d\tau}-(1+\delta V_c)\,[\delta E_{,xx}]^{-}-[\delta V_{,xx}]^{-}-\delta V_c\,[E_{ss}'']^{-}$$

$$+\frac{1}{2}\left(2-[V_{ss}']^{+}-[\delta V_{,x}]^{+}\right)U_c(\tau)-\frac{1}{2}\left([E_{ss}']^{+}+[\delta E_{,x}]^{+}\right)W_c(\tau)$$
$$-\frac{1}{2}\left([V_{ss}']^{-}[\delta E_{,x}]^{+}+[E_{ss}']^{-}[\delta V_{,x}]^{+}\right)=0, \tag{3.34}$$

$$x_c\,[\delta M_{,xx}]^{-}-2x_c\left(1-e^{\delta E_c}\right)[E_{ss}']^{-}+2x_c e^{\delta E_c}U_c(\tau)=0, \tag{3.35}$$

$$\left([E_{ss}']^{-}+U_c(\tau)\right)\delta V_c=0. \tag{3.36}$$

Equations (3.8) - (3.11), (3.19) - (3.21), and (3.32) - (3.36) consist of the full set of differential equations for the perturbations $\delta F^{\pm}(\tau,x)$ in the whole spacetime, which consists of three regions, $\Omega^{\pm}$ and Σ. Note that these equations are exact, and so far no approximation has been made. In Sec. V we shall consider their linearized perturbations.

IV. KINK STABILITY

Kink stability is the study of the perturbations of Eqs.(3.19) - (3.21), and (3.32) - (3.36) along the critical line $x=x_c$. To solve these equations for $\delta F^{\pm}(\tau,x)$, in addition to (3.12), other boundary conditions need to be given. Following Ori and Piran, we shall impose the following conditions: Assume that the perturbations turn on at the moment $t=t_0$ or $\tau=0$ [cf. Fig. 1], then we require

(A) the perturbations initially vanish in the interior,

$$\delta F^{-}(\tau=0,x)=0,\ x\in\Omega^{-}, \tag{4.1}$$

(B) the perturbations be continuous everywhere, and in particular across the sonic line,

$$[\delta F]^{-}=0,\ \ (x=x_c)\,, \tag{4.2}$$

(C) $\delta E^{\pm}{}_{,x}$ and $\delta V^{\pm}{}_{,x}$ be discontinuous at the sonic line,

$$U_c(\tau)=[\delta E_{,x}]^{-}\neq 0,\quad W_c(\tau)=[\delta V_{,x}]^{-}\neq 0,\ \ (x=x_c)\,. \tag{4.3}$$

From the above we first note that Eq.(4.1) remains true for all the $\tau>0$, because the perturbations cannot penetrate the sonic line from exterior into the interior, as we noticed in Sec. II, and $\delta F^{-}(x, \tau) = 0$ are indeed solutions of Eqs.(3.8) - (3.11) in Ω^{-}. Thus, in the following we need to consider only the perturbations in Ω^{+} as well as those along the sonic line $x = x_c$.

The studies of the perturbations in Ω^{+} will be considered in the next section, while in the rest of this section we shall consider only Eqs.(3.19) - (3.21) and (3.32) - (3.36). As first noticed by Ori and Piran, the evolutions of these two sets of equations are separable. In particular, Conditions (4.1) - (4.3) together with Eqs.(3.19) - (3.21) and (3.32) - (3.36) are already sufficient to determine the evolution of $[\delta F_{,x}]^{-}$ uniquely. To show this, let us first notice that Conditions (A) and (B) imply

$$\begin{aligned} \delta F_c(\tau) &= \delta F^{-}(\tau, x_c) = 0, \\ [\delta F_{,x}]^{-} &= \left.\frac{\partial \delta F^{+}(\tau, x)}{\partial x}\right|_{x = x_c^{+0}} = [\delta F_{,x}]^{+}. \end{aligned} \tag{4.4}$$

