

# Short Times Characterisations of Stochasticity in Nonintegrable Galactic Potentials <sup>\*</sup>

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This paper proposes a new, potentially useful, way in which to characterise the degree of stochasticity exhibited by orbits in a fixed time-independent galactic potential. This approach differs from earlier work involving the computation of Liapounov exponents in two ways, namely (1) by focusing on the statistical properties of ensembles of trajectories, rather than individual orbits, and (2) by restricting attention to the properties of these ensembles over time scales shorter than the age of the Universe,  $t_H$ . For many potentials, generic ensembles of initial conditions corresponding to stochastic orbits will evolve relatively quickly towards a time-independent invariant measure  $\Gamma$ , which is arguably the natural unit to consider if one is interested in self-consistent equilibria. The basic idea proposed here is to compute short time Liapounov characteristic numbers  $\chi(\Delta t)$  over time intervals  $\Delta t$  for orbits in an ensemble that samples this invariant measure, and to analyse the overall distribution of these  $\chi$ 's. This is done in detail for one model potential, namely the sixth order truncation of the Toda lattice potential. One especially significant conclusion is that time averages and ensemble averages coincide, so that the form of the distribution of short time  $\chi(\Delta t)$ 's for such an ensemble is actually encoded in the calculation of  $\chi(t)$  for a single orbit over long times  $t \gg t_H$ . The distribution of short time  $\chi$ 's is analysed as a function of the energy  $E$  of orbits in the ensemble and the length of the short time sampling interval  $\Delta t$ . For relatively high energies, the distribution is essentially Gaussian, the dispersion decreasing with time as  $t^{-p}$ , with an exponent  $0 < p < 1/2$  that depends on the energy  $E$ .

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## 1. MOTIVATION

Dating back to the pioneering work of Hénon and Heiles (1964) thirty years ago, considerable attention has focused on the study of orbits in nonintegral potentials which are thought to reflect the bulk mass distribution of a galaxy (see, e.g., Martinet & Pfenniger 1987 or Contopoulos & Grosbol 1989, and references cited therein). The Hamiltonian systems associated with these potentials differ from integrable Hamiltonian systems in that they may admit both regular and stochastic orbits.

In order to characterise the stochastic orbits, it is useful to introduce the concept of a Liapounov exponent. One way in which to define such Liapounov exponents (cf. Pesin 1977) is in a fashion related directly to the Kolmogorov entropy (cf. Chirikov 1979). However, it is quite natural physically and more convenient computationally to define these exponents instead in terms of the stability of individual stochastic trajectories (Bennetin et al 1976) with respect to small perturbations. Specifically, Liapounov exponents may be defined by the prescription

$$\chi \equiv \lim_{t \rightarrow \infty} \lim_{\delta r(0) \rightarrow 0} \frac{1}{t} \log \left( \frac{\delta r(t)}{\delta r(0)} \right), \quad (1)$$

where  $\delta r(0)$  and  $\delta r(t)$  denote respectively the configuration space deviations of two nearby orbits at times 0 and  $t$ . Of particular interest is the maximal Liapounov exponent corresponding to the most unstable perturbation, which can be computed numerically by selecting  $\delta r(0)$  at random. In everything that follows, the words “Liapounov exponent” will refer to this maximal Liapounov exponent.

This prescription involves a formal  $t \rightarrow \infty$  limit. Clearly, however, one cannot integrate forever, so that, as a practical matter, one is restricted to computing finite times estimates of  $\chi$ . The crucial question then is: For how long must one integrate to obtain reasonable estimates?

The precise answer to this question depends on the particular form of the potential. However, one knows from experience (cf. Contopoulos and Barbanis 1989) that, as a general rule, one must typically integrate over extremely long times, say  $t \geq 10^4 t_D$ , where  $t_D$  denotes a characteristic dynamical, or crossing, time. Unfortunately, though,  $t_D \sim 10^8$  yr for a galaxy like the Milky Way, so that this entails an integration for a period of time that is orders of magnitude longer than the age of the Universe. For this reason, the Liapounov exponent  $\chi$  does not provide a useful characterisation of the instability of individual trajectories on short time scales  $\leq t_H$ .

Motivated by this realisation, a number of different workers, notably Udry and Pfenniger (1988), have sought instead to define shorter time analogues of the Liapounov exponent which could provide a characterisation of the stochasticity of individual orbits on time scales  $\leq t_H$ . This is an extremely interesting idea. However,

these authors have not yet fully explored the potential implications of such an approach.

Most of the work hitherto on the problem of stochasticity in galactic potentials has been formulated in terms of the properties of individual orbits. However, even though it is convenient computationally to study such individual trajectories, there are solid physical motivations for working instead with ensembles of orbits. Perhaps the most obvious point is that individual orbits of stars are not accessible observationally: all that one can detect is an instantaneous snapshot of the overall surface brightness of a galaxy which, one hopes, traces the distribution of mass. Another obvious point is that one is really compelled to consider ensembles of orbits, rather than individual trajectories, if one is interested in incorporating stochastic orbits into a self-consistent model using Schwarzschild's (1979) method, or any variant thereof.

The object of this paper is to propose an alternative approach to the analysis of stochastic orbits which confronts these basic difficulties. Specifically, the aim is to provide useful characterisations of stochasticity appropriate for ensembles of orbits, focusing exclusively on relatively short time scales  $\leq t_H$ .

