A comparison of one and two dimensional models of transonic accretion discs around collapsed objects

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ABSTRACT: We construct models of the inner part of a transonic adiabatic accretion disc assuming constant specific angular momentum taking the vertical structure fully into account.

For comparison purposes, we construct the corresponding one dimensional viscous disc models derived under vertical averaging assumptions. The conditions under which a unique location for the critical/sonic point is obtained, given an appropriate set of exterior boundary conditions for these models, is also discussed. This is not unique if the standard 'α' prescription with viscous stress proportional to the angular velocity gradient is used.

We use a simple model to discuss the possible limitations on the form of the viscous stress arising from the requirement that viscous information must travel at a finite speed. Contrary to results in the existing literature, the viscous stress tends to be *increased* rather than reduced for the type of flows we consider in which the angular momentum and angular velocity gradients have opposite signs. However, finite propagation effects may result in a unique location for the sonic point.

We found good agreement between the radial flow and specific angular momentum profiles in the inner regions of the one dimensional models and those in the equatorial plane for

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corresponding two dimensional models which may be matched for a range of α between 0.1 and 10^{-4}.

1   Introduction

It has long been realized that the inner regions of accretion disc flows surrounding relatavistic objects have different properties from those pertaining to the case when the central object can be modelled as a Newtonian point mass.

In order to discuss such cases the approximation is often made that the flow may be treated as Newtonian but with the modification that the gravitational potential may be written in standard notation following Paczyński and Wiita (1980) as

$$\Psi = -\frac{GM}{\sqrt{r^2 + z^2} - r_G},$$

where r_G is the gravitational radius. In this potential the angular momentum of circular orbits has a maximum at $3r_G$ and so from a particle point of view orbiting material would be expected to become unstable and flow towards the centre attaining sonic velocities fairly rapidly. Hence a transonic flow is expected in the central regions (Abramowicz and Zurek, 1981).

Models of thick accretion discs with steady transonic flows were constructed by Paczyński (1980) who assumed that matter flowed over the surface of an essentially hydrostatic disc, much as a star in a close binary system overflows its Roche lobe. Later work assuming the accretion occurred entirely along the equatorial plane was carried out by Paczyński and Abramowicz (1982) and Różyczka and Muchotrzeb (1982). In this some assumptions are usually made so that the problem is effectively reduced to a one dimensionlal one.

More recently interest has focussed on so called 'slim discs' (Abramowicz, Czerny, Lasota and Szuszkiewicz, 1988). These studies reduce the problem to a one dimensional one, through a form of vertical averaging even though the discs are often geometrically thick. Radial advection is included and a form of the Shakura and Sunyaev (1973) prescription for the viscosity is often used. Modelling of accretion discs on this basis has been carried out more recently by Kato, Honma and Matsumoto (1988) and Chen and Taam (1993). It is one of the purposes of

this paper to construct two dimensional axisymmetric models of the inner region which take the vertical structure fully into account in order to compare the properties with those of the one dimensional models. We make the simplifying assumptions that the innermost regions have constant specific angular momentum and entropy. Apart from reasons of simplicity, the justification for this is that when the flow velocity approaches the sound speed, the inflow or advection timescale becomes significantly shorter than the viscous or thermal timescales for the region of interest (see Paczyński and Abramowicz, 1982) so justifying the assumptions. We remark that the inner regions of the one dimensional models with viscosity included are found to have a very slowly varying specific angular momentum profile.

Another issue raised recently by Narayan (1992) is the fact that one must contrain ones treatment of viscosity so that viscous information does not travel with supersonic velocities (see also Chen and Taam, 1993). Narayan (1992) suggests that the viscous stress should vanish when the flow velocity attains the sound speed. This may have important implications for models of transonic accretion discs esspecially when sonic velocities are attained outside $3r_G$.

In this paper we develop a simple model of the viscous process which has the propagation limitation built in. We find that the viscous stress does not vanish at a sonic point but actually tends to be increased if the angular velocity and angular momentum gradients are of opposite sign as occurs in the flows discussed here. However, limitations of finite propagation speed may affect issues associated with the uniqueness of the flow subject to specification of appropriate boundary conditions.

In section 2 we consider steady state accretion tori with constant specific angular momentum and entropy. We derive an equation for the velocity potential in the special case of zero vorticity. In section 3 we discuss the numerical solution of this equation. For the purpose of comparison of these models with one dimensional models, we consider such models, which include viscosity treated according to the usual Shakura and Sunyaev (1973) α prescription, derived from vertically averaged equations in section 4. We argue that specification of the mass accretion rate and constant entropy of the flow coupled with a specification that the flow have Keplerian rotation at some outer boundary radius does not yield a unique location for the critical point.

Further, we discuss, using a simple model, the possible effects of the limitation that viscous information should not be transmitted at a speed exceeding the sound velocity. We find that if this speed coincides with the sound speed, the lack of uniqueness in determination of the critical point discussed above may be removed.

In section 5 we describe our results for one and two dimensional steady state accretion flows. When the intersection point of the sonic surface with the equatorial plane in a two dimensional model coincides with the critical point in a one dimensional model and the sound velocities match there good agreement is found for the flow parameters.

Finally in section 6 we discuss our results.