Then, we can see that only the case of Eq.(3.19) is consistent with these expressions. Thus, in the following we shall discard the cases described by Eqs.(3.20) and (3.21). Hence, one can see that Eq.(3.36) is satisfied identically, while from Eqs.(3.33) - (3.35) we find

$$\frac{dW_c}{d\tau} + \frac{1}{4}\left(4 + [E_{ss}{}']^{+} - 3\,[V_{ss}{}']^{+} - 4\,[V_{ss}{}']^{-}\right) W_c - W_c^2 = 0, \tag{4.5}$$

where in writing this equation we had used Eq.(3.32). To study it, let us consider the two cases defined by Eqs.(2.37) - (2.38) separately.

Case A) $[V_{ss}{}']^{-} = 0$**:** In this case the internal and external solutions belong to the same type, and we have

$$F_{ss}^{+\,\prime}(x_c) = F_{ss}^{-\,\prime}(x_c) \equiv F_c{}', \tag{4.6}$$

for which Eq.(4.5) becomes

$$\frac{dW_c}{d\tau} = kW_c + W_c^2, \quad k \equiv (2V_c{}' - 1)\,. \tag{4.7}$$

This is exactly the equation obtained by Ori and Piran [1]. In particular, it was shown that when $V_c{}' \geq 1/2$ the self-similar solutions are unstable against the kink perturbations.

Case B) $[V_{ss}{}']^- \neq 0$**:** This is the case of Eq.(2.38), from which we find that Eq.(4.5) takes the same form as that of Eq.(4.7), but now with

$$k = [V_{ss}{}']^- . \tag{4.8}$$

The dynamics of small perturbations are determined by the linear part of Eq.(4.7),

$$\frac{dW_c}{d\tau} \simeq kW_c, \tag{4.9}$$

which has the solution,

$$W_c(\tau) \simeq W_0 e^{k\tau} = \begin{cases} \pm\infty, & k > 0, \\ W_0, & k = 0, \quad (\text{as } \tau \to \infty), \\ 0, & k < 0, \end{cases} \tag{4.10}$$

where W_0 is an integration constant and denotes the initial profile of the perturbation. Therefore, when $k \geq 0$ the corresponding self-similar solutions are unstable against kink perturbations, and when $k < 0$, they are stable. To find out the sign of k we need to consider the two cases where the external solution is Type 1 or 2 separately. When it is type 1, then the internal solution must be type 2, and then from Eqs.(2.22) and (2.23) we find

$$k = V_{ss}^{+}{}'(x_c) - V_{ss}^{-}{}'(x_c) = \frac{|x_c| - 2}{|x_c|} = \begin{cases} \geq 0, & x_c \leq -2, \\ < 0, & -1 > x_c > -2. \end{cases} \tag{4.11}$$

Thus, for this kind of matching, the self-similar solutions are stable for $-1 > x_c > -2$, and unstable for $x_c \leq -2$. When the external solution is type 2, the internal solution must be type 1. Then, the sign of k is just opposite to the last case, and we conclude that the corresponding self-similar solutions now are unstable for $-1 > x_c \geq -2$ and stable for $x_c < -2$.

It is interesting to notice that Eq.(4.10) allows the following general solution [1],

$$W_c(\tau) = \frac{kW_0 e^{k\tau}}{k + W_0\left(1 - e^{k\tau}\right)}, \tag{4.12}$$

which can diverge for certain choice of the initial profile W_0. Ori and Piran argued that this divergence could be the indication of the formation of a shock wave.

V. LINEAR PERTURBATIONS IN REGION Ω^+

In this section, we shall study the linear perturbations of the self-similar solutions given in Sec. III in the region Ω^+, where $|x| \geq |x_c|$, with $\delta F^+(\tau, x_c)$ and $\delta F^+{}_{,x}(\tau, x_c)$ obtained in the last section as the boundary conditions. Let us first write the perturbations $\delta F^+(\tau, x)$ as

$$\delta F^+(\tau, x) = \epsilon F_1(x) e^{k\tau}, \tag{5.1}$$

where ϵ is a very small quantity, and $F_1(x)$ denotes perturbations. It is understood that there may have many perturbation modes for different k. Then, the general linear perturbation will be the sum of these individual ones.