A consideration of stochasticity for individual orbits on short time scales is a very complicated proposition and, for this reason, has been avoided by most nonlinear dynamicists. However, even though the transient behavior of individual orbits is quite complex, the statistical properties of ensembles of orbits can, nevertheless, exhibit striking regularities which facilitate a relatively simple characterisation.

Three basic facts, described in detail below, facilitate this general approach. The first is that, for many potentials, including the model system analysed in this paper (Kandrup & Mahon 1994), generic ensembles of initial conditions corresponding to stochastic orbits will evolve on a relatively short time scale towards a particular distribution which is time-independent or, if not strictly time-independent, only exhibits subsequent variability on very long time scales. The approach towards a strictly time-independent invariant measure is in fact guaranteed for Hamiltonian systems in which the stochastic orbits are ergodic, although there is no reason theoretically to expect that the time scale associated with this approach is short.

The second fact is that, in many contexts, it is this invariant, or near-invariant, measure which constitutes the natural object in terms of which to analyse the statistical properties of stochasticity. In particular, it is reasonable to consider an ensemble of stochastic orbits that samples the invariant measure, and then compute the Liapounov characteristic numbers for the orbits in that ensemble over time intervals  $\Delta t \leq t_H$ . The statistical properties of the resulting *distribution of Liapounov characteristic numbers* then provides a useful characterisation of the overall degree of

stochasticity for the ensemble.

The final fact is that, for an ensemble that samples the invariant measure, ensemble averages and time averages may be equivalent, so that all the statistical information about stochasticity appropriate for such ensembles can actually be contained within the standard calculation of a Liapounov characteristic number  $\chi(t)$  for a single orbit over very long times  $t$ . For the case of a true invariant measure, this equivalence follows from the ergodic theorem.

These three facts indicate that it is especially natural to compute short time Liapounov characteristic numbers for the particular case of ensembles of orbits that sample the invariant measure. However, one can equally well choose to compute these characteristic numbers for other ensembles. Thus, for example, one might focus upon an ensemble initially localised in some small phase space region which eventually evolves towards the invariant measure, and then compute the Liapounov characteristic numbers for the orbits in that ensemble. Such a computation is potentially interesting in that it permits a characterisation of the overall stochasticity of a “nonequilibrium” ensemble as it “evolves towards equilibrium,” which can facilitate useful insights into the connection between stochasticity and relaxation.

Section 2 of this paper outlines the general tact to be used in providing short time characterisations of stochasticity for ensembles of orbits. The remaining sections then focus on an implementation of this approach for one particular model, namely the sixth order truncation of the Toda (1967a,b) lattice potential. This specific form was not selected because of the expectation that it provides an especially good approximation to the gravitational potential of any particular class of galaxies. Rather, in the spirit of the pioneering work of Hénon and Heiles (1964), the aim has been to choose a potential which, in the sense described in Section 3, appears to be generic; and to use that potential as a testing ground for various ideas.

Section 3 of the paper first demonstrates the concrete sense in which the information about short time measures of stochasticity is actually encoded in the long time calculations of  $\chi(t)$ , and then analyses the form of the distribution of short time Liapounov characteristic numbers associated with an ensemble of orbits that samples the invariant measure. Section 4 next outlines one concrete prescription for generating ensembles that sample this invariant measure by evolving initially localised ensembles of initial conditions, and then analyses the distribution of Liapounov characteristic numbers associated with these initially localised ensembles. Section 5 concludes by examining the form of these distributions as a function of the sampling interval  $\Delta t$ .

## 2. TRANSIENT DYNAMICS OF AN ENSEMBLE OF STOCHASTIC ORBITS

The standard formulation of nonlinear dynamics, as applied to galactic dynamics, entails a consideration of *asymptotic orbital dynamics*, focusing explicitly on the asymptotic, long time behavior of individual trajectories. The object here is to propose a new approach, *transient ensemble dynamics*. In this alternative approach, the principal focus is on the statistical properties of ensembles of orbits, rather than the details of individual trajectories. Moreover, the analysis is restricted entirely to properties of these orbits on short time scales  $\leq t_H$ . In this approach, asymptotic calculations over extremely long time intervals are eschewed as unphysical, except to the extent that they can provide information about the behavior of ensembles of orbits on short, astrophysically relevant, time scales.

As noted already, the transient behavior of individual orbits can be extremely difficult to characterise, so that a description of stochasticity in terms of transient orbital dynamics would typically prove quite complicated. However, there are intuitive reasons to believe that, nevertheless, the physically relevant aspects of the short time behavior can be characterised relatively easily if one chooses instead to focus on the statistical properties of ensembles of orbits.

The key feature entering into the proposed statistical description of stochastic orbits is the notion of an invariant measure. Mathematically, the invariant measure corresponds to a probability distribution which, if evolved into the future using the equations of motion, remains invariant (cf. Lichtenberg and Lieberman 1992). Applied to galactic dynamics, it corresponds to a time-independent phase space distribution for stochastic orbits moving in some specified galactic potential.