2 Tori with constant specific angular momentum

In axisymmetric tori with constant specific angular momentum the velocity $\mathbf{u} = (u_r, u_\varphi, u_z)$ in cylindrical coordinates (r, ϕ, z) is such that $l = r u_\phi$ is constant. We assume the polytropic equation of state,

$$P = \mathcal{K} \rho^\gamma. \tag{1}$$

Here P is the pressure, ρ is the density, $\mathcal{K}$ and γ are the polytropic constant and adiabatic index, respectively. The sound speed, c_s, is then a simple function of the density and can be written in the form

$$c_s^2 = \frac{dP}{d\rho} = \mathcal{K} \gamma \rho^{\gamma-1}.$$

The equation of motion governing the steady state may be written

$$\boldsymbol{\omega} \times \mathbf{u} = -\nabla \mathcal{B}, \tag{2}$$

where $\boldsymbol{\omega}$ is the vorticity and the Bernoulli function

$$\mathcal{B} = \frac{1}{2}\mathbf{u}^2 + \frac{c_s^2}{\gamma - 1} + \Psi,$$

with Ψ being the gravitational potential. For this we adopt the 'pseudo Newtonian potential' (Paczyński and Wiita, 1980) given by

$$\Psi = -\frac{GM}{\sqrt{r^2 + z^2} - r_G},$$

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where G is the gravitational constant, M is a mass of the central object and r_G is the gravitational radius. Paczyński and Wiita (1980) emphasized that all relatavistic effects which are relevant to the accretion process near the inner edge of a disk can be reproduced in the Newtonian formalism assuming the pseudo-Newtonian potential given above.

From equation (2) it follows that $\mathcal{B}$ is constant on stream lines and may be specified to vary arbitrarily from stream line to stream line. The fact that it is possible to specify an arbitrary function in this way, being associated with a non zero φ component of vorticity, results in a degree of arbitrariness in possible steady state solutions.

2.1   The zero vorticity case

In this paper we shall consider the case when $\mathcal{B}$ is constant. This corresponds to zero vorticity. This is the simplest assumption that can be made about $\mathcal{B}$ and it is also the one which is most likely to result in models that can be approximately described by one dimensional vertically averaged ones.

It must be noted that the ability to specify the Bernoulli function arbitrarily on stream lines introduces a high degree of arbitrariness into the two dimensional models. In principle one might for example be able to construct models with flow concentrated towards the disc surface (Paczyński, 1980) or towards the equatorial plane (Paczyński and Abramowicz, 1982) but such solutions would have a non zero component of vorticity in the φ direction which can only be specified as an entry condition for the flow under the assumption of the lack of importance of viscosity we make in the treatment of the inner region.

When the vorticity is zero, the meridional component of the velocity field may be expressed as the gradient of a potential, Φ, such that

$$(u_r, u_z) = \left(\partial\Phi/\partial r, \partial\Phi/\partial z\right).$$

For a steady flow, we also have the continuity equation, which for this flow reads

$$\nabla(\rho\nabla\Phi) = 0. \tag{3}$$

This may be combined with the Bernoulli equation,

$$\frac{1}{2}(\nabla\Phi)^2 + \frac{c_s^2}{\gamma - 1} + \Psi + \frac{l^2}{2r^2} = \mathcal{B} = \text{constant}$$

to give a single second order partial differential equation for the velocity potential Φ :

$$\nabla^2\Phi - \frac{1}{c_s^2}\left(\nabla(\Psi_T) + \nabla\left[\frac{1}{2}(\nabla\Phi)^2\right]\right)\cdot\nabla\Phi = 0, \tag{4}$$

where the total centrifugal plus gravitational potential is

$$\Psi_T = \Psi + \frac{l^2}{2r^2}.$$

In addition the density of the flow can be found from the Bernoulli equation which reads

$$\rho = \left[\frac{\gamma - 1}{\mathcal{K}\gamma}(\mathcal{B} - \Psi_T - \frac{1}{2}(\nabla\Phi)^2)\right]^{\frac{1}{\gamma - 1}}. \tag{5}$$

In order to discuss possible solutions of equation(4), we first consider properties of the potential Ψ_T for constant specific angular momentum l.

The equipotential surfaces have a saddle point or cusp if l is in the range given by $l_{mb} < l < l_{ms}$, where l_{mb} and l_{ms} are the values of the specific angular momentum at the marginally bound and marginally stable circular particle orbits respectively (Abramowicz et al. 1978). In order for accretion to take place it is expected that l be in the above range. In order to construct a steady state solution we first define a specific angular momentum in the above range and use an equipotential surface as a reference surface in order to construct a sonic surface.

2.2   The sonic surface

The accretion flow is expected to be transonic, starting at small subsonic speed at large radii and accelerating to attain the sonic speed along the sonic surface, being the generalization of a critical point in one dimension. In general a unique form for the sonic surface cannot be constructed in advance of knowing the solution, even if the other bounding surfaces are known (see Anderson, 1989). However, construction turns out to be simple in the special case when the flow is normal to the surface, (see Anderson, 1989 for reasons why this might be a preferred case). A surface of this form may be constructed by first selecting a bounding equipotential surface through which the sonic surface passes orthogonally. We suppose this to be a zero density surface where the flow approaches zero velocity, the sound speed being

zero. If the equation of the sonic surface with the above properties is $r = r(z)$, on the surface we must have

$$\frac{dr}{dz} = -\frac{\partial \Phi}{\partial z} \Big/ \frac{\partial \Phi}{\partial r}.$$

Differentiating this equation with respect to z and using equations (4) and (5), together with the fact that on the surface $u_r^2 + u_z^2 = c_s^2$, we find the following second order differential equation for the sonic surface:

$$\frac{\frac{d^2 r}{dz^2}}{1 + \left(\frac{dr}{dz}\right)^2} = -\frac{\left(\frac{\partial \Psi_T}{\partial r} - \frac{\partial \Psi_T}{\partial z}\frac{dr}{dz}\right)}{2\frac{\gamma-1}{\gamma+1}(\mathcal{B} - \Psi_T)} + \frac{1}{r}.$$

Here $\mathcal{B}$ is constant and equal to the value of Ψ_T on the bounding equipotential surface. The sonic surface that passes orthogonally through the chosen bounding zero density surface as well as the equatorial plane can be constructed uniquely by solving this equation. The zero density surface of the torus, sonic surface, the equator $(z = 0)$ and a bounding surface $r = $ constant on which the inflow velocity is small are taken to define the flow domain. We assume reflection symmetry in z for the flow below the equator. At the sonic surface we set the outflow velocity equal to the sound speed and normal to the surface.

