A General Analytic Formula for the Spectral Index of the Density Perturbations produced during Inflation

Misao Sasaki

Department of Earth and Space Science

Osaka University

Toyonaka 560

Japan

Ewan D. Stewart*

School of Physics and Chemistry

Lancaster University

Lancaster, LA1 4YB

U.K.

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Abstract

The standard calculation of the spectrum of density perturbations produced during inflation assumes that there is only one real dynamical degree of freedom during inflation. However, there is no reason to believe that this is actually the case. In this paper we derive general analytic formulae for the spectrum and spectral index of the density perturbations produced during inflation.

*Present address: Research Center for the Early Universe, School of Science, University of Tokyo, Bunkyo-ku, Tokyo 113, Japan. Email: eds@utaphp1.phys.s.u-tokyo.ac.jp

1 Introduction

During inflation [1] vacuum fluctuations on scales less than the Hubble radius in scalar fields with effective masses much less than the Hubble parameter[1] are magnified into classical perturbations in the scalar fields on scales larger than the Hubble radius. These classical perturbations in the scalar fields can then change the number of e-folds of expansion and so lead to classical curvature/density perturbations after inflation. These density perturbations are thought to be responsible for the formation of galaxies and the large scale structure of the observable Universe as well as, in combination with the gravitational waves produced during inflation, for the anisotropies in the cosmic microwave background.

The standard calculation [7, 8, 9, 10, 11] of the spectrum of density perturbations produced during inflation assumes that there is only one real dynamical degree of freedom during inflation. Although this is the case in most of the models of inflation constructed up to now, it is by assumption rather than prediction. When one tries to construct models of inflation [3, 5, 12] that might arise naturally in realistic models of particle physics, such as the low energy effective supergravity theories derived from superstrings, one often gets more than one dynamical degree of freedom during inflation. The standard calculation is then generally not applicable.

In this paper we derive general analytic formulae for the spectrum and spectral index of the density perturbations produced during inflation. This work is based on earlier work by Starobinsky [13]. See also [14]. While this work was in slow preparation three other related papers [15, 16, 17] appeared in the archives. After this work was completed another related paper [18] appeared in the astro-ph archive.

2 Gravity

We assume the gravitational part of the action to be

$$S = -\frac{1}{2} \int R \sqrt{-g} \, d^4 x \,. \tag{1}$$

It seems unlikely that non-Einstein gravity is relevant to inflation because the inflation that inflated the observable Universe beyond the Hubble radius must have occurred at an energy scale well below the Planck scale. We therefore do not consider it. We set $M_{\mathrm{Pl}} = 1/\sqrt{8\pi G} = 1$ throughout the paper.

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[1]All scalar fields generically acquire effective masses at least of the order of the Hubble parameter in the early Universe [2, 3, 4]. However, this can be naturally avoided for some scalar fields during inflation in certain classes of supergravity theories [5, 6].

2.1 The background

The background metric is

$$ds^2 = dt^2 - a(t)^2 \delta_{ij} dx^i dx^j \ . \tag{2}$$

Two important quantities are the Hubble parameter

$$H = \frac{\dot{a}}{a} \tag{3}$$

and the number of e-folds of expansion

$$N = \int H dt \ . \tag{4}$$

2.2 $\mathcal{R}$

Scalar linear perturbations to the metric can be expressed most generally as [19, 20, 9]

$$ds^2 = (1 + 2A)dt^2 - 2\partial_i B \, dx^i dt - a(t)^2 \left[(1 + 2\mathcal{R})\delta_{ij} + 2\partial_i\partial_j E\right] dx^i dx^j \ . \tag{5}$$

We follow [20] most closely. $\mathcal{R}$ is the intrinsic curvature perturbation of the constant time hypersurfaces. On comoving hypersurfaces, $\dot{\mathcal{R}}_c = HA_c$. On flat hypersurfaces, $\mathcal{R}_f \equiv 0$.

2.3 $\mathcal{N}$

Let $\{\Sigma(t)\}$ be a foliation of spacetime with hypersurfaces $\Sigma(t)$ labeled by a certain coordinate time t and let v^μ be the unit vector field normal to $\Sigma(t)$. Then $\theta = v^\mu_{\ ;\mu}$ is the volume expansion rate of the hypersurfaces along the integral curve $\gamma(\tau)$ of v^μ. For each integral curve, define

$$\mathcal{N} = \int_{\gamma(\tau)} \frac{1}{3}\theta d\tau \ , \tag{6}$$

where τ is the proper time along the curve.

