

# $E = mc^2$ Without Relativity

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## ABSTRACT

The equivalence of mass and energy is indelibly linked with relativity, both by scientists and in the popular mind. Here I prove that  $E = mc^2$  by demanding momentum conservation of an object that emits two equal electromagnetic wave packets in opposite directions in its own frame. In contrast to Einstein's derivation of this equation, which applies energy conservation to a similar thought experiment, the new derivation employs no effects that are greater than first order in  $v/c$  and therefore does not rely on results from Special Relativity. In addition to momentum conservation, it uses only aberration of starlight and the electromagnetic-wave momentum-energy relation  $p_\gamma = E_\gamma/c$ , both of which were established by 1884. In particular, no assumption is made about the constancy of the speed of light, and the derivation proceeds equally well if one assumes that light is governed by a Galilean transformation. In view of this, it is somewhat puzzling that the equivalence of mass and energy was not derived well before the advent of Special Relativity. The new derivation is simpler and more transparent than Einstein's and is therefore pedagogically useful.

*Subject headings:* history and philosophy of astronomy – relativity

## 1. Introduction

Einstein (1905b) derived the equivalence of mass and energy ( $E = mc^2$ ) by considering an object of mass  $m$  that simultaneously emits two electromagnetic packets, each with energy  $\Delta E/2$  in opposite ( $\hat{\mathbf{x}}$  and  $-\hat{\mathbf{x}}$ ) directions. By momentum conservation in the rest frame of the object, it does not change its velocity after emission. Seen from a frame moving at velocity  $-v\hat{\mathbf{x}}$  (i.e., along the axis defined by the emissions), the two packets are each Doppler shifted (in opposite directions), so that the total energy of these packets is higher in the moving frame than the rest frame by  $\Delta E(\gamma - 1)$ , where  $\gamma = (1 - v^2/c^2)^{-1/2}$ . Einstein (1905b)

argued that by energy conservation, the object must lose energy in the moving frame by an amount that is greater than what it loses in the rest frame by exactly this difference. Einstein (1905a) had already shown in his earlier paper introducing Special Relativity, that the kinetic energy of a mass  $m$  moving at velocity  $v$  is  $E_k = mc^2(\gamma - 1)$ . Appealing to this result, Einstein (1905b) concluded that the mass of the emitting object must decline from  $m$  to  $m' = m - \Delta m$ , where  $\Delta m = \Delta E/c^2$ .

Here I show that the same result can be derived from conservation of momentum, without invoking any results from Special Relativity. That is, the derivation uses only effects that are first order in  $v/c$ , and does not employ the second-order effects that characterize Special Relativity.

## 2. Derivation of $E_0 = m_0c^2$

Consider as above an object emitting the two electromagnetic packets that, viewed in its rest frame are equal and opposite. By momentum conservation, the dual ejection leaves the object at rest in this frame. See Figure 1. Now consider the same event from a frame that is moving *perpendicular* (with velocity  $-v\hat{\mathbf{z}}$ ) relative to the emission directions. By symmetry, the wave packets are still equal, but they are no longer opposite: because of the aberration of starlight (first discovered by James Bradley in 1729), the packets will both appear to be moving slightly upward, at an angle  $\theta = v/c$ . Denote the emitted energies of the packets *in the moving frame* by  $\Delta E/2$ , and denote the mass of object in this frame before and after emission by  $m$  and  $m'$ .

By a variety of arguments elaborated below, the magnitude of the momenta of the two packets in this frame are

$$p_{\gamma,\pm} = \frac{\Delta E}{2c}. \quad (1)$$

Hence, because of aberration of starlight, the vertical components of these momenta will be (to first order in  $v/c$ )

$$(p_{\gamma,\pm})_z = \frac{\Delta E}{2c} \frac{v}{c}. \quad (2)$$

Equating the total  $z$ -momentum in the moving frame before and after emission yields,

$$mv = m'v + (p_{\gamma,+})_z + (p_{\gamma,-})_z = \left(m' + \frac{\Delta E}{c^2}\right)v, \quad (3)$$

which can be solved to obtain,

$$\Delta m = m - m' = \frac{\Delta E}{c^2}. \quad (4)$$

Note that in carrying out this derivation, I explicitly ignored terms higher than first order in  $(v/c)$ , in particular when I adopted  $p_z = p(v/c)$ . Hence, the result strictly applies only in the limit  $v \rightarrow 0$ , i.e., in the rest frame. This can be expressed as an equivalence between energy and rest-mass,

$$E_0 = m_0 c^2. \tag{5}$$

I address the question of how this result can be generalized to moving bodies in § 3.