There are two important features about the invariant measure which are relevant in the present context. The first is that, for many time-independent potentials, generic ensembles of initial conditions corresponding to stochastic orbits will evolve on relatively short time scales towards an invariant measure. An initially localised ensemble of stochastic orbits will disperse and eventually, via a sort of phase mixing, evolve towards a distribution which fills all of the accessible phase space, with a relative weight for different phase space regions that is given by the invariant measure. The second feature is that the invariant measure defines the natural ensemble of stochastic orbits to consider if one is interested in time-independent equilibrium configurations. In particular, it constitutes the natural building block for the construction of self-consistent models which include stochastic orbits.

If one is interested in models of galaxies that do not exhibit strict spherical or axisymmetry, it is in general impossible to construct analytic equilibria. For that rea-

son, one is typically forced to proceed numerically via Schwarzschild’s (1979) method, or some variant thereof. The idea underlying this method is straightforward, at least in principle. What one must do is (1) specify some time-independent gravitational potential  $\Phi$ , (2) generate a large number of orbits in that potential, and then (3) select an ensemble of orbits which yields a mass distribution  $\rho$  that generates  $\Phi$  self-consistently as a solution to the Poisson equation,  $\nabla^2\Phi = 4\pi G\rho$ . The basic point now is clear. If the self-consistent configuration is to constitute a true equilibrium, it must have the property that, when evolved into the future using the self-consistent equations of motion, its form remains unchanged. However, when considering stochastic orbits, the only way to insure that this be true is to demand that the orbits be chosen to sample the time-independent invariant measure.

As a concrete example, consider a nonintegrable two degree of freedom system characterised by a time-independent Hamiltonian  $H$ . Because  $H$  is independent of time, the energy  $E$  of any given orbit is conserved, so that evolution is restricted to a three-dimensional hypersurface of constant  $E$ . Suppose further that there exist no additional conserved quantities. Generically, this constant energy hypersurface will then contain both islands of regular orbits and a surrounding sea of stochastic orbits. The stochastic regions may be divided into disjoint regions by invariant tori. Suppose, however, that the stochastic regions are all connected, so that any given orbit can access all of the stochastic portions of the constant energy hypersurface. The evolution towards an invariant measure then corresponds to the fact that a generic ensemble of initial conditions, corresponding to stochastic orbits of energy  $E$ , will evolve towards an invariant distribution  $\Gamma(E)$ , the form of which depends only on the energy and is independent of all other details.

The two basic questions should now be clear: How rapid and efficient is this evolution towards the invariant measure, and how should one characterise the stochastic properties of this invariant measure? Answering the first of these questions involves probing the rate at which a generic ensemble evolves towards an “equilibrium.” Answering the second involves probing the properties of that equilibrium.

At least for some model potentials, one knows that generic ensembles of initial conditions corresponding to stochastic orbits can evidence a very rapid coarse-grained evolution towards an invariant measure. For example, Kandrup & Mahon (1994) have shown that, for the sixth order truncation of the Toda lattice potential, generic ensembles of fixed energy  $E$  evolve exponentially towards an invariant measure on a time scale which, in physical units, is substantially shorter than  $t_H$ . Moreover, they have shown that the rate at which ensembles of energy  $E$  evolve towards the invariant measure  $\Gamma(E)$  is directly related to the value of the Liapounov exponent

$\chi(E)$ . There is no guarantee that a similar behavior will be observed for all potentials. In particular, there may be other systems in which an invariant measure does not even exist. However, in such cases it would seem impossible to construct a time-independent self-consistent model which incorporates stochastic orbits.

Given the importance of the invariant measure  $\Gamma$ , it is natural to search for useful characterisations of the statistical properties of orbits that sample  $\Gamma$ . One way in which to characterise this measure is to probe the overall degree of instability exhibited by orbits within the ensemble on physically relevant time scales  $\Delta t \leq t_H$ . Precisely this information is contained within a calculation of short time Liapounov characteristic numbers

$$\chi(\Delta t) \equiv \lim_{\delta r(0) \rightarrow 0} \frac{1}{\Delta t} \log \left( \frac{\delta r(\Delta t)}{\delta r(0)} \right). \quad (2)$$

Given the values of  $\chi(\Delta t)$  for all the orbits in the ensemble, one can of course compute  $N(\chi(\Delta t))$ , the distribution of Liapounov characteristic numbers, which depends on both the form of the invariant measure, and hence the energy, and on the duration of the sampling interval. One would expect that the form of this distribution should depend on the invariant measure since the values of the Liapounov exponents depend on the energy. Moreover, the distribution must depend on the sampling interval  $\Delta t$  since, as one samples for progressively longer times, one must eventually converge towards a distribution that is infinitely sharply peaked about the conventional Liapounov exponent.

Even though this prescription differs substantively from the standard approach involving asymptotic orbital dynamics, the overall conclusions may not be all that different. The basic reason for this is that time averages computed for an individual orbit can coincide with ensemble averages generated from a collection of orbits, provided that the ensemble is so chosen as to sample the invariant measure. This would imply in particular that the mean Liapounov characteristic number  $\bar{\chi}$  associated with the distribution  $N(\chi(\Delta t))$  should coincide with the standard Liapounov exponent  $\chi$ , as calculated in the usual way from a single orbit for very long times. As discussed more carefully in Sections 3 - 5, this is actually true for at least one model potential, namely the truncated Toda potential (cf. Kandrup & Mahon 1994).