2.4 $\mathcal{R}$ and $\mathcal{N}$

From [20]

$$\frac{1}{3}\theta = H\left(1 - A + \frac{1}{H}\dot{\mathcal{R}} + \frac{1}{3H}\frac{1}{a^2}\partial^i\partial_i S_g\right) \ , \tag{7}$$

where $S_g = a^2\dot{E} - B$. (If one Fourier expands S_g, $S_g = \sigma_g/q$, where $q = k/a$, in the notation of [20].) Assuming that the anisotropic stress perturbation is negligible,

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which is the case for scalar field, radiation or dust perturbations, then the spatial trace-free part of the Einstein equations gives

$$\frac{1}{H}\dot{S}_g + S_g = \frac{1}{H}(A + \mathcal{R}) \,. \tag{8}$$

From this equation we see that S_g is at most of order A/H or $\mathcal{R}/H$ so that it is clear that the last term in Eq. (7) is negligible compared with the other terms on superhorizon scales, that is for $q^2 \ll H^2$ when the perturbations are Fourier expanded. From now on we work on superhorizon scales, and so we get

$$\frac{1}{3}\theta \simeq H\left(1 - A + \frac{1}{H}\dot{\mathcal{R}}\right) \,. \tag{9}$$

Also, from Eq. (5),

$$d\tau = (1 + A)\, dt \,. \tag{10}$$

Therefore

$$\mathcal{N} = \int_{\gamma(\tau)} H\left(1 - A + \frac{1}{H}\dot{\mathcal{R}}\right)(1 + A)\, dt = \int_{\gamma(\tau)} \left(H + \dot{\mathcal{R}}\right) dt \,, \tag{11}$$

and so

$$\delta N \equiv \mathcal{N} - N = \Delta\mathcal{R} \,. \tag{12}$$

In particular, if we choose a foliation such that the initial hypersurface is flat and the final one is comoving, we get

$$\delta N(\Sigma_f(t_1), \Sigma_c(t_2); \gamma(\tau)) = \mathcal{R}_c(t_2) \tag{13}$$

for a given curve $\gamma(\tau)$.

Now take t_1 to be some time during inflation soon after the relevant scale has passed outside the horizon and t_2 to be some time after complete reheating[2] when $\mathcal{R}_c$ has become constant. The relevant scale is assumed to be still well outside the horizon at $t = t_2$. Then one may regard $\mathcal{N}$ as a function of the field configuration $\phi^a(t_1, x^i)$ on $\Sigma(t_1)$ and the time t_2,

$$\mathcal{N} = \mathcal{N}\left(\phi^a(t_1, x^i), t_2\right) \,. \tag{14}$$

Note that in general $\mathcal{N}$ depends on both $\phi^a(t_1)$ and $\dot{\phi}^a(t_1)$, but as t_1 is during inflation we use the slow roll approximation to eliminate the dependence on $\dot{\phi}^a(t_1)$. Therefore

$$\mathcal{R}_c(t_2, x_2^i) = \delta N = \frac{\partial N}{\partial \phi^a}\delta\phi_f^a(t_1, x_1^i) \,, \tag{15}$$

where x_1^i and x_2^i are the spatial coordinates of $\gamma(\tau)$ on $\Sigma_f(t_1)$ and $\Sigma_c(t_2)$, respectively. One can of course choose the spatial coordinates on the hypersurfaces by the condition $B = 0$ to make $x_1^i = x_2^i$. Since the perturbations in both θ and the density are negligibly small on comoving hypersurfaces on superhorizon scales, $\Sigma_c(t_2)$ may be regarded as a hypersurface of constant Hubble parameter or constant energy density.

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[2] We do not consider the case of isocurvature perturbations that persist until the present.

3 Scalar fields

We assume the scalar field part of the action to be

$$S = \int \left[\frac{1}{2} h_{ab} g^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^b - V(\phi) \right] \sqrt{-g} \, d^4x \,. \tag{16}$$

where $g_{\mu\nu}$ is the spacetime metric and h_{ab} is the metric on the scalar field space.

