Linearization Instability of Gravitational Waves Interacting with Matter in General Relativity

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Abstract

The gravitational wave solutions obtained from a perturbation about conformally flat backgrounds in Einstein gravity are investigated. A perturbation theory analysis of the Lesame, Ellis and Dunsby results, based on a covariant approach, shows that for gravitational waves interacting with irrotational dust, the equations are linearization unstable. The gravitational wave equations based on the Weyl curvature tensor must be solved by non-perturbative methods. The significance of this result for gravitational wave calculations and experiments is discussed.

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I. INTRODUCTION

Perturbations of spatially homogeneous and isotropic universes have been investigated by several authors, beginning with the work of Lifshitz [1], Bonnor [2], Lifshitz and Khalatnikov [3], Ehlers [4], Hawking [5], Sachs and Wolfe [6], D'Eath [7] and Moncrief [8]. More recently, a fully gauge invariant formulation of perturbation theory has been presented by Bruni, Dunsby, and Ellis [9]. By using the lemma of Stewart and Walker [10], they derived covariant equations from the Bianchi and Ricci identities, which are gauge invariant with respect to a zero-order conformally flat background, such as the Friedmann, Robertson, and Walker model (FRW). Following this work, Lesame, Ellis, and Dunsby [11,12] proved that a tetrad frame exists in which the shear tensor and its covariant time derivative are diagonalizable, if and only if the divergence of the magnetic Weyl tensor vanishes. Moreover, using the Ricci constraints they showed that the magnetic part of the Weyl tensor also vanishes. This result is applied to a perturbative analysis of the constraint equations and shows that it leads to a linearization instability in the perturbative expansions about a conformally flat spacetime. The implications for gravitational wave calculations in GR and for future gravitational wave experiments is considered in the following.

II. FIELD EQUATIONS AND IDENTITIES

We shall assume Einstein's field equations in the form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu},$$
(1)

where Λ is the cosmological constant, and $T_{\mu\nu}$ is the stress-energy momentum tensor of the matter, described by a fluid. The Riemann and Ricci tensors are given by

$$u_{\mu;[\nu\sigma]} = 2R^{\alpha}{}_{\mu\sigma\nu}u_{\alpha}, \quad R_{\mu\nu} = R^{\ \alpha}_{\mu\nu\alpha}. \tag{2}$$

We have for a perfect fluid:

$$T_{\mu\nu} = \rho u_{\mu} u_{\nu} + p h_{\mu\nu}, \tag{3}$$

where ρ is the density, p is the pressure, u_{μ} is the velocity of the fluid, $u_{\mu}u^{\mu} = -1$, and $h_{\mu\nu} = g_{\mu\nu} + u_{\mu}u_{\nu}$ is the projection operator into the hyperplane orthogonal to $u_{\mu} : h_{\mu\nu}u^{\nu} = 0$. The gradient of the velocity vector is decomposed as

$$u_{\mu;\nu} = \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3}h_{\mu\nu}\theta - \dot{u}_{\mu}u_{\nu},$$
(4)

where $\dot{u}_{\mu} = u_{\mu;\nu}u^{\nu}$ is the acceleration, $\theta = u_{\mu}^{;\mu}$ is the expansion, $\sigma_{\mu\nu} = u_{(\alpha;\beta)}h^{\alpha}_{\mu}h^{\beta}_{\nu} - \frac{1}{3}h_{\mu\nu}\theta$ is the shear, and $\omega_{\mu\nu} = u_{[\alpha;\beta]}h^{\alpha}_{\mu}h^{\beta}_{\nu}$ is the rotation of the flow lines u_{μ} .

The Weyl curvature tensor can be decomposed into the Ricci tensor $R_{\mu\nu}$ and the Riemann tensor $R_{\mu\nu\alpha\beta}$:

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + g_{\mu[\beta}R_{\alpha]\nu} + g_{\nu[\alpha}R_{\beta]\mu} + \frac{1}{3}Rg_{\mu[\alpha}g_{\beta]\nu},\tag{5}$$

and $C_{\mu\nu\alpha\beta} = C_{[\mu\nu][\alpha\beta]}$.