To the first order of ϵ, the substitution of Eq.(5.1) into Eqs.(3.8) - (3.11) yields

$$(V_{ss} - x) V_1{}' + E_1{}' + (V_{ss}{}' - k) V_1 + \frac{1}{x} M_1 = 0, \tag{5.2}$$

$$(V_{ss} - x) E_1{}' + V_1{}' + E_{ss}{}' V_1 - k E_1 = 0, \tag{5.3}$$

$$x M_1{}' + M_1 - e^{E_{ss}} E_1 = 0, \tag{5.4}$$

$$x M_1{}' + k M_1 - \frac{1}{x} e^{E_{ss}} (V_{ss} E_1 + V_1) = 0, \quad \left(x \in \Omega^+\right), \tag{5.5}$$

where, without causing any confusions, in writing the above equations we had dropped the superscript "+".

Physically acceptable solutions of the above equations must satisfy some boundary conditions at the sonic line $x = x_c$ as well as at the spatial infinity $x = \infty$. At $x = x_c$, the boundary conditions are the solutions obtained in the last section along $x = x_c$. From Eqs.(3.15), (3.32), (4.4) and (4.10) we find that these conditions read

$$V_1(x_c) = E_1(x_c) = M_1(x_c) = 0, \tag{5.6}$$

$$V_1{}'(x_c) = -E_1{}'(x_c) = W_0, \quad M_1{}'(x_c) = 0, \quad (x = x_c), \tag{5.7}$$

where W_0 is the integration constant given in Eq.(4.10). In the neighborhood of $x = x_c$, the functions $F_{ss}(x)$ behave as those given by Eqs.(2.22) and (2.23).

On the other hand, at the spatial infinity $x = \infty$ (or $r = \infty$), we require that the perturbations $F_1(x)$ should not grow faster than the background solutions $F_{ss}(x)$, that is,

$$F_1(x) \text{ grows slower than } \; F_{ss}(x), \quad \text{as } x \to \infty, \tag{5.8}$$

where $F_1 \equiv \{E_1, V_1, M_1\}$. The asymptotic behavior of $F_{ss}(x)$ is given by [9],

$$\begin{aligned} V_{ss}(x) &= V_\infty - \frac{(D_\infty - 2)}{x} + \frac{V_\infty}{x^2} + \frac{[4V_\infty + (D_\infty - 2)(D_\infty - 6)]}{6x^3} + O\left(x^{-4}\right), \\ E_{ss}(x) &= \ln(D_\infty) - \frac{(D_\infty - 2)}{2x^2} + O\left(x^{-4}\right), \end{aligned} \tag{5.9}$$

as $x \to \infty$, where D_∞ and V_∞ are two arbitrary constants.

The asymptotic behavior of $M_{ss}(x)$ as $x \to x_c$ and $x \to \infty$ can be obtained from Eqs.(2.19), (2.22), (2.23) and (5.9).

Since $F_{ss}(x)$ is well-defined in the region $x \in (-\infty, -|x_c|)$, one can see that there always exist solutions to Eqs.(5.2)- (5.4), which in general has three free parameters. However, Eq.(5.5) represents a constraint, and shall reduce the number of the free parameters from three to two. On the other hand, one can show that $M_1(x_c) = 0$ holds, provided that $V_1(x_c) = E_1(x_c) = 0$ is true. Thus, the boundary conditions given by Eq.(5.6) are already sufficient to determine the two free parameters uniquely. Then, an important question rises: Do Eqs.(5.7) and (5.8) give further restrictions on these parameters? In particular, once all the conditions of Eqs.(5.6) - (5.8) are taken into account, do the values of the spectrum k still remain the same as those given by Eq.(4.7) or (4.8)? If not, is it possible that $Re(k)$ cannot be positive any more, and that all the unstable modes found in the last section are due to the incomplete treatment of the problem?