In this way the parameters defining a solution are the selected constant specific angular momentum and the bounding zero density surface . Alternatively one may use the point on the equatorial plane through which the sonic surface passes (identified with the critical point in one dimensional flows) $r = r_*$, and the value of $GM/(r_* c_s^2)$ there as the parameters defining a model. The solutions can be scaled to give any mass accretion rate because they are invariant to an arbitrary scaling of the density.

3   Numerical method for the two dimensional tori

In order to solve the equilibrium equation for Φ and at the same time find the velocity of the fluid, we rewrite equation (4) in the form of a diffusion type equation:

$$\frac{\partial \Phi}{\partial t} = D(r, z, t) \left[\nabla \left(\rho \nabla \Phi\right)\right],$$

where $D(r,z)$ is a diffusion coefficient, which in general can be a function of r, z and t. The form of this can be chosen for numerical convenience subject to the requirement of numerical stability. We solved this equation in the elliptic (subsonic) regime of the computational grid by solving the diffusion equation as an initial value problem, integrating forward until a steady state was achieved. A two stage procedure was used. During the first stage, the equations were advanced with ρ fixed at the value it would have were the flow sonic. Once a steady state had been reached, the evolution was continued for the second stage during which the density was calculated from equation(5) until a final steady state was attained. It is required to specify boundary conditions on the boundary of the computational domain. These were that the velocity is sonic and normal to the sonic surface, that the inflow be normal at the exterior boundary, that the flow be tangential at the bounding equipotential surface and symmetry with respect to reflection in the equatorial plane. The solution is undertaken on an equally spaced computational grid with between 40 and 50 grid points in the subsonic region of the flow on the equatorial plane. It is straightforward apart from the complication that because the density and therefore the sound speed go to zero somewhat interior to the bounding surface that would occur if there was hydrostatic equilibrium, there must be a supersonic transition slightly interior to this surface. But note that this transition occurs at low absolute speeds. The region between this other sonic surface and the bounding surface is small and we treated it by not allowing the density to fall below the value taken on when the flow is sonic. In this way a transition to a hyperbolic region is prevented. However, the location of this new bounding sonic surface could be found in this way and the solution continued into the hyperbolic domain by initial value techniques. As this region was small, having a negligible effect on the interior, and in reality it would depend on details of the proper boundary condition, we did not calculate it in detail.

Before discussing the solutions for two dimensional tori that were obtained numerically, we describe the one dimensional models which correspond to them. In this regard we note that the full disk can only be approximated as a constant angular momentum torus in the innermost region. Further out there must be material with a higher specific angular momentum and viscous effects to allow it to accrete. With reference to one dimensional disc models we shall argue below that once the two parameters r_* and $GM/(r_* c_s^2)$ are specified and if

the angular velocity gradient at the critical point corresponds to constant specific angular momentum, there is a unique standard " α" viscosity parameter corresponding to the torus.

4 Basic equations for one dimensional flow

The basic equations for one dimensional flow with the time dependence retained are the equation of motion

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} = -\frac{1}{\Sigma} \frac{\partial P}{\partial r} + r\Omega^2 - \frac{\partial \Psi}{\partial r}. \tag{6}$$

Here u is the radial velocity, Σ is the surface density and P is now the vertically integrated pressure which we shall assume to be a function of both Σ and r. The latter dependence on r may come about through the process of vertical averaging (see below). A local sound speed c_s is then defined through

$$c_s^2 = \left(\frac{\partial P}{\partial \Sigma} \right)_r.$$

The continuity equation is

$$\frac{\partial \Sigma}{\partial t} + u \frac{\partial \Sigma}{\partial r} = -\frac{\Sigma}{r} \frac{\partial(ur)}{\partial r}. \tag{7}$$

The specific angular momentum $l(r,t) = r^2 \Omega(r,t)$ satisfies

$$\frac{\partial l}{\partial t} + u \frac{\partial l}{\partial r} = \frac{1}{(r\Sigma)} \frac{\partial(r^2 \tau_{r\varphi})}{\partial r}. \tag{8}$$

Here $\tau_{r\varphi}$ is the $r\varphi$ component of the viscous stress tensor. This is usually taken to be given by $\tau_{r\varphi} = \tau_{r\varphi 0} = \nu \Sigma r (\partial \Omega/\partial r)$, with ν being the kinematic viscosity. For the Shakura and Sunyaev (1973) α prescription, we have $\nu = \alpha c_s^2/\Omega_K$, Ω_K being the circular orbit frequency.

4.1 The question of causality

In an important paper, Narayan (1992) has pointed out that the conventional formulation of viscosity as a diffusion process allows propagation of information at infinite speed and so could lead to unphysical results in transonic flows which attain velocities exceeding the maximum velocity of the information carriers which produce the viscosity. Unphysical results might arise through unrealizable diffusive communication between different parts of the fluid.