The "electric" and "magnetic" components of the Weyl tensor are defined by

$$E_{\mu\nu} \equiv E_{(\mu\nu)} = C_{\mu\alpha\nu\beta} u^{\alpha} u^{\beta}, \qquad (6)$$

and

$$H_{\mu\nu} \equiv H_{(\mu\nu)} = \frac{1}{2} \epsilon_{\mu\gamma}{}^{\alpha\beta} C_{\alpha\beta\nu\sigma} u^{\gamma} u^{\sigma}, \qquad (7)$$

where $\epsilon_{\mu\nu\alpha\beta}$ is the Levi-Cevita symbol, $E_{\mu\nu}$ and $H_{\mu\nu}$ satisfy: $E_{\mu\nu}u^{\nu} = 0, E_{\mu}^{\mu} = 0$ and $H_{\mu\nu}u^{\nu} = 0, H_{\mu}^{\mu} = 0.$

The Bianchi identities read:

$$R_{\mu\nu[\alpha\beta;\sigma]} = 0, \tag{8}$$

while the Ricci identities are given in (2).

III. PROPAGATION AND CONSTRAINT EQUATIONS

We shall assume that the vorticity, $\omega_{\mu\nu}$, and the pressure, p, are negligible. Then, the Bianchi identities give Maxwell-type equations [5,9]:

$$h^{\nu}{}_{\mu}E_{\nu\alpha;\beta}h^{\alpha\beta} - \epsilon_{\mu\nu\alpha\beta}u^{\nu}\sigma^{\alpha}{}_{\rho}H^{\beta\rho} = \frac{8\pi}{3}Gh^{\nu}{}_{\mu}\rho_{;\nu}, \qquad (9a)$$

$$h^{\nu}{}_{\mu}H_{\nu\alpha;\beta}h^{\alpha\beta} - \epsilon_{\mu\nu\alpha\beta}u^{\nu}\sigma^{\alpha}{}_{\rho}E^{\beta\rho} = 0, \qquad (9b)$$

$$\dot{E}_{\mu\nu} + E_{\mu\nu}\theta + h^{\gamma}{}_{(\mu}\epsilon_{\nu)\alpha\beta\rho}u^{\alpha}H_{\gamma}{}^{\beta;\rho} - E^{\alpha}{}_{(\mu}\sigma_{\nu)\alpha}$$
$$-\epsilon_{\mu\alpha\beta\sigma}\epsilon_{\nu\rho\tau\delta}u^{\alpha}u^{\rho}\sigma^{\beta\tau}E^{\sigma\delta} = -4\pi G\rho\sigma_{\mu\nu}, \qquad (9c)$$

$$\dot{H}_{\mu\nu} + H_{\mu\nu}\theta - h^{\gamma}{}_{(\mu}\epsilon_{\nu)\alpha\beta\rho}u^{\alpha}E_{\gamma}{}^{\beta;\rho} - H^{\alpha}{}_{(\mu}\sigma_{\nu)\alpha} - \epsilon_{\mu\alpha\beta\sigma}\epsilon_{\nu\rho\tau\delta}u^{\alpha}u^{\rho}\sigma^{\beta\tau}H^{\sigma\delta} = 0,$$
(9d)

where we have used $\dot{u}_{\mu} = 0$ for p = 0.

The expansion parameter θ obeys the Raychaudhuri equation:

$$\dot{\theta} + \frac{1}{3}\theta^2 + 2\sigma^2 + 4\pi G\rho = 0, \tag{10}$$

where $\sigma^2 = \frac{1}{2} \sigma^{\mu\nu} \sigma_{\mu\nu}$. The shear tensor satisfies the evolution equation:

$$\dot{\sigma}_{\mu\nu} + \sigma_{\mu\alpha}\sigma^{\alpha}{}_{\nu} - \frac{2}{3}\theta\sigma_{\mu\nu} + E_{\mu\nu} = 0.$$
(11)