More generally, one might expect that the information required for the standard calculation of the Liapounov exponent  $\chi$  via a long time integration actually contains within it the same information as is contained within the short time distribution  $N(\chi(\Delta t))$ . Given a sequence of estimates  $\{\chi(t_i)\}$ , ( $i = 1, 2, \dots$ ), computed at fixed intervals  $t_{i+1} - t_i = \Delta t$ , one can extract a sequence of short time Liapounov charac-

teristic numbers  $\chi(\Delta t_i)$  via the obvious prescription

$$\chi(\Delta t_i) \equiv \frac{\chi(t_i + \Delta t)[t_i + \Delta t] - \chi(t_i)t_i}{\Delta t}. \quad (3)$$

The critical point then is that, if time and ensemble averages agree, the distribution of Liapounov characteristic numbers  $\chi(\Delta t_i)$  generated in this way should coincide (to within statistical errors) with the distribution  $N(\chi(\Delta t))$ .

In the remaining sections of this paper, this general picture will be confirmed in detail for one specific model potential, namely the sixth order truncation of the Toda lattice. What this means is that the standard Liapounov exponent  $\chi$ , defined formally as a  $t \rightarrow \infty$  limit, actually has physical meaning on short time scales as well.  $\chi$  measures the mean stochasticity for an ensemble of orbits that samples the invariant measure. This implies that earlier work (cf. Contopoulos & Barbani 1989) tracking the time evolution of Liapounov characteristic numbers for long times can be exploited to extract information about the short times distribution  $N(\chi(\Delta t))$ .

It is also clear that the Liapounov exponent  $\chi$  is only one moment of  $N(\chi(\Delta t))$ , which can also be supplemented by a consideration of higher moments. For example, a calculation of the dispersion  $\sigma_\chi$  gives a measure of the degree to which individual Liapounov characteristic numbers deviate from the mean. Moreover, the overall shape of the distribution can provide information about “stickiness” associated with islands of regular orbits. More precisely, one knows that stochastic orbits can sometimes get trapped temporarily in the neighborhood of an island, and one might expect that, if trapped in that neighborhood,  $\chi(\Delta t)$  will be substantially smaller than the mean value  $\bar{\chi}$ .

The equivalence of time and ensemble averages discussed above means that one can hope to apply at least one basic idea from ergodic theory to systems which contain both regular and stochastic orbits. Ergodic systems are systems in which the entire phase space is stochastic, and in which time and ensemble averages agree, provided that the ensemble used to construct the average is chosen appropriately (cf. Lichtenberg and Lieberman 1992). Thus, e.g., the standard ergodic hypothesis underlying equilibrium thermodynamics implies that, for an isolated Hamiltonian system, the natural ensemble corresponds to a microcanonical distribution, i.e., a uniform sampling of the constant energy hypersurface. Realistic galactic potentials do not seem to define ergodic systems, since even the most chaotic potentials which have been envisioned contain at least some regular orbits, and are characterised by an invariant measure which is not microcanonical (cf. Kandrup & Mahon 1994). However, despite deviating from ergodicity in these respects, they do appear to satisfy the important condition that, for the stochastic orbits, time and ensemble averages



agree, provided that one works with ensembles that sample the invariant measure.

When considering regular orbits, it is physically well motivated to consider the properties of individual trajectories since, typically, infinitesimal changes in initial conditions will not lead to gross qualitative changes in the subsequent evolution. Thus, in particular, initially nearby trajectories remain nearby and the overall shape of an orbit is typically stable towards small perturbations. However, for the case of stochastic orbits this is no longer true. The fact that these orbits have positive Liapounov exponent implies that two initially nearby trajectories will typically diverge exponentially. In this case, it is arguably more appropriate to follow the evolution of an initial phase space element  $\Delta\mathbf{z}(t_0)$  than the evolution of a single material point. The obvious fact then is that the distribution  $N(\chi(\Delta t))$  provides a characterisation of the average divergence of nearby trajectories and, as such, information about the subsequent spreading of an initially localised phase space element.

All of this has focused on the role of Liapounov characteristic numbers in characterising the behavior of “equilibrium” ensembles that sample the invariant measure. However, one can also consider short time Liapounov characteristic numbers in the context of the approach towards equilibrium. Numerically, realisations of the invariant measure can be constructed by choosing any “random” ensemble of initial conditions corresponding to stochastic orbits and evolving these initial data forward in time until, statistically, they approach a sampling of a time-independent distribution. However, while evolving these initial conditions into the future, one can simultaneously compute short times Liapounov characteristic numbers  $\chi(\Delta t)$ , which can provide a useful statistical characterisation of the overall stochasticity of the initial conditions as they evolve towards the invariant measure.

### 3. EQUIVALENCE OF TEMPORAL AND ENSEMBLE AVERAGES AND THE DISTRIBUTION OF SHORT TIME $\chi$ 'S

The sixth order truncation of the Toda lattice corresponds to a Hamiltonian system of the form

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y), \quad (4)$$

where

$$\begin{aligned} V(x, y) = & \frac{1}{2}(x^2 + y^2) + x^2y - \frac{1}{3}y^3 + \frac{1}{2}x^4 + x^2y^2 + \frac{1}{2}y^4 \\ & + x^4y + \frac{2}{3}x^2y^3 - \frac{1}{3}y^5 + \frac{1}{5}x^6 + x^4y^2 + \frac{1}{3}x^2y^4 + \frac{11}{45}y^6. \end{aligned} \quad (5)$$

This defines a two degree of freedom system in terms of the canonical pairs  $\{x, p_x\}$  and

$\{y, p_y\}$ . The potential  $V(x, y)$  can be derived formally from the true Toda potential

$$V(x, y) = \frac{1}{24} \left\{ \exp[2(3)^{1/2}x + 2y] + \exp[-2(3)^{1/2}x + 2y] + \exp(-4y) \right\} - \frac{1}{8} \quad (6)$$

by a Taylor series in  $x$  and  $y$ , truncated at sixth order.