Using an ilustrative model of the diffusion process with a finite propagation speed, Narayan (1992) concludes that the $r\varphi$ stress should vanish at a sonic/critical point, where $u^2 = c_s^2$, in a one dimensional accretion flow. Here we provide a different but similar example, within a Newtonian framework, which uses all the basic equations. The full set of equations is of hyperbolic type and so information propagates at finite speeds. The example shows that for flows in which the angular velocity and specific angular momentum gradients are of opposite sign, the viscous stress is, if anything increased, rather than reduced at the critical point. Furthermore, when the specific angular momentum gradient is very small when the radial velocities are significant, such as might be expected for the flows considered here, the corrections due to finite propagation effects become small so that the conventional Shakura and Sunyaev (1973) treatment should be reasonable. However, constraints arising from causality considerations may be important when the question of the uniqueness of steady state solutions is discussed. When the angular velocity gradient is very large, the conclusions derived from this simplified model are similar to those of Narayan (1992).

The basic equations of our simplified model are equations (6), (7) and (8) together with an equation for $\tau_{r\varphi}$ of the form

$$\frac{\partial \tau_{r\varphi}}{\partial t} + u\frac{\partial \tau_{r\varphi}}{\partial r} = \frac{(\tau_{r\varphi 0} - \tau_{r\varphi})}{\tau}. \tag{9}$$

This states that the stress relaxes locally to its equilibrium value on a relaxation timescale τ which may be an arbitrary function of the state variables but not their gradients. We stress that this formalism has been introduced to ensure that information propagates at a finite speed in the system rather than to provide a realistic formulation of viscosity. It is thus adequate to demonstrate that zero viscous stress at the sonic point cannot be inferred from any limitation in the propagation speed of information. Equations (6), (7), (8) and (9) form a system of four simultaneous first order partial differential equations which may be written in the form:

$$\frac{\partial U_i}{\partial t} + \sum_{j=1}^{j=4} A_{ij}\frac{\partial U_j}{\partial r} + S_i = 0. \ (i = 1, 2, 3, 4) \tag{10}$$

Here $U^T = (\Sigma, u, l, \tau_{r\varphi})$, $S_1 = \Sigma u/r$, $S_2 = \frac{1}{\Sigma}\left(\frac{\partial P}{\partial r}\right)_\Sigma - r\Omega^2 + \frac{\partial \Psi}{\partial r}$, $S_3 = -2\tau_{r\varphi}/\Sigma$, $S_4 = (\tau_{r\varphi} + 2\nu\Sigma l r^{-2})\tau^{-1}$. The diagonal elements of the matrix (A) are all equal to u, $A_{12} = $

Σ, $A_{21} = c_s^2/\Sigma$, $A_{34} = -r/\Sigma$, $A_{43} = -\nu\Sigma/(\tau r)$, with all other elements being zero. The system is hyperbolic if, as for our set of equations, the eigenvalues of A are all real. The characteristics are rays with propagation speeds given by these eigenvalues. In our case these are the two sonic speeds $u + c_s, u - c_s$ and the two viscous speeds $u + c_v$, $u - c_v$, with $c_v = \sqrt{\nu/\tau}$, being the speed of propagation of viscous information. Therefore information cannot propagate faster than a finite speed in the model system. But note that the viscous speed, which we assume to be comparable to the sound speed, becomes infinite when the relaxation time is zero. When $c_v = c_s$, we have $\alpha = \Omega_K \tau$.

4.2 The steady state

We now examine the steady state in which $\frac{\partial}{\partial t} \equiv 0$. The continuity equation gives on integration

$$2\pi \Sigma r u = -c_1, \tag{11}$$

c_1 being the constant accretion rate. Equation (8) gives on integration

$$c_1 l(r) + 2\pi r^2 \tau_{r\varphi} = c_1 c_2, \tag{12}$$

where c_2 is the constant rate of flow of angular momentum through a circle of radius r. Equation (9) gives

$$\tau_{r\varphi} = \nu \Sigma r (d\Omega/dr) - \tau u \frac{d\tau_{r\varphi}}{dr}. \tag{13}$$

From the above two equations it follows, after differentiating the first and using the result to eliminate $\frac{d\tau_{r\varphi}}{dr}$, that:

$$\tau_{r\varphi} = \frac{\nu \Sigma r \left((d\Omega/dr) - (u^2/(c_v^2 r^2))(dl/dr)\right)}{(1 - (2u\tau/r))}. \tag{14}$$

This is the Shakura and Sunyaev (1973) viscosity prescription modified by additional terms involving the radial velocity. We first remark that for $u \sim -c_s$, and $\tau \leq \Omega^{-1}$, $|u\tau/r| \leq H/r$, H being the disk thickness. Thus for most purposes the denominator in equation (14) may be approximated as unity (it could also be absorbed by a redefinition of α incorporating radial dependence).

We note that corrections arising from propagation effects are small if $|dl/dr|$ is small such the flow has almost constant specufic angular momentum even when $u^2 = c_s^2$. Furthermore

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the viscous stress is increased if the gradients of angular velocity and angular momentum are of opposite sign. But note that an interesting situation arises at a sonic point in the physically reasonable case when $c_v = c_s$. Equation (14) may be written

$$\tau_{r\varphi} = \frac{\nu \Sigma r \left((1 - (u^2/(c_s^2)))(d\Omega/dr) - (2\Omega u^2)/(rc_s^2) \right)}{(1 - (2u\tau/r))}. \tag{15}$$

From this it follows that at a sonic point:

$$\tau_{r\varphi} = -\frac{2\nu \Sigma \Omega}{\left(1 + \frac{2c_s\tau}{r}\right)}. \tag{16}$$

We see that the stress is independent of the angular velocity gradient and takes on the same form as it would do in the case of constant specific angular momentum. Equation(16) shows that the angular momentum flux is then fixed at the sonic point by the magnitude of Ω.