The Ricci constraint equations obtained from (2) are given by

$$h^{\mu}{}_{\nu}\left(\frac{2}{3}\theta^{;\nu} - h^{\beta}{}_{\alpha}\sigma^{\nu\alpha}{}_{;\beta}\right) = 0, \qquad (12)$$

and

$$H_{\mu\nu} = -h^{\sigma}{}_{\mu}h^{\rho}{}_{\nu}\sigma_{(\sigma}{}^{\tau;\delta}\epsilon_{\rho)\kappa\tau\delta}u^{\kappa}.$$
(13)

IV. THE BACKGROUND SPACETIME AND THE CONSTRAINT EQUATIONS

Since the unperturbed background spacetime is conformally flat, the Weyl tensor vanishes, as do the projected fields $E_{\mu\nu}$, $H_{\mu\nu}$, the shear tensor $\sigma_{\mu\nu}$ and the vorticity tensor $\omega_{\mu\nu}$. Moreover, $u_{\mu} = \tau_{;\mu}$, where τ measures the proper time along the world lines. The spacetime is homogeneous and isotropic with 3-surfaces of constant curvature. For the FRW universe, the metric is

$$ds^2 = -d\tau^2 + R^2 d\sigma^2, \tag{14}$$

where $R = R(\tau)$ and $d\sigma^2$ is the line element of space with zero, unit positive or negative curvature.

It has been shown by Hawking [5] that for p = 0 the rotation satisfies: $\omega = \omega_0/R^2$, so that the rotation dies away as the universe expands, in accord with our assumption that we can neglect $\omega_{\mu\nu}$.

Let us define

$$\Sigma_{\mu\beta\rho} = \epsilon_{\mu\nu\alpha\beta} u^{\nu} \sigma_{\rho}{}^{\alpha}.$$

We can write equation (9a) and (9b) as

$$(\operatorname{div} E)_{\mu} - \Sigma_{\mu\beta\sigma} H^{\beta\sigma} = \frac{8\pi}{3} G h^{\nu}{}_{\mu} \rho_{;\nu}, \qquad (15)$$

and

$$(\operatorname{div} H)_{\mu} - \Sigma_{\mu\beta\sigma} E^{\beta\sigma} = 0.$$
(16)

By considering the proper time derivative of the constraint equation (12):

$$h^{\mu\nu}[\frac{2}{3}(\theta_{;\nu})\cdot -h^{\beta}{}_{\alpha}(\sigma^{\alpha}{}_{\nu;\beta})\cdot] = 0, \qquad (17)$$

and using a tetrad frame, Lesame et al., [11] obtained:

$$\sigma_1 H_1 = \sigma_2 H_2 = \sigma_3 H_3, \quad (1 + \lambda + \lambda^2) \sigma H = 0, \tag{18}$$

where we used the notation $\sigma_i = \sigma_{ii}$ (no sum), $H_i = H_{ii}$ (no sum) (i = 1, 2, 3) and λ is a constant. Moreover, for the diagonal case:

$$\sigma_1 = \sigma, \quad \sigma_2 = \lambda \sigma,$$

and

$$H_1 = \lambda H, \quad H_2 = H.$$

Also, the trace-free condition: $H_{\mu}{}^{\mu} = 0$ gives

$$\sigma_3 = -(1+\lambda)\sigma, \quad H_3 = -(1+\lambda)H.$$

For arbitrary λ and $\sigma \neq 0$, it follows that H = 0. For $1 + \lambda + \lambda^2 = 0$, the values of λ are complex, but in tetrad form:

$$|\sigma|^{2} = \frac{1}{2}(1+\lambda+\lambda^{2})\sigma^{2} = 0, \quad |H|^{2} = (1+\lambda+\lambda^{2})H^{2} = 0$$

so both the shear and the magnetic Weyl tensor are zero, which also follows for the case when $\sigma = 0$, since this implies that the spacetime is FRW for which E = H = 0. This leads to the Lesame, Ellis and Dunsby theorem [11,12]:

For irrotational dust, the divergence of the magnetic Weyl tensor vanishes:

 $(\operatorname{div} H)_{\mu} = 0$ for a shear tensor that is diagonalizable in a tetrad

frame, if and only if the magnetic Weyl tensor vanishes, $H_{\mu\nu} = 0$.