Before addressing these questions, we first consider the problem concerning gauge modes. Under the coordinate transformations (2.7), we have

$$\begin{aligned} &\tau \to \tau + \epsilon e^{\tau}, \quad x \to x\left(1 + \epsilon e^{\tau}\right), \\ &F_{ss}(x) \to F_{ss}(x) + \epsilon F_1^{(g)}(x) e^{\tau}, \end{aligned} \tag{5.10}$$

where

$$F_1^{(g)}(x) \equiv x{F'}_{ss}(x). \tag{5.11}$$

Comparing it with Eq.(5.1) we find that the gauge mode corresponds to the case where $k = 1$. This mode is not physical, and completely due to the gauge transformations (2.7). Thus, in the following we shall discard this case and consider only the case where $k \neq 1$. Then, from Eqs.(5.4) and (5.5) we find

$$M_1 = -\frac{1}{(1-k)x}\left[(V_{ss} - x)\, E_1 + V_1\right] e^{E_{ss}}. \tag{5.12}$$

Combining it with Eqs.(5.2) and (5.3) we find

$$E_1{}' = \frac{1}{(V_{ss} - x)^2 - 1}\left(AV_1 + BE_1\right), \tag{5.13}$$

$$V_1{}' = \frac{1}{(V_{ss} - x)^2 - 1}\left(CV_1 + DE_1\right), \tag{5.14}$$

where

$$\begin{aligned}
A &\equiv (V_{ss}{}' - k) - (V_{ss} - x)\, E_{ss}{}' - \frac{1}{(1-k)x^2}e^{E_{ss}},\\
B &\equiv (V_{ss} - x)\left[k - \frac{1}{(1-k)x^2}e^{E_{ss}}\right],\\
C &\equiv E_{ss}{}' - (V_{ss} - x)\left[(V_{ss}{}' - k) + \frac{1}{(1-k)x^2}e^{E_{ss}}\right],\\
D &\equiv \frac{(V_{ss} - x)^2}{(1-k)x^2}e^{E_{ss}} - k.
\end{aligned} \tag{5.15}$$

In the following we shall consider the asymptotic behavior of E_1 and V_1 at $x = x_c$ and $x = \infty$ separately. First, in the neighborhood of x_c we expand them as

$$\begin{aligned}
E_1 &= E_1^0 + E_1^1\,(x - x_c) + R_E^1(x, x_c),\\
V_1 &= V_1^0 + V_1^1\,(x - x_c) + R_V^1(x, x_c).
\end{aligned} \tag{5.16}$$

Substituting the above expressions into Eqs.(5.13) and (5.14) we find that

$$\begin{aligned}
E_1^0 &= \frac{(1-k)\left[(k-2)x_c - 2\right] - 2}{k(1-k)x_c + 2}V_1^0,\\
E_1^1 &= -V_1^1 + kE_1^0 + \frac{1 + x_c}{x_c}V_1^0, \quad \text{(Type 1)},
\end{aligned} \tag{5.17}$$

for type 1 solutions, and

$$E_1^0 = \frac{k\left[(1-k)x_c - 2\right]}{k(1-k)x_c + 2} V_1^0,$$
$$E_1^1 = -V_1^1 + kE_1^0 - \frac{1}{x_c} V_1^0, \quad \text{(Type 2)}, \tag{5.18}$$

for type 2 solutions, where V_1^0 and V_1^1 are arbitrary constants in both types. Applying the boundary conditions (5.6) and (5.7) to the above solutions we find that

$$E_1^0 = V_1^0 = 0, \quad E_1^1 = -V_1^1 = -W_0, \tag{5.19}$$

for a given k.