The Hamiltonian system defined by the true Toda potential (5) is integrable, but it is believed that all truncations at third and higher order are nonintegrable. In particular, one knows that the sixth order truncation is nonintegrable, and, that, for energies above a critical value  $E \approx 0.80$ , there exist both regular and stochastic orbits (cf. Contopoulos and Polymitis 1987). The Toda lattice and its truncations are all special in the sense that they manifest a discrete  $2\pi/3$  rotational symmetry. One knows (cf. Udry & Pfenniger 1988) that special symmetries of this form can have important effects on the form of the regular orbits and the relative abundance of regular and stochastic orbits. However, experience would suggest that, even though breaking this, or any other, symmetry will typically increase the relative abundance of stochastic orbits, the behavior of the stochastic orbits in this potential should be qualitatively similar to stochastic orbits in other potentials with different or less symmetry.

Having chosen to consider a truncated Toda potential, it were perhaps appropriate to justify the particular truncation at *sixth* order. This is in fact easily done: If one is concerned with the evolution towards an invariant measure, one must consider a potential in which the orbits are confined to a compact phase space region. This is achieved by demanding that  $V(x, y) \rightarrow \infty$  as  $x$  and  $y \rightarrow \infty$ , which requires an even order truncation. One also wishes to consider the lowest order truncation possible, since this will minimise the time required in effecting the numerical computations. The second order truncation is of course integrable, and hence unacceptable. The fourth order truncation is not integrable, as proved by Yoshida et al (1988) and demonstrated explicitly by Udry and Martinet (1990). However, albeit nonintegrable, this lower order truncation exhibits relatively little chaos, perhaps because the leading order quartic terms in the potential (as well as the quadratic terms) are strictly axisymmetric.

In order to study stochastic orbits in a given potential, one must first ascertain the location of the stochastic regions as a function of energy  $E$ . For sixteen different values of energy between  $E = 10$  and  $E = 200$ , surfaces of section were generated, plotting  $y$  and  $p_y$  at successive points where randomly chosen orbits pass through the value  $x = 0$ . This is a useful choice of section because the Toda potential is symmetric under a reflection  $x \rightarrow -x$ , so that each orbit must repeatedly intersect the  $x = 0$  hyperplane. These surfaces of section were then used to select initial data

corresponding to ten different stochastic orbits for each of the sixteen selected values of the energy.

For each of these ten different choices of initial conditions, Liapounov exponents were computed in the standard way (cf. Bennetin et al 1976). This was done by introducing a small initial perturbation  $\delta x = 10^{-10}$  into each orbit, continually solving the variational equations for the perturbation as the unperturbed orbit is evolved, and then renormalising the evolved perturbation back to a total amplitude

$$\delta z = (\delta x^2 + \delta y^2 + \delta p_x^2 + \delta p_y^2)^{1/2} = 10^{-10} \quad (7)$$

at intervals  $\Delta t = 10$ . Successive computations of  $\chi(t)$  were made at the same time intervals, with the calculation for each orbit proceeding for a total time  $t = 10^4$ . The value  $\chi(t = 10^4)$  was then interpreted as providing an estimate for the true Liapounov exponent, which is of course defined only in the  $t \rightarrow \infty$  limit. It was found thereby that, for orbits of fixed energy, the value of the Liapounov exponent is independent of the initial conditions, so that one can speak of a unique  $\chi(E)$ .

The surfaces of section were also used as an aid to construct three localised ensembles of initial conditions of fixed energy  $E$  within the stochastic phase space regions. Each cell was constructed as follows: One started by selecting a point  $\{y, p_y\}$  on the  $x = 0$  surface of section which is displaced significantly from any large islands of regular orbits. This point was then used to define the center of a cell of specified size  $\Delta y$  and  $\Delta p_y$ . A uniform sampling of this cell via a rectangular grid served to select 400 pairs  $\{y, p_y\}$ , and these pairs were then used to generate an ensemble of initial conditions  $\{x, y, p_x, p_y\}$ , setting  $x = 0$  and

$$p_x \equiv \left\{ 2[E - V(x = 0, y)] - p_y^2 \right\}^{1/2} > 0. \quad (8)$$

For energies  $E \geq 50$ ,  $\Delta y = 0.2$  and  $\Delta p_y = 0.72$ . For lower energies,  $\Delta y$  and  $\Delta p_y$  were chosen to be one half or one quarter as large.

As described elsewhere (Kandrup & Mahon 1994), when these ensembles of initial conditions were evolved into the future, they were found to evidence a coarse-grained, exponential approach towards an invariant measure on a time scale  $t \ll 100$ . For this reason, the collections of 400 phase space coordinates at time  $t = 100$  were taken as constituting random samplings of the invariant measure  $\Gamma(E)$ . Short time  $t = 100$  Liapounov characteristic numbers were computed for these random samplings of the invariant measure, with  $\chi(t)$  for each orbit again being recorded at  $\Delta t = 10$  intervals. The resulting distribution of Liapounov characteristic numbers,  $N(\chi(t = 100))$ , was then analysed to extract the first and second moments,  $\bar{\chi}(t = 100)$  and  $\sigma_\chi(t = 100)$ .