V. LINEAR GRAVITATIONAL WAVE EQUATION

We shall now expand the fields $E_{\mu\nu}$, $H_{\mu\nu}$ and the tensor $\sigma_{\mu\nu}$ in a power series with respect to a small parameter $\epsilon \ll 1$:

$$E_{\mu\nu} = \epsilon E_{\mu\nu}^{(1)} + \epsilon^2 E_{\mu\nu}^{(2)} + O(\epsilon^3), \qquad (19a)$$

$$H_{\mu\nu} = \epsilon H_{\mu\nu}^{(1)} + \epsilon^2 H_{\mu\nu}^{(2)} + O(\epsilon^3), \qquad (19b)$$

$$\sigma_{\mu\nu} = \epsilon \sigma_{\mu\nu}^{(1)} + \epsilon^2 \sigma_{\mu\nu}^{(2)} + O(\epsilon^3).$$
(19c)

Let us consider the first-order equations resulting from Eqs. (9c) and (9d):

$$\dot{E}^{(1)}_{\mu\nu} + E^{(1)}_{\mu\nu}\theta + h^{\gamma}{}_{(\mu}\epsilon_{\nu)\alpha\beta\rho}u^{\alpha}H_{\gamma}{}^{(1)\beta;\rho} = -4\pi G\rho\sigma^{(1)}_{\mu\nu},$$
(20)

$$\dot{H}^{(1)}_{\mu\nu} + H^{(1)}_{\mu\nu}\theta - h^{\gamma}{}_{(\mu}\epsilon_{\nu)\alpha\beta\rho}u^{\alpha}E_{\gamma}{}^{(1)\beta;\rho} = 0.$$
(21)

By multiplying (20) by $\partial/\partial \tau = u^{\alpha} \nabla_{\alpha}$ and (21) by $h^{\mu}{}_{(\rho} \epsilon_{\beta})^{\gamma \nu \tau} u_{\gamma} \nabla_{\tau}$, we get [5]

$$\ddot{E}^{(1)}_{\mu\nu} - \Delta^2 E^{(1)}_{\mu\nu} + \frac{7}{3} \dot{E}^{(1)}_{\mu\nu} \theta + E^{(1)}_{\mu\nu} (\dot{\theta} + \frac{4}{3}\theta^2 + \frac{8\pi}{3}G\rho) + 8\pi G\sigma^{(1)}_{\mu\nu} (\frac{1}{3}\rho + \frac{1}{2}\dot{\rho}) = 0,$$
(22)

where

$$\Delta^2 E_{\mu\nu} = (E^{(1)}_{\alpha\beta;\tau} h^{\alpha}{}_{\delta} h^{\beta}{}_{\rho} h^{\tau}{}_{\gamma})_{;\sigma} h^{\gamma\sigma} h^{\delta}{}_{\mu} h^{\rho}{}_{\nu}$$

is the Laplacian operator in the hypersurface $\tau = \text{constant}$. It has the eigenfunction expansion:

$$E_{\mu\nu} = \Sigma_i a^{(i)} K^{(i)}_{\mu\nu}$$

with $\dot{K}^{(i)}_{\mu\nu} = 0.$

For a non-expanding congruence u^{α} and empty Minkowski spacetime, Eq. (22) reduces to the wave equation

$$\Box E_{\mu\nu} = 0. \tag{23}$$

VI. LINEARIZATION INSTABILITY

Equating coefficients of powers of ϵ , we obtain from Eqs. (9a) and (9b):

$$(\operatorname{div} E^{(1)})_{\mu} = \frac{8\pi}{3} G h^{\nu}{}_{\mu} \rho_{;\nu},$$
 (24a)

$$(\operatorname{div} H^{(1)})_{\mu} = 0,$$
 (24b)

$$(\operatorname{div} E^{(2)})_{\mu} - \Sigma^{(1)}_{\mu\nu\sigma} H^{(1)\nu\sigma} = \frac{8\pi}{3} G h^{\nu}{}_{\mu} \rho_{;\nu}, \qquad (24c)$$