### 2.1. Energy and Momentum of Light Packets

In the derivation, I used the relation between the energy  $E_\gamma$  and momentum  $p_\gamma$  for (monodirectional) electromagnetic fields,

$$p_\gamma = \frac{E_\gamma}{c}. \tag{6}$$

Of course, this can be derived from Special Relativity, but the orientation here is to derive equation (4) with no recourse to Relativity, nor to concepts of a similar vintage, such as photons.

Jackson (1975) recapitulates Poynting (1884)’s manipulations of Maxwell’s equations to derive the electromagnetic energy flux density  $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{H}$ , where  $\mathbf{E}$  and  $\mathbf{H}$  are the electric and magnetic fields. He then develops a similar manipulation of Maxwell’s equations (together with the Lorentz force law) to derive the momentum density  $\mathbf{g} = \mathbf{E} \times \mathbf{B}/4\pi c$ , where  $\mathbf{B}$  is the magnetic induction. Combining these two equations for monodirectional electromagnetic waves in free space yields equation (6). This shows that this relation rests directly on the Maxwell/Lorentz equations, although whether anyone actually derived the expression for  $\mathbf{g}$  prior to the simplification of vector notation is not clear.

However, Boltzmann (1884) already uses  $P = u/3$  for isotropic electromagnetic radiation in his thermodynamic derivation of Stefan’s law. Here  $P$  is the pressure and  $u$  is the energy density. This expression already implies  $p = E/c$  for monodirectional electromagnetic waves.

## 3. Generalization to $E = mc^2$

As emphasized in § 2, by carrying out the derivation only to first order in  $(v/c)$ , I ultimately restricted its validity to bodies at rest. Put differently, if the true relation between mass and energy had the form,  $E = mc^2(1 + \kappa(v/c)^2 + \dots)$ , the derivation would have proceeded exactly the same way. There are two paths to generalizing the result to moving bodies.

The first is to adopt the results of Special Relativity. This is the approach of Brown (2003), who derived  $E = mc^2$  using momentum conservation when light is emitted in an arbitrary direction. In Special Relativity, equation (2) is exact, so the derived relation between mass and energy is exact to all orders in  $(v/c)$ . This approach is pedagogically useful: like Einstein’s derivation, it makes use of Special Relativity, but it is simpler and more direct.

However, as a historical and logical exercise, one may also ask how equation (5) could have been generalized if it had been discovered prior to Special Relativity. Such a generalization follows from a simple thought experiment. Imagine a box filled with warm gas, whose thermal energy ultimately resides in the kinetic energy of the atoms. At the time, this picture was controversial but at least some physicists (e.g., Boltzmann) held to it. Light is emitted from two holes in the box, similarly to the situation in § 2. The energy of the light packets is drawn from the kinetic energy of the atoms in the box, some of which now move more slowly. By equation (4), the box has lost not only energy, but also mass. However, since the box contains no inter-atom potential energy, the mass (i.e., inertia) of the box must be the sum of the mass (inertia) of the atoms in it. As the number of these has not changed, the mass of some of the atoms must have been reduced by exactly the amount of reduced mass of the box, which is exactly the same as the kinetic energy lost from these atoms divided by  $c^2$ . That is, kinetic energy also contributes to inertia.

### 3.1. Derivation of Relativity From $E = mc^2$

Up to this point, I have derived  $E = mc^2$  without ever making use of Einstein (1905a)’s postulate that  $c$  is the same in all frames of reference, nor of any of the results that he derived from this postulate. I now show that Special Relativity, including the universality of  $c$ , can be derived from this equation.

First, Feynman (1963) shows that  $E = mc^2$  leads to the growth of inertia with velocity,  $m = m_0\gamma$ . To permit clarification of a subtle point, I repeat that derivation here, beginning with the Newtonian equation relating force to the increase of kinetic energy,  $F = dE/dx$ . Using the definitions,  $F = dp/dt$ ,  $p = mv$ ,  $v = dx/dt$ , this can be written  $dE/dx = d(mv)/dt$ , or

$$\frac{dE}{dt} = v^2 \frac{dm}{dt} + mv \frac{dv}{dt}. \quad (7)$$

Substituting in the just derived  $E = mc^2$  yields

$$\frac{d(mc^2)}{dt} - \frac{dm}{dt}v^2 = \frac{m}{2} \frac{dv^2}{dt}. \quad (8)$$

At this point, there may be some question as to whether the one may pull “ $c$ ” out of the derivative, since it has not yet been shown to be “constant”. But  $c$  is a *constant* in any one frame: the point that has not yet been addressed is whether it is *invariant* under frame changes. In the present case, the observer is not changing frames: it is the mass that is accelerating. The quantities  $E$ ,  $m$ ,  $v$ , and  $c$  are all as measured in the observer frame, which is inertial. We then obtain,

$$\frac{dm}{m} = \frac{d(v^2)}{2(c^2 - v^2)}, \quad (9)$$

whose solution is

$$m = \frac{m_0}{\sqrt{1 - (v/c)^2}}, \quad (10)$$

where  $m_0$  is an integration constant, which we identify with the rest mass.