3.1 The background

The background scalar fields are spatially homogeneous

$$\phi^a = \phi^a(t) \,. \tag{17}$$

The following formula will be useful

$$\frac{\partial N}{\partial \phi^a} \dot{\phi}^a = -H \,. \tag{18}$$

The background equation of motion for the scalar fields is

$$\frac{D\dot{\phi}^a}{dt} + 3H\dot{\phi}^a + h^{ab} V_{,b} = 0 \,, \tag{19}$$

where $DX^a = dX^a + \Gamma^a{}_{bc} X^b d\phi^c$ and $\Gamma^a{}_{bc} = \frac{1}{2} h^{ad} (h_{db,c} + h_{dc,b} - h_{bc,d})$. We assume that the scalar potential is sufficiently flat, i.e. satisfies

$$V^{;a} V_{;a} \ll V^2 \qquad \text{and} \qquad \sqrt{V^{;ab} V_{;ab}} \ll V \,, \tag{20}$$

where the semicolon denotes the covariant derivative in the scalar field space. Then the scalar field dynamics will rapidly approach slow roll given by

$$3H\dot{\phi}^a \simeq -h^{ab} V_{,b} \,, \tag{21}$$

or

$$\frac{\dot{\phi}^a}{H} \simeq -\frac{V^{;a}}{V} \,. \tag{22}$$

We assume that slow roll has been attained for all epochs of interest.

3.2 $\delta\phi^a$

The equation of motion for the Fourier modes of scalar field perturbations on flat hypersurfaces is

$$\frac{D^2 \delta\phi^a_{\mathbf{k}}}{dt^2} + 3H \frac{D\delta\phi^a_{\mathbf{k}}}{dt} - R^a{}_{bcd} \dot{\phi}^b \dot{\phi}^c \delta\phi^d_{\mathbf{k}} + q^2 \delta\phi^a_{\mathbf{k}} + V^{;a}{}_{;b} \delta\phi^b_{\mathbf{k}} = \frac{1}{a^3} \frac{D}{dt} \left(\frac{a^3}{H} \dot{\phi}^a \dot{\phi}^b \right) h_{bc} \delta\phi^c_{\mathbf{k}} \tag{23}$$

where $\mathbf{k}$ is the comoving wavenumber, $q = |\mathbf{k}|/a$, and the scalar field space curvature $R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ce}\Gamma^e{}_{db} - \Gamma^a{}_{de}\Gamma^e{}_{cb}$. Let $\Delta\phi$ denote the characteristic curvature radius of the scalar field space. We assume $h_{ab}\dot\phi^a\dot\phi^b/(\Delta\phi)^2 \ll H^2$ or

$$\frac{V^{;a}V_{;a}}{V^2} \ll (\Delta\phi)^2 . \tag{24}$$

This is automatic if $\Delta\phi \sim 1$ as is typically the case. Then modes well outside the horizon, i.e. with $q^2 \ll H^2$, satisfy the slow roll equation of motion

$$3H\frac{D\delta\phi^a_{\mathbf{k}}}{dt} - R^a{}_{bcd}\dot\phi^b\dot\phi^c\delta\phi^d_{\mathbf{k}} + V^{;a}{}_{;b}\delta\phi^b_{\mathbf{k}} = 3\dot\phi^a\dot\phi^b h_{bc}\delta\phi^c_{\mathbf{k}} . \tag{25}$$

4   Calculation of the spectral index

Fourier expansion of Eq. (15) gives

$$\mathcal{R}_{\mathbf{k}}(t_2) = \frac{\partial N}{\partial\phi^a}\delta\phi^a_{\mathbf{k}}(t_1) . \tag{26}$$

For scalar field perturbations generated from vacuum fluctuations during inflation

$$\delta\phi^a_{\mathbf{k}} = \sum_\alpha \phi^{a\alpha}_k a^\alpha_{\mathbf{k}} , \tag{27}$$

where $a^\alpha_{\mathbf{k}}$ is a classical[3] random variable satisfying

$$\langle a^\alpha_{\mathbf{k}} a^{\beta\dagger}_{\mathbf{l}}\rangle = \delta^{\alpha\beta}\delta^3(\mathbf{k} - \mathbf{l}) , \tag{28}$$