This is in contrast to the situation when the viscosity prescription with $u = 0$ is used. Then the angular momentum flux may be adjusted by altering the angular velocity gradient at the sonic point, albeit by a small amount if α is small. Within this prescription, when $d\Omega/dr = -2\Omega/r$, the angular momentum flux is also fixed by Ω at the sonic point and the standard α prescription is essentially unmodified by finite propagation effects. We shall refer to this condition at the sonic point as the natural boundary condition.

In fact equation (15) implies the existence of a critical point at the sonic point and that equation (16) should be satisfied there to avoid a singularity and that indeed freedom to specify the angular velocity gradient should be lost. However, the technicalities of additional critical point analysis (scarcely justified by the sophistication of the model) can be avoided by noting that for our solutions, when the natural boundary condition is used, equation (16) is essentially satisfied in any case and the angular momentum profiles are so flat that limitations due to finite propagation effects may be ignored. Note too that this is reasonable on physical grounds because viscous effects are small in any case when the angular momentum profile is flat.

For the simple model discussed here, we have seen that when the viscous propagation speed is equal to the sound speed, freedom to adjust the angular momentum flux by varying the angular velocity gradient at the critical point is lost. This flux is always the same as when the natural boundary condition is used. We shall see that this is important for the issue of uniqueness of steady state solutions.

When the mass flow rate, α and the enropy of an adiabatic steady state solution are specified and the natural boundary condition is used, the solution is apparently unique. However, if there is freedom to adjust the angular momentum flux by varying the angular velocity gradient at the critical point, as might arise if $c_v > c_s$, the solution is no longer unique (see below).

Finally, we comment that when the angular velocity gradient is *very* steep, we have approximately

$$\tau_{r\varphi} = \frac{\nu\Sigma r \left(1 - \frac{u^2}{c_s^2}\right)(d\Omega/dr)}{\left(1 - \frac{2u\tau}{r}\right)}. \tag{17}$$

This is of a similar form to that given by Narayan (1992).

In the numerical work presented below, we adopt the standard Shakura and Sunyaev (1973) form: $\tau_{r\varphi} = \tau_{r\varphi 0}$, noting that we expect modifications due to finite propagation effects to be negligible, when the natural boundary condition is used.

For the steady state, the equation of motion reads:

$$u\frac{du}{dr} = -\frac{1}{\Sigma}\frac{dP}{dr} - \frac{d\Psi}{dr} + \frac{l^2}{r^3}, \tag{18}$$

The vertically integrated pressure may be taken to obey a polytropic equation of state $P = K\Sigma^{1+1/n}$, n being the polytropic index. The polytropic factor K may be taken to be constant, or a function of r to reflect vertical averaging. This dependence can be found from the scalings for an adiabatic equation of state: $P \propto H\rho^\gamma$, $\Sigma \propto \rho H$, and $\Omega_K^2 H^2 \propto c_s^2 \propto \rho^{\gamma-1}$, with $\Omega_K = \sqrt{GM/(r(r-r_G)^2)}$ being the Keplerian angular velocity for circular orbits. Eliminating H and ρ from these gives

$$K = K_0 \Omega_K^{\frac{2(\gamma-1)}{(\gamma+1)}},$$

K_0 being constant and $1 + 1/n = (3\gamma - 1)/(\gamma + 1)$. Combining the equation of motion with the continuity equation and the above equation of state gives

$$\left(u - \frac{c_s^2}{u}\right)\frac{du}{dr} = -\frac{GM}{(r - r_G)^2} + \frac{c_s^2}{r}\left(1 - \frac{nr}{(n+1)K}\frac{dK}{dr}\right) + \Omega^2 r. \tag{19}$$

This exhibits a critical point when the flow speed reaches the sound speed. At such a point the right hand side of equation (19) must vanish. The nature of the critical point when K is constant has been discussed by Papaloizou and Szuszkiewicz (1993) who find that for physically acceptable solutions it is of saddle type.

4.3 Dimensionless form of the equations

In order to discuss solutions of the steady state equations further, we adopt the usual "α" viscosity:

$$\tau_{r\varphi} = \tau_{r\varphi 0} = \nu \Sigma r (d\Omega/dr),$$

with $\nu = \alpha c_s^2/\Omega_K$. We use this together with equations (11), (12) and (19).

We find it convenient to adopt as units of radius and velocity r_* and u_* being the radius at the critical point and the absolute magnitude of the velocity velocity there, respectively. Then we define the dimensionless variables $v = -u/u_*$, $x = r/r_*$, $\omega = \Omega r_*/u_*$, $\omega_K = \Omega_K r_*/u_*$ and $x_G = r_G/r_*$. Note that because we are dealing with inflow v is positive. The square of the dimensionless sound speed is $\tilde{c}_s^2 = \tilde{K}/(vx)^{1/n}$, with $\tilde{K} = K(x)/K(1)$. In addition we define $\tilde{c}_2 = c_2/(u_* r_*)$, and $m = GM/(r_* u_*^2)$ as the Bondi parameter.