The first striking result derived from such an analysis is that the mean value  $\bar{\chi}$  associated with the short time Liapounov characteristic numbers coincides, to within

statistical uncertainties, with the long time estimate of the Liapounov exponent  $\chi$ . This is illustrated in Fig. 1, which exhibits the estimated values of the Liapounov exponents computed in both ways. To generate this Figure, the ten long time estimates at each energy  $E$  were first averaged together to obtain a mean Liapounov exponent. The values obtained thereby were then plotted as small triangles, with error bars that represent the standard deviations associated with the means. The three larger diamonds for each value of  $E$  represent calculations of the mean values  $\bar{\chi}$  associated with the short times distributions,  $N(\chi(t = 100))$ , generated from ensembles of 400 orbits.

The second significant result derived from this analysis is that the long time calculation of  $\chi$  for a single stochastic orbit actually contains within it the same information as the distribution of short time Liapounov characteristic numbers derived for an ensemble of orbits. To extract this information, the values of  $\chi(t)$  for a single orbit were partitioned as in Eq. (3) into intervals  $\Delta t = 100$  to extract a different set of short time estimates  $\chi(\Delta t_i)$ . The key point then is that the distribution of  $\chi(\Delta t_i)$ 's generated in this way is almost identical to  $N(\chi(\Delta t))$ , the distribution of short time  $\chi$ 's generated from the ensembles of orbits that sample the invariant measure.

Fig. 2 illustrates the forms of these distributions for four different values of energy, namely  $E = 150, 75, 30,$  and  $20$ . These distributions were generated by binning the data into intervals  $\delta\chi = 0.05$  and normalising the resulting binned distribution so that the most populous bin is assigned the value  $N(\chi) = 1$ .

The first obvious point to be inferred from this Figure is that, as noted already, except for some small differences at very low values of  $\chi$  the two distributions are essentially the same. The second point is that, at least at relatively high energies, both distributions are extremely well approximated by a Gaussian form. However, for energies below a value  $E \approx 75 - 100$ , the distribution begins to acquire a statistically significant short  $\chi$  tail. When the energy is decreased further to a value as small as  $E \approx 40 - 60$ , this tail then acquires a secondary peak. However, at very low energies,  $E \approx 10 - 20$ , the two peaks eventually merge into a single distribution which deviates significantly from a Gaussian.

The observed structure at small values of  $\chi$  seems to be associated with the increasing predominance of regular orbits at lower energies. It is well known (cf. MacKay et al 1984a,b, Lau et al 1991) that stochastic orbits that stray too close to islands of regularity tend to get trapped for relatively long periods of time around these islands, and it would appear that, while in these regions, their Liapounov characteristic numbers are substantially reduced.

The differences between the two distribution at small values of  $\chi < 0.1 - 0.15$  arise

because the alleged sampling of the invariant measure is contaminated by a small number of regular orbits. In constructing these samplings, one selected an initial phase space region which seemed, at least superficially, to be comprised completely of stochastic orbits. However, these stochastic regions actually contain a nonzero measure of regular orbits embedded in the stochastic sea.

The basic conclusion, therefore, is that, because of the equivalence of time and ensemble averages, an evaluation of  $\chi(t)$  for a single orbit over long times contains the same information as the distribution of short time Liapounov characteristic numbers associated with an ensemble of orbits that samples the invariant measure.

#### 4. LIAPOUNOV CHARACTERISTIC NUMBERS AND THE APPROACH TOWARDS AN INVARIANT MEASURE

The preceding section focused on the form of the distribution of Liapounov characteristic numbers associated with an initial ensemble that samples the invariant measure. However, as discussed in Section 2, one can also consider the distribution  $N(\chi)$  associated with other ensembles as well. In particular, it is natural to compute the distribution of Liapounov characteristic numbers for the initially localised ensembles considered in Section 3 as they evolve towards the invariant measure. Doing this enables one to understand how the evolution towards an invariant measure correlates with the overall stochasticity of the orbits, a problem already considered in a slightly different context by Kandrup & Mahon (1994).

The results for one particular energy, namely  $E = 50$ , are illustrated in Fig. 3. The data plotted in this Figure were generated by selecting the three initially localised ensembles discussed in Section 3, and computing  $\chi(t)$  for each of the 400 orbits in each ensemble at  $t = 10$  intervals for a total time of  $t = 200$ . The resulting  $\chi$ 's were then partitioned into collections of short time estimates  $\chi(\Delta t_i)$  at  $\Delta t = 10$  intervals. Fig. 3 exhibits the mean  $\bar{\chi}(\Delta t_i)$  and the associated dispersions  $\sigma_{\chi}(\Delta t_i)$  for these three different ensembles.

In this Figure, the first interval  $\Delta t = 100$  is interpreted as corresponding to the period during which the ensemble evolves towards the invariant measure. The phase space coordinates at  $t = 100$  correspond to the sampling of the invariant measure used for the computations in Section 3, and the remaining  $\Delta t = 100$  are thus interpreted as corresponding to the evolution of an initial ensemble that samples the invariant measure.