$$(\operatorname{div} H^{(2)})_{\mu} - \Sigma^{(1)}_{\mu\nu\sigma} E^{(1)\nu\sigma} = 0,$$
 (24d)

$$(\operatorname{div} E^{(n)})_{\mu} - \sum_{i=1}^{n-1} \Sigma^{(i)}_{\mu\nu\sigma} H^{(n-i)\nu\sigma} = \frac{8\pi}{3} G h^{\nu}{}_{\mu} \rho_{;\nu}, \qquad (24e)$$

$$(\text{div } H^{(n)})_{\mu} - \sum_{i=1}^{n-1} \Sigma^{(i)}_{\mu\nu\sigma} E^{(n-i)\nu\sigma} = 0, \qquad (24\text{f})$$

and from Eqs. (9c) and (9d):

$$\dot{E}^{(1)}_{\mu\nu} + E^{(1)}_{\mu\nu}\theta + h^{\gamma}{}_{(\mu}\epsilon_{\nu)\alpha\beta\rho}u^{\alpha}H_{\gamma}{}^{(1)\beta;\rho} = -4\pi G\rho\sigma^{(1)}_{\mu\nu}, \qquad (25a)$$

$$\dot{H}^{(1)}_{\mu\nu} + H^{(1)}_{\mu\nu}\theta - h^{\gamma}{}_{(\mu}\epsilon_{\nu)\alpha\beta\rho}u^{\alpha}E_{\gamma}{}^{(1)\beta;\rho} = 0, \qquad (25b)$$

$$\dot{E}^{(2)}_{\mu\nu} + E^{(2)}_{\mu\nu}\theta + h^{\gamma}{}_{(\mu}\epsilon_{\nu)\alpha\beta\rho}u^{\alpha}H_{\gamma}{}^{(2)\beta;\rho} - E^{(1)\alpha}{}_{(\mu}\sigma^{(1)}{}_{\nu)\alpha} -\epsilon_{\mu\alpha\beta\sigma}\epsilon_{\nu\rho\tau\delta}u^{\alpha}u^{\rho}\sigma^{(1)\beta\tau}E^{(1)\sigma\delta} = -4\pi G\rho\sigma^{(2)}_{\mu\nu}, \qquad (25c)$$
$$\dot{H}^{(2)}_{\mu\nu} + H^{(2)}_{\mu\nu}\theta - h^{\gamma}{}_{(\mu}\epsilon_{\nu)\alpha\beta\rho}u^{\alpha}E_{\gamma}{}^{(2)\beta;\rho} - H^{(1)\alpha}{}_{(\mu}\sigma^{(1)}{}_{\nu)\alpha} -\epsilon_{\mu\alpha\beta\sigma}\epsilon_{\nu\rho\tau\delta}u^{\alpha}u^{\rho}\sigma^{(1)\beta\tau}H^{(1)\sigma\delta} = 0, \qquad (25d)$$

$$\dot{E}_{\mu\nu}^{(n)} + E_{\mu\nu}^{(n)}\theta + h^{\gamma}{}_{(\mu}\epsilon_{\nu)\alpha\beta\rho}u^{\alpha}H_{\gamma}{}^{(n)\beta;\rho} - \sum_{i=1}^{n-1}E^{(i)\alpha}{}_{(\mu}\sigma^{(n-i)}{}_{\nu)\alpha} -\epsilon_{\mu\alpha\beta\sigma}\epsilon_{\nu\rho\tau\delta}u^{\alpha}u^{\rho}\sum_{i=1}^{n-i}\sigma^{(i)\beta\tau}E^{(n-i)\sigma\delta} = -4\pi G\rho\sigma_{\mu\nu}^{(n)},$$
(25e)
$$\dot{H}_{\mu\nu}^{(n)} + H_{\mu\nu}^{(n)}\theta - h^{\gamma}{}_{(\mu}\epsilon_{\nu)\alpha\beta\rho}u^{\alpha}E_{\gamma}{}^{(n)\beta;\rho} - \sum_{i=1}^{n-1}H^{(i)\alpha}{}_{(\mu}\sigma^{(n-i)}{}_{\nu)\alpha} -\epsilon_{\mu\alpha\beta\sigma}\epsilon_{\nu\rho\tau\delta}u^{\alpha}u^{\rho}\sum_{i=1}^{n-1}\sigma^{(i)\beta\tau}H^{(n-i)\sigma\delta} = 0.$$
(25f)