α runs over the number of scalar field components, and $\phi^{a\alpha}_k$ is real and satisfies

$$\sum_\alpha \phi^{a\alpha}_k \phi^{b\alpha}_k = \frac{H^2}{2k^3}\left(h^{ab} + \epsilon^{ab}\right) \tag{29}$$

where, assuming the slow roll conditions Eqs. (20) and (24), ϵ^{ab} is small and slowly varying with respect to a at fixed k/a. Therefore the spectrum of curvature perturbations is given by

$$
\begin{aligned}
\frac{2\pi^2}{k^3}P_{\mathcal{R}}\delta^3(\mathbf{k} - \mathbf{l}) &= \langle\mathcal{R}_{\mathbf{k}}(t_2)\mathcal{R}^\dagger_{\mathbf{l}}(t_2)\rangle &(30)\\
&= \frac{\partial N}{\partial\phi^a}\frac{\partial N}{\partial\phi^b}\langle\delta\phi^a_{\mathbf{k}}(t_1)\delta\phi^{b\dagger}_{\mathbf{l}}(t_1)\rangle &(31)\\
&= \sum_\alpha \frac{\partial N}{\partial\phi^a}\frac{\partial N}{\partial\phi^b}\phi^{a\alpha}_k\phi^{b\alpha}_k\delta^3(\mathbf{k} - \mathbf{l}) , &(32)
\end{aligned}
$$

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[3]We are outside the horizon so everything is classical.

or

$$P_{\mathcal{R}} = \frac{k^3}{2\pi^2} \sum_\alpha \frac{\partial N}{\partial \phi^a} \frac{\partial N}{\partial \phi^b} \phi_k^{a\alpha} \phi_k^{b\alpha} \,. \tag{33}$$

The spectral index is then given by

$$n_{\mathcal{R}} - 1 = \frac{d \ln P_{\mathcal{R}}}{d \ln k} = 3 + \frac{k^3}{2\pi^2 P_{\mathcal{R}}} \sum_\alpha \frac{\partial N}{\partial \phi^a} \frac{\partial N}{\partial \phi^b} \frac{D\left(\phi_k^{a\alpha} \phi_k^{b\alpha}\right)}{\partial \ln k} \,. \tag{34}$$

Note that $\sum_\alpha \phi_k^{a\alpha} \phi_k^{b\alpha}$, which is given by Eq. (29), is to be evaluated at a fixed time t_1. The k-dependence of the spectrum (apart from the trivial k^3 factor) is hidden in the small term ϵ^{ab}. We will exploit the fact that ϵ^{ab} is small and slowly varying with respect to a at fixed k/a in order to evaluate it as follows.

$$\left. \frac{D}{\partial \ln k} \right|_{a=\text{constant}} = \left. \frac{D}{\partial \ln a} \right|_{k/a=\text{constant}} - \left. \frac{D}{\partial \ln a} \right|_{k=\text{constant}} \,, \tag{35}$$

Eq. (29) gives

$$\left. \frac{D}{\partial \ln a} \left(\sum_\alpha \phi_k^{a\alpha} \phi_k^{b\alpha} \right) \right|_{k/a=\text{constant}} = -\left(3 - 2\frac{\dot{H}}{H^2} \right) \sum_\alpha \phi_k^{a\alpha} \phi_k^{b\alpha} \,, \tag{36}$$

and the slow roll equation of motion Eq. (25) gives

$$\left. \frac{D\phi_k^{a\alpha}}{\partial \ln a} \right|_{k=\text{constant}} = \left(\frac{\dot{\phi}^a \dot{\phi}^b}{H^2} h_{bd} + \frac{1}{3} R^a{}_{bcd} \frac{\dot{\phi}^b \dot{\phi}^c}{H^2} - \frac{V^{;a}{}_{;d}}{V} \right) \phi_k^{d\alpha} \,. \tag{37}$$