From this point, it is straightforward to derive the other relations of Special Relativity by well-known arguments. For example, as a fast train passes by, a passenger and a bystander each throw tennis balls transverse to the motion of the train (with equal strength) in such a way that they hit and each bounces back directly to its respective thrower. The balls must each return at the speed they were launched or the train passenger could detect her own motion. Thus, they must have equal and opposite momenta. The bystander reckons that the passenger’s ball is more massive and therefore concludes it’s transverse velocity is smaller, which can only be true if time passes more closely. By similar traditional arguments, one can go on to derive length contraction, etc. In this way, one can prove that the speed of light is the same in all frames of reference rather than assuming it.

#### 4. Discussion

While the result obtained here is obviously not new, there are three reasons for establishing this result using a new derivation. First, the expression  $E = mc^2$  is zeroth order in  $v/c$ , in sharp contrast to the majority of results from Special Relativity, which are second order. It seems more elegant, therefore, to derive this expression using first-order arguments, rather than relying on second-order expressions. Second, because the derivation is more elegant, it has pedagogical value, i.e., it is easier to transmit to students. Third, because the derivation is independent of Special Relativity, it raises the question of why  $E = mc^2$  was not derived earlier than 1905. In particular, the elements needed to derive it (momentum conservation, aberration of starlight, and the proportionality between electromagnetic energy and momentum) were all in place by 1884. Indeed, once one realizes that electromagnetic waves have momentum (even if one does not yet know the exact expression for this quantity), it follows

immediately from momentum conservation and aberration of starlight that a light-emitting object must lose mass.

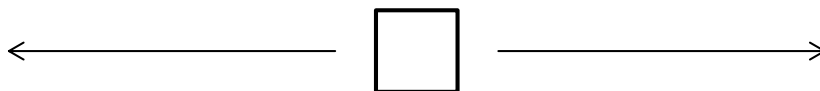
As reviewed by Pais (1982), during the 25 years before Special Relativity there were many efforts to express the mass of particles in terms of their energy divided by  $c^2$ . But these differed from the arguments given here (and that I have argued could have been given at least as early as 1884) by two important features. First, they generally centered around evaluations of the ultimately rather nebulous electromagnetic self-energy of charged particles rather than the kinetic properties of all matter (charged or neutral). Second, these evaluations did not recognize (at least explicitly) that when an object emitted energy, it also lost mass. Indeed, the very complexity of the arguments developed in this era compared to the absolute simplicity of the derivation in § 2 makes it even more puzzling why no one hit on the latter.

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## REST FRAME



## MOVING FRAME

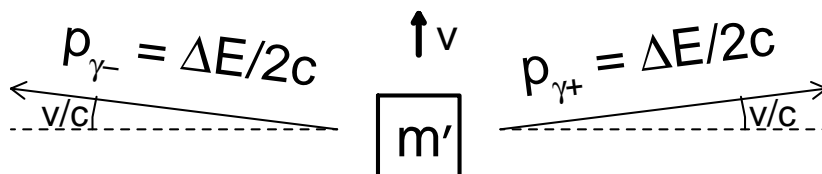


Fig. 1.— Thought experiment proving  $E = mc^2$  from momentum conservation and without appealing to Special Relativity. An object, originally of mass  $m$ , is pictured just after emitting two electromagnetic wave packets, each of energy  $\Delta E/2$ . Top panel: by momentum conservation, the object remains at rest. Bottom panel: experiment viewed from frame moving downward at velocity  $v$ , in which the packets have energy  $\Delta E/2$  and so momenta  $p_{\gamma,\pm} = \Delta E/2c$ . The object appears to be moving upward at  $v$ , both before and after emission. By aberration of starlight, the wave packets now travel slightly upward at an angle  $\theta = v/c$ , and so have vertical components to their momenta  $(p_{\gamma,\pm})_z = \Delta E v/2c^2$ . From momentum conservation in the  $z$  direction,  $mv = m'v + (p_{\gamma,+})_z + (p_{\gamma,-})_z = (m' + \Delta E/c^2)v$ . That is, the object has lost a mass  $\Delta m = m - m' = \Delta E/c^2$ .