It is possible to derive a single nonlinear second order differential equation for ω from the above equations which may be written

$$\mathcal{F} = \frac{1}{n_1}\left(v^{1-1/n} - \frac{\tilde{K}}{x^{1/n}v^{1+2/n}}\right)\frac{dv^{n_1}}{dx} + \frac{m}{(x-x_G)^2} - \frac{\tilde{K}}{v^{1/n}x^{n_1}}\left(1 - \frac{x}{n_1\tilde{K}}\frac{d\tilde{K}}{dx}\right) - \omega^2 x = 0, \quad (20)$$

where, $n_1 = 1 + 1/n$ and

$$v^{n_1} = \frac{\alpha\tilde{K}}{\omega_K x^{1/n-2}(\tilde{c}_2 - \omega x^2)}\frac{d\omega}{dx}. \quad (21)$$

At the critical point where $x = v = \tilde{K} = 1$, equation (20) requires that the dimensionless angular velocity ω be given by

$$\omega_* = \sqrt{\frac{m}{(1-x_G)^2} - 1 + \frac{n}{(n+1)}\left(\frac{d\tilde{K}}{dx}\right)_{x=1}}.$$

Equation (21) then gives

$$\tilde{c}_2 = \left(\frac{\alpha}{\omega_K}\frac{d\omega}{dx}\right)_{x=1} + \omega_*. \quad (22)$$

We have also found it convenient to work in terms of a quantity c_N related to $\tilde{c}_2$ through

$$\tilde{c}_2 = -\frac{2c_N\alpha\omega_*}{\omega_K(1)} + \omega_*. \quad (23)$$

Thus when the natural boundary condition is used $\tilde{c}_N = 1$.

4.4 Parameters specifying a solution

Equation (20) after use of (21) constitutes a second order differential equation for ω. After specification of the Bondi parameter m, x_G, α and $\tilde{c}_2$, both ω and its first derivative are specified at the critical point. For the critical points considered here these conditions specify a unique solution that is subsonic for $x > 1$ (Papaloizou and Szuszkiewicz, 1993). In order to be physically acceptable this solution must match onto a Keplerian or near Keplerian disk at some specified large radius. In general this requires that one of the parameters such as α be adjusted as an eigenvalue in order to fulfill this requirement. In this way we see that specification of m, x_G and $\tilde{c}_2$ determines a value of α required to satisfy the boundary conditions.

However the parameter x_G which can be regarded as specifying the location of the critical point may also be varied, other parameters being kept fixed, so determining this quantity as a function of α.

On physical grounds we expect that for adiabatic solutions of the type we are considering, we should be able to specify the accretion rate and the entropy with some degree of arbitrariness. The solutions can always be adjusted to give any accretion rate by scaling the constant c_1, while the Bondi parameter m may be used to adjust the entropy. If these are determined in this way, the above discussion shows that when $\tilde{c}_2$ is fixed the location of the critical point is then determined as a function of α.

On dynamical grounds we expect the solutions considered here to be of nearly constant specific angular momentum when there are significant velocities and the discussion of the requirement of a finite propagation speed of viscous information given above then suggests that $\tilde{c}_2$ be chosen to satisfy the natural boundary condition such that

$$\left(\frac{d\omega}{dx}\right)_{x=1} = -2\omega_*.$$

Under these conditions $\tilde{c}_2$ is no longer free and we expect that for a given accretion rate, entropy and α, the solution together with the location of the critical point are specified uniquely *at least locally in parameter space*. However, if we allow freedom in the specification of $\tilde{c}_2$ this is no longer precisely so. For a given m and x_G there is a possible range of α

corresponding to the range in $\tilde{c}_2$ or equivalently for a given α there is a possible variation in x_G or the location of the critical point.

Variation of $\tilde{c}_2$ causes the angular velocity gradient to differ from that appropriate to constant specific angular momentum at the critical point. Because the dynamics requires that the angular momentum profile does not differ much from a constant, this situation is restored in a fairly narrow boundary layer when the boundary condition for $\tilde{c}_2$ differs from the natural one. In practice, for a fixed m, the variation in α allowed for a given x_G, or the variation in x_G found for a given α consequent on varying $\tilde{c}_2$ by a maximum reasonable amount, is found to be small.

Adopting the natural boundary condition, the above discussion indicates that specification of m and α results in a unique critical point location given through x_G.

Such a solution can then be matched to a two dimensional solution with constant specific angular momentum which has the same value of m and which has a sonic surface which passes through the equatorial plane at the same location as the critical point in the one dimensional case.

4.5 Numerical methods for the one dimensional case

Numerical solutions in the one dimensional case have been generated by two methods. In the first, the domain between the critical point and the outer boundary was divided into N equally spaced zones, with values of N up to 18000 having been used. Equations (20) and (21) were then written in a centered finite difference form. On specification of c_N, m, x_G and α, the solution can be found by stepping outwards from the critical point. However, as discussed above, the condition that the angular velocity match on to the Keplerian value at some outer boundary radius, in general cannot be satisfied unless α takes on a specific characteristic value. This method was most satisfactory when the critical point was inside $3r_G$ and m was not too large. When m is large, the disk thickness is very much less than the radius, and the equations are stiff. This has the effect that with the above method it is awkward to extend the solution to very large outer boundary radii. However, the characteristic value of α corresponding to given x_G and c_N is very accurately determined.

An alternative approach is to use a relaxation method (Papaloizou and Szuszkiewicz 1993) such as for example solving the diffusion type equation derived from equation(20)

$$\frac{\partial \omega}{\partial t} = \mathcal{D}\mathcal{F} \tag{24}$$

where $\mathcal{D}$ is a diffusion coefficient which can be a function of x and t and which can be chosen for numerical convenience.

This can be solved in a computational domain extending from the critical point to the outer boundary. In this case ω is specified at the domain end points. For fixed α, m and x_G this is enough to specify the solution but then $\tilde{c}_2$ is determined in the course of solution through equation (21). This has the disadvantage that in order for a solution to exist, once x_G and m have been specified, α must be in the range given through the maximum permitted variation of $\tilde{c}_2$ (see below). This method which does not have the stiffness difficulties of the first was found to work best when the critical point was outside $3r_G$. Some solutions obtained by the first method were checked using this method.