Substituting into Eq. (34) then gives

$$n_{\mathcal{R}} - 1 = 2\frac{\dot{H}}{H^2} - 2\frac{k^3}{2\pi^2 P_{\mathcal{R}}} \sum_\alpha \frac{\partial N}{\partial \phi^a} \left(\frac{\dot{\phi}^a \dot{\phi}^b}{H^2} h_{bd} + \frac{1}{3} R^a{}_{bcd} \frac{\dot{\phi}^b \dot{\phi}^c}{H^2} - \frac{V^{;a}{}_{;d}}{V} \right) \phi_k^{d\alpha} \phi_k^{e\alpha} \frac{\partial N}{\partial \phi^e} \,. \tag{38}$$

Now from Eq. (29)

$$\sum_\alpha \phi_k^{a\alpha} \phi_k^{b\alpha} \simeq \frac{H^2}{2k^3} h^{ab} \,, \tag{39}$$

and so from Eq. (33)

$$P_{\mathcal{R}} = \left(\frac{H}{2\pi} \right)^2 h^{ab} \frac{\partial N}{\partial \phi^a} \frac{\partial N}{\partial \phi^b} \,, \tag{40}$$

therefore from Eq. (38)

$$n_{\mathcal{R}} - 1 = 2\frac{\dot{H}}{H^2} - 2\frac{\frac{\partial N}{\partial \phi^a} \left(\frac{\dot{\phi}^a \dot{\phi}^d}{H^2} + \frac{1}{3} R^a{}_{bc}{}^d \frac{\dot{\phi}^b \dot{\phi}^c}{H^2} - \frac{V^{;ad}}{V} \right) \frac{\partial N}{\partial \phi^d}}{h^{ef} \frac{\partial N}{\partial \phi^e} \frac{\partial N}{\partial \phi^f}} \,. \tag{41}$$

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Therefore, from Eqs. (18) and (22),

$$n_{\mathcal{R}} - 1 \;=\; 2\frac{\dot{H}}{H^2} - 2\frac{1 + \frac{\partial N}{\partial \phi^a}\left(\frac{1}{3}R^{abcd}\frac{V_{;b}V_{;c}}{V^2} - \frac{V^{;ad}}{V}\right)\frac{\partial N}{\partial \phi^d}}{h^{ef}\frac{\partial N}{\partial \phi^e}\frac{\partial N}{\partial \phi^f}} \,, \tag{42}$$

$$\;=\; \frac{\left[2(\ln V)^{;a}{}_{;b} + \left(\frac{2}{3}R^a{}_{cbd} - h^a{}_b h_{cd}\right)(\ln V)^{;c}(\ln V)^{;d}\right]N_{;a}N^{;b}}{N_{;e}N^{;e}} \,. \tag{43}$$

5   Summary

The spectrum of gravitational waves produced during inflation is [21]

$$P_g = \left(\frac{H}{2\pi}\right)^2 \,, \tag{44}$$

where, here and below, the right hand side of the equation is to be evaluated at the time when the relevant scale passed through the Hubble radius during inflation, i.e. at $aH = k$. The spectrum of curvature perturbations produced during inflation is

$$P_{\mathcal{R}} = \left(\frac{H}{2\pi}\right)^2 h^{ab}\frac{\partial N}{\partial \phi^a}\frac{\partial N}{\partial \phi^b} \,, \tag{45}$$

where N is the number of e-folds of expansion from a given point in scalar field space during inflation to some reference energy density or Hubble parameter during radiation domination, i.e. after complete reheating.[4] The spectral index of the gravitational waves is [8]

$$n_g = 2\frac{\dot{H}}{H^2} \,. \tag{46}$$

The spectral index of the curvature perturbations is

$$n_{\mathcal{R}} - 1 \;=\; 2\frac{\dot{H}}{H^2} - 2\frac{1 + \frac{\partial N}{\partial \phi^a}\left(\frac{1}{3}R^{abcd}\frac{V_{;b}V_{;c}}{V^2} - \frac{V^{;ad}}{V}\right)\frac{\partial N}{\partial \phi^d}}{h^{ef}\frac{\partial N}{\partial \phi^e}\frac{\partial N}{\partial \phi^f}} \,, \tag{47}$$

$$\;=\; \frac{\left[2(\ln V)^{;a}{}_{;b} + \left(\frac{2}{3}R^a{}_{cbd} - h^a{}_b h_{cd}\right)(\ln V)^{;c}(\ln V)^{;d}\right]N_{;a}N^{;b}}{N_{;e}N^{;e}} \,. \tag{48}$$

Using Eq. (18) we see that inflation predicts

$$\frac{P_g}{P_{\mathcal{R}}} \leq |n_g| \,, \tag{49}$$

so that the ratio of gravitational wave to curvature perturbation contributions to the cosmic microwave background anisotropy satisfies $R \lesssim 5|n_g|$. [5] Note that

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[4]We do not consider the case of isocurvature perturbations that persist until the present.