On the other hand, setting $\epsilon \equiv 1/x$ we can expand F_1 at the neighborhood of $\epsilon = 0$ as,

$$E^1 = e_1^0 + e_1^1 \epsilon + O\left(\epsilon^2\right),$$
$$M^1 = m_1^0 + m_1^1 \epsilon + O\left(\epsilon^2\right),$$
$$V^1 = v_1^0 + v_1^1 \epsilon + O\left(\epsilon^2\right). \tag{5.20}$$

Inserting these expressions and the ones given by (5.9) into Eqs.(5.2)-(5.5) we find that all the coefficients, f_1^i, are zero, unless $k = 0$, for which we have

$$m_1^0 = D_\infty\left(e_1^0 - v_1^0\right),$$
$$e_1^1 = 0, \quad v_1^1 = -D_\infty e_1^0,$$
$$m_1^1 = D_\infty\left(D_\infty - V_\infty e_1^0\right) e_1^0, \quad \text{(Type 1)}, \tag{5.21}$$

for type 1, and

$$m_1^0 = D_\infty\left(e_1^0 - v_1^0\right),$$
$$e_1^1 = 0, \quad v_1^1 = -D_\infty e_1^0,$$
$$m_1^1 = D_\infty\left(D_\infty - V_\infty e_1^0\right) e_1^0, \quad \text{(Type 2)}, \tag{5.22}$$

for type 2, where e_1^0 and v_1^1 are arbitrary. However, in each of the cases conditions (5.8) are satisfied for any given k.

In review all the above, we conclude that *the considerations of the perturbations outside the sonic line $x = x_c$ and their matching to the ones obtained along the line do not impose any additional conditions on the spectrum, k, of the kink perturbations, obtained by considering the linear perturbations only along the sonic line.*

VI. DISCUSSIONS AND CONCLUDING REMARKS

In this paper we first gave a general review over the weak discontinuities of a fluid along a sonic line by providing a concrete example, and showed how such a discontinuity can propagate along the line. Then, in Sec. II we summarized the main properties of the self-similar solutions of an isothermal spherical self-similar flow. In particular, we investigated the discontinuities of these solutions across the sonic line, which is useful and necessary for the studies of the kink stability to be considered in the following sections. In Sec. III, using distribution theory we first developed a general formula of perturbations in the background of a self-similar solution. For such a development, we assumed that the background is at least C^2 outside of the sonic line and C^0 across it. The field equations were divided into three different sets, one in the region $x > x_c$, one in the region $x < x_c$, and the other along the sonic line $x = x_c$. Any perturbations must satisfy all of these equations. In Sec. IV, following Ori and Piran [1] we considered the differential equations along the sonic line and found their explicit solutions. When the background is smooth across the sonic line, we re-derived the results of Ori and Piran. When the background is not smooth, we found the stability criterion for the self-similar solutions. In particular, if the external solution is type 1 and the internal is type 2, the solution is stable for $-2 < x_c < -1$, and not stable for $x_c \leq -2$. If the external solution is type 2 and the internal is type 1, it is just the other way around. That is, it is stable for $x_c < -2$, and not stable for $-2 \leq x_c < -1$.

Up to this point, it was not clear whether the solutions of the perturbations obtained along the sonic line can be matched to the ones outside the sonic line. Since the perturbations in the region $|x| < |x_c|$ is assumed to be zero identically, the ones in the region $|x| > |x_c|$ are necessarily non-zero. Otherwise, it is impossible to get a non-zero discontinuity across the sonic line. To answer the above question, we studied the linear perturbations in the region $|x| > |x_c|$, by taking the solutions obtained along the sonic line as the boundary conditions. By properly imposing other boundary conditions at the spatial infinity, we were able to show that there always exist linear perturbations in the region $|x| > |x_c|$, which satisfy all the boundary conditions for a given k, where k is the spectrum of the perturbations obtained along the sonic line. As a result, the studies of perturbations along the sonic line carried out before are consistent with the complete treatment of the problem in the whole spacetime. In particular, the spectrum of perturbations obtained along the sonic line remains the same even after the perturbations in the whole spacetimes are considered.

Finally, we note that it would be very interesting to generalize the above studies to the relativistic case, a subject that is under our current investigations.

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