From this Figure several conclusions are apparent. The first is that even at very early times, when the ensemble deviates significantly from a sampling of the invariant measure, the values of  $\bar{\chi}(\Delta t_i)$  and  $\sigma_{\chi}(\Delta t_i)$  do not deviate all that much from the

“equilibrium” values associated with the invariant measure. At early times,  $\bar{\chi}$  tends to be somewhat larger than the value associated with the invariant measure. This is easily understood by observing that the initial ensemble was located in a phase space region particularly far from any large islands of regular orbits where, overall, one might expect a higher degree of stochasticity. At early times, one also observes that  $\sigma_{\chi}$  is somewhat smaller than the value associated with the invariant measure. This is again easily understood by observing that the initial ensemble is constructed from a collection of phase space points that are relatively close together and significantly displaced from any large islands.

It is also apparent that, by a time  $t = 100$ ,  $\bar{\chi}$  and  $\sigma_{\chi}$  have asymptoted towards constant values, this corroborating the expectation that the phase space coordinates at  $t = 100$  can be interpreted at least appropriately as constituting a random realisation of the invariant measure. For higher values of energy, the approach towards the invariant measure is even more rapid. Indeed, as discussed more extensively in Kandrup & Mahon (1994), there exists a direct one-to-one correlation between the exponential approach towards an invariant measure and the value of the Liapounov exponent. Specifically, as was demonstrated in especial detail for energies  $10 \leq E \leq 75$ , increasing  $E$  increases both (1) the value of the Liapounov exponent  $\chi(E)$  associated with stochastic orbits of that energy *and* (2) the exponential rate  $\Lambda(E)$  associated with the approach towards the invariant measure, in such a fashion that the ratio  $\mathcal{R}(E) \equiv \Lambda/\chi$  is approximately constant, independent of energy. The fact that the mean  $\bar{\chi}(t)$  associated with the evolution of the initially localised ensemble is rather close in value to the Liapounov exponent  $\chi$ , as defined in a  $t \rightarrow \infty$  limit, explains why there can exist a direct connection between the “equilibrium”  $\chi$  and the “nonequilibrium” evolution towards an invariant measure.

Finally, it should perhaps be noted that the first points in Fig. 3 at  $t = 10$  may be somewhat suspect. Specifically, the values of  $\chi(t = 10)$  may be significantly influenced by the particular choice of initial perturbation, namely  $\delta x = 10^{-10}$  and  $\delta y = \delta p_x = \delta p_y = 0$ .

## 5. THE EFFECTS OF A VARYING SAMPLING TIME

In this Section, attention focuses on how the form of the distribution of short time  $\chi$ 's extracted from a single long time integration, or generated from an ensemble that samples the invariant measure, varies as a function of the sampling interval  $\Delta t$ . One anticipates physically that, as the length of the sampling interval increases, the mean Liapounov characteristic number  $\bar{\chi}$  will provide an increasingly better characterisation of the overall stochasticity, and that the dispersion  $\sigma_{\chi}$  associated with the distribution

will tend to zero for  $\Delta t \rightarrow \infty$ .

Given a long time calculation of  $\chi(t)$  for a single orbit, one can extract short time  $\chi(\Delta t_i)$ 's for various choices of sampling interval and compare the results. Fig. 4a exhibits the values of  $\chi(\Delta t_i)$  for one long time integration of an orbit with  $E = 150$ , for a sampling interval  $\Delta t = 10$ . It is clear that  $\chi(\Delta t_i)$  shows rapid large amplitude variability, so that the degree of stochasticity exhibited by the orbit at time  $t$  may be substantially different at times  $t \pm \Delta t$ . However, if one considers a longer sampling interval, the variability is substantially reduced. This is illustrated in Fig. 4b, which analyses the same data for sampling intervals  $\Delta t = 100$ . This Figure was generated from Fig. 4a by taking time averages of successive collections of ten intervals with  $\Delta t = 10$ .

The differences between Figs. 4a and 4b can be quantified by specifying the maximum and minimum values of  $\chi(\Delta t_i)$ , as well as the standard deviation about the mean. For the shorter time sampling, the minimum and maximum values of  $\chi$  are  $-0.125$  and  $1.670$ , and the standard deviation is  $0.2997$ . For the longer time sampling, the minimum and maximum values are  $0.078$  and  $1.053$ , and the standard deviation is approximately half as large, namely  $0.1449$ . However, even though the longer time sampling is smoother, nontrivial structures are still observed. In particular, there is a significant dip in the value of  $\chi$  between  $t = 4000$  and  $4300$  which is observed in both figures, where  $\chi(\Delta t_i)$  decreases appreciably to a value  $< 0.1$ .

To further elucidate these sorts of differences, one can also examine the form of the distribution  $N(\chi(\Delta t))$  as a function of the sampling interval. Figures 5a and 5b illustrate the form of  $N(\chi(\Delta t))$  for two different energies,  $E = 150$  and  $50$ , and three different sampling intervals,  $\Delta t = 10, 40$ , and  $100$ . These figures were constructed by partitioning each of the ten  $t = 10^4$  calculations into a collection of intervals of fixed length  $\Delta t$ , combining the data for all ten orbits, and constructing a binned distribution with  $\Delta\chi = 0.05$ . For both values of the energy, it is clear that the width of the distribution decreases as the sampling interval becomes longer. It is also clear that, for each choice of sampling interval, the relative width of the low energy distribution is slightly larger than that of the high energy distribution. For  $E = 150$ , the distribution is essentially Gaussian for the two larger sampling intervals, whereas, for  $E = 50$  one sees indications of a low  $\chi$  tail.