By inserting the infinite power series in ϵ for $H_{\mu\nu}$ into the result of the Lesame, Ellis and Dunsby theorem: $(\text{div}H)_{\mu} = 0$ and $H_{\mu\nu} = 0$, we find that $H_{\mu\nu}$ vanishes in every order. Then, Eq. (24b) leads to the result:

$$H_{\mu\nu}^{(1)} = 0, (26)$$

and to first-order we get

$$(\operatorname{div} E^{(1)})_{\mu} = \frac{8\pi}{3} G h^{\nu}{}_{\mu} \rho_{;\nu},$$
 (27a)

$$\dot{E}^{(1)}_{\mu\nu} + E^{(1)}_{\mu\nu}\theta = -4\pi G\rho\sigma^{(1)}_{\mu\nu},\tag{27b}$$

$$\operatorname{curl} E^{(1)}_{\mu\nu} = h^{\gamma}{}_{(\mu}\epsilon_{\nu)\alpha\beta\rho} u^{\alpha} E_{\gamma}{}^{(1)\beta;\rho} = 0.$$
(27c)

In second-order, we obtain

$$(\operatorname{div} E^{(2)})_{\mu} = \frac{8\pi}{3} G h^{\nu}{}_{\mu} \rho_{;\nu},$$
 (28a)

$$(\operatorname{div} H^{(2)})_{\mu} - \Sigma^{(1)}_{\mu\nu\sigma} E^{(1)\nu\sigma} = 0,$$
 (28b)

$$\dot{E}^{(2)}_{\mu\nu} + E^{(2)}_{\mu\nu}\theta + h^{\gamma}{}_{(\mu}\epsilon_{\nu)\alpha\beta\rho}u^{\alpha}H_{\gamma}{}^{(2)\beta;\rho} - E^{(1)\alpha}{}_{(\mu}\sigma^{(1)}{}_{\nu)\alpha} -\epsilon_{\mu\alpha\beta\sigma}\epsilon_{\nu\rho\tau\delta}u^{\alpha}u^{\rho}\sigma^{(1)\beta\tau}E^{(1)\sigma\delta} = -4\pi G\rho\sigma^{(2)}_{\mu\nu}, \qquad (28c)$$

$$\dot{H}^{(2)}_{\mu\nu} + H^{(2)}_{\mu\nu}\theta - h^{\gamma}{}_{(\mu}\epsilon_{\nu)\alpha\beta\rho}u^{\alpha}E_{\gamma}{}^{(2)\beta;\rho} = 0.$$
(28d)

Thus, neither the first-order nor the second-order of perturbation theory predict the same evolution as the rigorous theory. This means that there is a linearization instability and the gravitational wave equations can only be solved by non-perturbative methods. It is not possible to derive the standard linear wave equation (23) in empty spacetime, because of the absence of a curl $H^{(1)}$ term in the first-order equations. This confirms the Lesame et al., conclusions [11]. However, our analysis reveals that the breakdown of the linearization scheme follows inevitably from the Lesame, Ellis and Dunsby theorem, when a power series solution of the equations is sought.