[5]We follow [16] in using the factor of ~ 5. The results of this paper, and in particular Eq. (49), are not valid for an inflationary model that cannot be realized using any number of interacting scalar fields minimally coupled to Einstein gravity [22].

the spectral index depends on the curvature of the space of scalar fields as well as the potential, though as (in the opinion of EDS) realistic models of inflation tend to give $(V'/V)^2 \ll |V''/V|$ this may be difficult to observe. An interesting point is that in models with more than one dynamical degree of freedom there is generally not a unique inflationary trajectory and so the initial conditions might play a role in determining the spectrum and hence be observable.

Acknowledgements

References

[1] A. D. Linde, Particle Physics and Inflationary Cosmology (Harwood Academic, Switzerland, 1990); E. W. Kolb and M. S. Turner, The Early Universe (Addison-Wesley, New York, 1990).

[2] M. Dine, W. Fischler and D. Nemeschansky, Phys. Lett. 136B (1984) 169; G. D. Coughlan, R. Holman, P. Ramond and G. G. Ross, Phys. Lett. 140B (1984) 44; O. Bertolami and G. G. Ross, Phys. Lett. B183 (1987) 163.

[3] E. J. Copeland, A. R. Liddle, D. H. Lyth, E. D. Stewart and D. Wands, Phys. Rev. D49 (1994) 6410.

[4] M. Dine, L. Randall and S. Thomas, Phys. Rev. Lett. 75 (1995) 398.

[5] E. D. Stewart, Phys. Rev. D51 (1995) 6847.

[6] M. K. Gaillard, H. Murayama and K. A. Olive, Phys. Lett. B355 (1995) 71.

[7] S. W. Hawking, Phys. Lett. 115B (1982) 295; A. A. Starobinsky, Phys. Lett. 117B (1982) 175; A. H. Guth and S.-Y. Pi, Phys. Rev. Lett. 49 (1982) 1110; J. M. Bardeen, P. J. Steinhardt and M. S. Turner, Phys. Rev. D28 (1983) 679; D. H. Lyth, Phys. Rev. D31 (1985) 1792; V. F. Mukhanov, JETP Lett. 41 (1985) 493; M. Sasaki, Prog. Theor. Phys. 76 (1986) 1036.

[8] A. R. Liddle and D. H. Lyth, Phys. Lett. B291 (1992) 391.

[9] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rep. 215 (1992) 203.

[10] E. D. Stewart and D. H. Lyth, Phys. Lett. B302 (1993) 171.

[11] A. R. Liddle and D. H. Lyth, Phys. Rep. 231 (1993) 1.

[12] E. D. Stewart, Phys. Lett. B345 (1995) 414.

[13] A. A. Starobinsky, JETP Lett. 42 (1985) 152.

[14] L. A. Kofman and D. Yu. Pogosyan, Phys. Lett. B214 (1988) 508; D. S. Salopek, J. R. Bond and J. M. Bardeen, Phys. Rev. D40 (1989) 1753.

[15] A. A. Starobinsky and J. Yokoyama, gr-qc/9502002.

[16] D. Polarski and A. A. Starobinsky, Phys. Lett. B356 (1995) 196.

[17] J. García-Bellido and D. Wands, gr-qc/9506050.

[18] D. S. Salopek, astro-ph/9506146.

[19] J. M. Bardeen, Phys. Rev. D22 (1980) 1882.

[20] H. Kodama and M. Sasaki, Prog. Theor. Phys. Supp. 78 (1984) 1.

[21] A. A. Starobinsky, JETP Lett. 30 (1979) 683.

[22] A. A. Starobinsky, Sov. Astron. Lett. 9 (1983) 302; Sov. Astron. Lett. 11 (1985) 133; private communication.