5   Numerical Results

We have constructed two dimensional model tori with different values of the constant specific angular momentum and vertical thickness. Each torus is characterized by a value of m and x_G given in Table 1. For each of these models the outer boundary was taken at $r = 4r_G$ which was far enough out for the inflow velocity to be small there, making it seem reasonable that the disk can be regarded as essentially static beyond.

The results we describe below are qualititatively similar for all of the models we examined so we shall focus on two examples for illustrative purposes. We show the velocity and density structure appropriate to models 3 and 12 in figures 1 and 2 respectively. Model 3 represents a moderately thick disc. The inner edge of this disc, where the radial inflow reaches the sound speed, is located at the radius $\approx 2.5r_G$. The maximum density occurs around $3.5r_G$. Model 12 is an example of a genuinely thick accretion disc. Its inner edge is close to $2.05r_G$ and the density maximum is outside $4r_G$.

For each of the two dimensional models we have constructed a one dimensional viscous equivalent with the same values of m and x_G. As argued above if c_N, which controls the angular velocity gradient at the critical point is fixed, there is a unique value of α for which the angular velocity takes on the Keplerian value at some outer radius, here taken to be at $10r_G$. We denote the value of α corresponding to $c_N = 1$, the natural boundary condition, by α_{max}. In addition we denote the value of α corresponding to $c_N = 0$ or zero angular velocity gradient at the critical point by α_{min}.

The range between α_{max} and α_{min} is spanned continuously as c_N is varied from 1 to 0. Values of c_N less than 0 correspond to the situation where the angular velocity gradient turns over. If variation in c_N is permitted this results in a range of possible x_G for fixed m and α, (see the above discussion of uniqueness). We remark that α_{max} and α_{min} do not normally differ by more than twenty percent. In addition, other parameters being fixed, these values are somewhat sensitive to the location of the outer boundary and tend to decrease as this moves outwards. The sensitivity is greater for smaller values of m. The angular momentum profile interior to $4r_G$ is very flat with $l \propto r^{\sim 0.04}$. This is fully consistent with the idea that viscous effects are weak in this region and justifies the approximation of constant specific angular momentum. The one dimensional models do not depend very much on whether vertical averaging is taken into account in the equation of state ($K = K(x)$) or a straight forward polytropic equation of state ($K = $ constant) is used.

Values of α_{max} and α_{min} for some values of x_G and m are given for the former case in table 2 and the latter case in table 3.

It is seen that a range of $7.6 \times 10^{-6} \leq \alpha \leq 1.7 \times 10^{-1}$ can be spanned by varying m and x_G. The specific angular momentum for one dimensional models (with vertical averaging taken into account in the equation of state) and two dimensional models are compared in figures 3 and 4. In figure 3 we show the distributions of specific angular momentum in the model 3 torus and in the corresponding one dimensional vertically integrated models with c_N equal to zero and one respectively. For $c_N = 0$ a very narrow, but clearly pronunced boundary layer exists. In this layer, the specific angular momentum rises steeply, eventually attaining an almost constant value. In figure 4 we plot the same curves but for model 12.

The radial velocities in the one dimensional models are not very sensitive to whether

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Table 1: The various two dimensional model calculations: m is the Bondi parameter measured where the sonic surface intersects the equatorial plane at $r = r_*$. The equipotential surfaces have a cusp at $r = r_G r_{eq}$, h is the ratio of the vertical distance between the equatorial plane and the zero density surface of the torus, to the radius and $x_G = r_G/r_*$. In addition we show α_{max} and α_{min}, the values of α appropriate to one dimensional models, with vertical averaging taken into account in the equation of state, with the same x_G and m. The former value corresponds to the natural boundary condition while the latter corresponds to zero angular velocity gradient at the critical point. For each one dimensional model, the exterior boundary was taken at $10 r_G$.

Model	m	x_G	r_{eq}	h	α_{max}	α_{min}
1	302	0.34	2.9	0.1	6.03×10^{-2}	2.98×10^{-2}
2	83	0.39	2.9	0.2	1.26×10^{-2}	1.07×10^{-2}
3	75	0.39	2.7	0.2	1.40×10^{-2}	1.17×10^{-2}
4	21	0.42	2.7	0.4	1.81×10^{-2}	1.50×10^{-2}
5	63	0.43	2.4	0.2	3.65×10^{-3}	3.48×10^{-3}
6	16	0.43	2.7	0.5	1.58×10^{-3}	1.34×10^{-2}
7	18	0.45	2.4	0.4	9.01×10^{-3}	8.18×10^{-3}
8	13	0.45	2.4	0.5	9.94×10^{-3}	8.97×10^{-3}
9	55	0.46	2.2	0.2	5.67×10^{-4}	5.63×10^{-4}
10	51	0.48	2.1	0.2	2.59×10^{-4}	2.58×10^{-4}
11	15	0.48	2.2	0.4	3.50×10^{-3}	3.38×10^{-3}
12	11	0.49	2.2	0.5	4.02×10^{-3}	3.86×10^{-3}

Table 2: Values of α_{max} and α_{min}, for a sample of one dimensional models with various x_G and m for the case when vertical averaging was taken into account in the equation of state. In each case $\gamma = 4/3$ and the external radius was taken at $10r_G$.