It is natural to ask whether the systematic decrease in the width of the distribution, as probed by the dispersion, is exponential in time, or whether it is better fit by a power law. The answer is that, for all values of the energy, the dispersion is well fit by a power law  $\sigma_\chi \propto \Delta t^{-p}$ , where  $p$  is a positive constant. The goodness of fit is illustrated in Fig. 6 for three different energies,  $E = 50, 125$ , and  $175$ .

One might also ask how the value  $p$  depends on the energy  $E$ . It is difficult to extract a precise estimate of  $p$  for any given energy because the best fit value depends sensitively on whether or not there exists one or two initial conditions where the orbit spends a large amount of time in a region of especially low  $\chi$ . However, one can still extract a best fit value of  $p$  for each energy. The result of such an analysis is exhibited in Fig. 7. Because of the large scatter in this plot, it is difficult to extract a precise functional form for  $p(E)$ . However, two trends are unambiguous: First, it is clear that, overall, the value of  $p$  increases with increasing energy, and second, it would appear that the value of  $p$  approaches 0.5 at large energy.

A value of  $p = 0.5$  is relatively easy to explain. The distribution  $N(\chi(\Delta t))$  for large  $\Delta t$  can be viewed as a convolution of a large number of distributions  $N(\chi(\Delta t_i))$  for shorter time intervals  $\Delta t = 10$ . Suppose however, that the values of  $\chi(\Delta t_i)$  for successive  $\Delta t = 10$  intervals are completely uncorrelated. In this case, the central limits theorem guarantees that the long time distribution will converge towards a Gaussian with a dispersion that scales as  $k^{-1/2}$ , where  $k$  denotes the total number of short time intervals. The fact that this number grows linearly in time would then imply a dispersion that scales as  $t^{-1/2}$ . If these successive intervals are not completely uncorrelated, then the dispersion should decrease more slowly. Fig. 7 therefore indicates that, at low energies, there is substantial correlation between successive  $\Delta t = 10$  intervals, but that this correlation decreases for higher energies.

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**Fig. 1.** Liapounov exponents as a function of energy  $E$  estimated in two different ways. The small triangles represent  $\chi$ , calculated in the usual way by following ten different orbits for a total time  $t = 10^4$ . The error bars represent the associated standard deviations. The larger diamonds represent calculations of the mean  $\bar{\chi}(t)$  for ensembles of 400 orbits over a period  $t = 100$ .

**Fig. 2.** (a) The binned distribution of Liapounov characteristic numbers  $\chi(\Delta t)$  for  $\Delta t = 100$ , calculated in two different ways for orbits with  $E = 100$ . The solid curve represents the distribution obtained from an initial ensemble that sampled the invariant measure. The dashed curve represents the distribution obtained from partitioning the curve  $\chi(t)$  for  $t = 10000$  into 100 segments of length  $\Delta t = 100$ , and construction  $\chi(\Delta t_i)$  for each segment. (b) The same for  $E = 75$ . (c) The same for  $E = 30$ . (d) The same for  $E = 20$ .

**Fig. 3.** The mean  $\bar{\chi}(\Delta t_i)$ , and the associated dispersion  $\sigma_{\chi}(\Delta t_i)$ , computed at  $\Delta t = 10$  intervals for a total time  $t = 200$  for three initially localised ensembles of 400 stochastic orbits with  $E = 50$ . The evolved phase space coordinates at  $t = 100$  were interpreted as constituting random realisations of the invariant measure, and used as initial data for the calculations described in Section 3. The solid lines represent the values of  $\bar{\chi}$  and  $\sigma_{\chi}$  associated with the invariant measure.

**Fig. 4.** The Liapounov characteristic number  $\chi(t)$  for one orbit with  $E = 100$  partitioned into a collection of short times estimates  $\chi(\Delta t)$ , with (a)  $\Delta t = 10$  and (b)  $\Delta t = 100$ .

**Fig. 5.** (a) The distribution of short times  $\chi(\Delta t)$  for  $E = 150$ , extracted from the long time  $\chi(t)$ . The solid curve represents a sampling interval  $\Delta t = 100$ , the dot-dashed line  $\Delta t = 40$ , and the dashed line  $\Delta t = 10$ . (b) The same for  $E = 50$ .

**Fig. 6.** The dispersion  $\sigma_{\chi}$  as a function of sampling interval  $\Delta t$  for energies  $E = 50$  (bottom curve), 125 (middle curve), and 175 (top curve). The solid curves represent least squares power law fits  $\sigma_{\chi}(\Delta t) \propto \Delta t^{-p}$ . The curves for  $E = 50$  and  $E = 175$  were displaced respectively upwards and downwards by  $\sigma_{\chi} = 0.125$  to provide a less cluttered diagram.

**Fig. 7.** The exponent  $p$  associated with the least squares fit  $\sigma_{\chi} \propto \Delta t^{-p}$ , plotted as a function of energy  $E$ .