VII. CONSEQUENCES FOR GRAVITATIONAL WAVE EXPERIMENTS

Considerable work has been devoted to studying the linearization stability of gravitational wave perturbation theory in Minkowski spacetime [8,13,14]. It demonstrated that Cauchy stability can be proved and that the linear approximation of the field equations can be expected to be stable for the empty space Einstein equations. However, the problem of perturbative stability for cosmological solutions, such as the FRW solution of the Einstein field equations, is a more difficult issue to resolve. The full Einstein field equations have (under certain conditions) a Cauchy solution which is stable, but potential difficulties arise when a perturbative analysis of the equations is performed due to the role of the Bianchi and Ricci constraints. This is borne out by the results of Lesame et al., who consistently retain the Ricci constraints in their analysis by demanding that they hold in the time evolution of the initial data. A central issue is that the Lesame et al., analysis is based on a fully gauge invariant (covariant) perturbation theory. As first stressed by Hawking [5], a non-gauge invariant analysis based on metric perturbations is not reliable and can yield misleading results. He therefore suggested that the perturbation theory be based on an analysis of the curvature tensors.

In the proposed gravitational wave experiments such as the LIGO/VIRGO or LISA projects [15], which attempt to detect gravitational radiation directly, the gravitational

waves arrive at the detector after travelling through the universe described by a cosmological model such as the standard FRW model. In these models there is always a nonvanishing density ρ i.e., the estimated density of the smoothed out fluid of the universe is $\rho \sim 10^{-29} \,\mathrm{g\,cm^{-3}}$. Even though this is a small density it will play an important role in the perturbative stability analysis, because the linearization instability of the gravitational wave equations in the presence of an irrotational dust is expected to be singular, in the sense that the solutions of the gravitational wave equations become singular in the limit of a conformally flat FRW solution. Thus, only non-pertubative or non-linear numerical solutions of the gravitational wave equations, based on a gauge invariant set of equations within a cosmological model scenario, can be trusted to produce reliable physical predictions for gravitational wave experiments.

The proposed strong gravitational wave sources which one hopes can produce detectable radiation, such as the coalescence of neutron star and stellar mass black hole binaries in distant galaxies or colliding black holes, involve strong gravitational fields with non-vanishing matter and shear in the close vicinity of the sources, requiring a non-perturbative solution of the wave equations.

Although the conclusions drawn about the linearization instability of the cosmological gravitational wave equations by Lesame et al., and in the present work, is based on irrotational dust, such an approximation is expected to be very reliable in the present universe, since the vorticity and pressure are vanishingly small. Therefore, the linearization instability of the gravitational wave solutions travelling through a medium discussed here, *plays an important role for realistic gravitational wave calculations and experiments in the present universe.*

Perhaps, this kind of linearization instability of the gravitational wave equations, arises from a breaking of the conformal symmetry of the cosmological model e.g., the FRW model which is conformally invariant with a constant curvature. Further work should be carried out to understand the mechanism associated with this symmetry breaking, which is responsible for producing the linearization instability. This mechanism may be related to the result obtained by Kundt and Trümper and Szekeres [16,17], that in an exact treatment of GR, no Petrov type N irrotational dust solutions exist.

VIII. CONCLUSIONS

A perturbative analysis of the Weyl curvature Maxwell-type equations and the associated constraints about a conformally flat background, which was chosen to be represented by the FRW model, led to a linearization instability when the Lesame, Ellis and Dunsby theorem was invoked. This confirmed the conclusions about linearization instability inferred by Lesame et al., [11,12].

It is important to emphasize that this instability is *inevitable* already at first-order in a conventional power series expansion of the Weyl curvature equations in a small parameter ϵ , and reflects itself in the higher order equations. The Ricci constraints must be imposed at the lowest order, which results in $H^{(1)}_{\mu\nu}$ vanishing and the breakdown of the power series perturbation theory. Thus, the perturbative instability is a *generic feature* of any consistent expansion about an FRW backround, based on the covariant Weyl curvature approach. Perhaps, a reinvestigation of the Cauchy perturbative instability of gravitational wave equations, expanded about an irrotational dust solution, is needed to fully understand the implications of this result.

Since gravitational wave experiments are conducted for strong gravitational field sources with non-vanishing matter density and shear, and since the gravitational waves that the experiments hope to detect travel through an irrotational dust in the present universe, then the whole issue of the validity of the perturbative calculations of gravitational wave solutions should be treated with caution. This may become an important topic of study in numerical calculations of gravitational waves in the near-linear regime [18].

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