m	x_G	α_{max}	α_{min}
10	0.32	1.67×10^{-1}	6.24×10^{-2}
100	0.35	6.51×10^{-2}	3.39×10^{-2}
10	0.35	1.00×10^{-1}	5.08×10^{-2}
100	0.40	7.45×10^{-3}	6.71×10^{-3}
10	0.40	4.29×10^{-2}	3.00×10^{-2}
100	0.45	2.71×10^{-4}	2.71×10^{-4}
10	0.45	1.44×10^{-2}	1.25×10^{-2}
100	0.50	7.61×10^{-6}	7.61×10^{-6}
10	0.50	3.18×10^{-3}	3.08×10^{-3}

Table 3: Values of α_{max} and α_{min}, for a sample of one dimensional models with various x_G and m for the case when vertical averaging was not taken into account in the equation of state. In each case $\gamma = 4/3$ and the external radius was taken at $10r_G$.

m	x_G	α_{max}	α_{min}
10	0.32	1.23×10^{-1}	5.51×10^{-2}
100	0.35	5.96×10^{-2}	3.26×10^{-2}
10	0.35	7.88×10^{-2}	4.46×10^{-2}
100	0.40	8.01×10^{-3}	7.23×10^{-3}
10	0.40	3.55×10^{-2}	2.64×10^{-2}
100	0.45	4.71×10^{-4}	4.70×10^{-4}
10	0.45	1.31×10^{-2}	1.17×10^{-2}
100	0.50	2.41×10^{-5}	2.33×10^{-5}
10	0.50	3.54×10^{-3}	3.43×10^{-3}

$c_N = 0$ or $c_N = 1$ is used. This is illustrated in figures 5 and 6 in which the radial velocities are plotted for the one dimensional models corresponding to models 3 and 12 respectively for a small range of radii near to the critical point.

In figures 7 and 8 radial velocity profiles in one and two dimensional cases are compared for models 3 and 12 respectively. For the two dimensional tori, we plot the radial velocity measured on the equatorial plane. We plot the radial velocities for the corresponding one dimensional models with and without vertical averaging being taken into account in the equation of state. There is good agreement between all three of the radial velocity profiles for each model. This indicates that the results are not very sensitive to the precise mode of vertical averaging.

6   Discussion

In this paper we constructed steady state non static accretion tori with constant specific angular momentum and entropy. In general there is some degree of arbitrariness in this procedure because the Bernoulli integral may be chosen to vary in an arbitrary way on stream lines. This corresponds to differing specifications for the φ component of vorticity which needs to be specified as an entry condition on the flow. For simplicity we have restricted attention to the case of zero vorticity. In this case we derived an equation for the velocity potential and solved it to give solutions appropriate to the inner transonic region of an accretion disc.

For the purpose of comparison of these models with models obtained in a one dimensional approximation, we considered such models, including viscosity treated according to the usual Shakura and Sunyaev (1973) α prescription.

An important issue is whether specification of the mass accretion rate, entropy (or sound speed at the critical point) and the constraint that the material have the specific angular momentum appropriate to a Keplerian disc at some specified outer boundary radius is enough to fix the location of the critical point for a given value of α.

We found that if the usual α viscosity prescription is used in which the viscous stress is proportional to the angular velocity gradient, the location of the critical point is not

determined uniquely, although the possible spread in locations might be small. Note too that previous work by Abramowicz *et al* (1988) used a viscosity prescription in which the viscous stress was simply proportional to the pressure. In this case one expects a unique determination for the location of the critical point (at least locally in parameter space.) The extra freedom in our case arises from the possibility of varying the viscous stress at the critical point through changing the angular velocity gradient.

In order to investigate this situation further, we discussed, using a simple model, the possible effects of the limitation that viscous information should not be transmitted at a speed exceeding the sound velocity (Narayan, 1992). We found that contrary to some previous work, the viscous stress does not vanish at the critical point but that it tends to be increased if as in our models, the angular momentum and angular velocity gradients have opposite sign. In addition models such as the ones we constructed with very flat specific angular momentum profiles throughout (ie with no boundary layer) are barely affected by this propagation constraint.

However, if the speed associated with the propagation of viscous information coincides with the sound speed, the ability to change the viscous stress through varying the angular velocity gradient at the critical point together with the lack of uniqueness in determining its location may be removed (at least in our simple model).

In general we found good agreement between flow parameters determined from the one and two dimensional models. When the intersection point of the sonic surface with the equatorial plane in a two dimensional model coincides with the critical point in a one dimensional model and the sound velocities match there good agreement is found for the radial velocity profile measured on the equatorial plane in the two dimensional case and the radial velocity profile found in the one dimensional case.

Figure captions

Figure 1: The density structure (gray scale levels) and the velocity field (arrows) for the model 3 torus with $m = 75$ and $x_G = 0.39$. The dashed curve is the sonic surface.

Figuere 2: As in figure 1 but for model 12 which has with $m = 11$ and $x_G = 0.49$.

Figure 3: The specific angular momentum distribution, given in units of $\sqrt{GMr_G}$, for the model 3 torus (dashed line) and corresponding one dimensional models (dot-dashed and solid lines). The dot-dashed line represents the solution with the natural boundary condition $c_N = 1$ and the solid line the solution with $c_N = 0$.

Figure 4: As in figure 3 but for model 12.

Figure 5: The magnified region near the critical point showing the comparison between the radial velocities in the one dimensional models obtained with $c_N = 1$ (dot-dashed curve) and $c_N = 0$ (solid curve) for model 3.

Figure 6: As in figure 5 but for model 12.

Figure 7: The radial velocity in units of the sound speed at the critical point is plotted for the model 3 torus (dashed curve), for the one dimensional model with vertical averaging taken into account in the equation of state (solid curve) and for the two dimensional polytrope (dot-dashed curve):

Figure 8: As in figure 7 but for model 12.

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