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The spectrum of cosmological perturbations produced by a multi-component inflaton to second order in the slow-roll approximation

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Abstract

We derive general analytic formulae for the power spectrum and spectral index of the curvature perturbation produced during inflation driven by a multi-component inflaton field, up to the second order in the slow-roll approximation. We do not assume any specific properties of the potential or the metric on the scalar field space, except for the slow-roll condition, Einstein gravity, and the absence of any permanent isocurvature modes.

Key words: cosmology; inflation; perturbations
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1 Introduction

Inflation of the early universe [1] magnifies microscopic quantum fluctuations in the inflaton field ϕ into macroscopic classical perturbations in space-time and matter. The latter are supposed to be the seeds that grow to become the rich structures, such as galaxies or clusters of galaxies, that are observed today. Thus the power spectrum P and spectral index n predicted by a model of inflation can be tested from observation of the large-scale structures, and it is therefore important to calculate them as accurately as possible.

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Standard calculations [2] of P and n work to lowest order in the slow-roll approximation, and assume that ϕ has only one dynamical degree of freedom. However, the latter assumption has no theoretical or observational justification. Previously, Stewart & Lyth [3] computed P and n up to the second order in the slow-roll approximation, in the single component inflaton case. Sasaki & Stewart [4] derived general formulae for P and n in the multi-component inflaton case, but only up to the first order in the slow-roll approximation. In this paper, we calculate P and n for a multi-component inflaton up to the second order in the slow-roll approximation. The method of our calculation also refines the earlier one in Ref.[4]. We use the units $c = \hbar = 8\pi G = 1$.

2 Homogeneous background and slow-roll approximation

Let ϕ^a be the multiple scalar fields that slowly roll on the potential $V(\phi)$ during inflation. We start from the action of the form:

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}h_{ab}g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b - V(\phi) - \frac{1}{2}R\right] \tag{1}$$

where $g_{\mu\nu}$ and R are the metric and curvature scalar in space-time, and h_{ab} is the metric on the ϕ-space (which may be curved). In an exactly homogeneous universe, we have $\partial_i\phi^a = 0$ $(i = 1, 2, 3)$ and

$$ds^2 := g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a^2(t)\delta_{ij}dx^i dx^j \tag{2}$$

(we assume that the background universe is spatially flat). Then, from Eqs.1 and 2, $\dot\phi^a := \partial_0\phi^a(t)$ obeys the equation

$$\ddot\phi^a + 3H\dot\phi^a + V^{,a} = 0 \tag{3}$$

where $\ddot\phi^a := (D/dt)\dot\phi^a := \dot\phi^b\nabla_b\dot\phi^a$, ∇_a is the covariant derivative operator associated with h_{ab}, and $H := \dot a/a$ is the Hubble parameter. Raising and lowering of indices $abc\cdots$ are always done by h_{ab}. Also we have from the Einstein equation

$$3H^2 = \frac{1}{2}\dot\phi^a\dot\phi_a + V\,, \tag{4}$$

$$\dot H = -\frac{1}{2}\dot\phi^a\dot\phi_a\,. \tag{5}$$

We assume that the potential $V(\phi)$ has a sufficiently gentle slope:

$$V_{,a}V^{,a} \ll V^2, \quad (V_{;ab}V^{;ab})^{1/2} \ll V\,,\ \text{etc.} \tag{6}$$

(we list all the assumptions we need in this paper more rigorously in Appendix A). Then, from Eqs.3 and 4, $\phi^a(t)$ soon approaches a slowly rolling state given by

$$\frac{\dot\phi^a}{H} \simeq -\frac{V^{,a}}{3H^2} \simeq -\frac{V^{,a}}{V}\,. \tag{7}$$

Let us define

$$\alpha := -\frac{\dot H}{H^2}\,, \quad \beta := \frac{\ddot\phi^a\dot\phi_a}{H\dot\phi^a\dot\phi_a}\,. \tag{8}$$

These are small quantities ($\ll 1$) of the same order in the slow-roll approximation. They are slowly varying and their time derivatives (divided by H) are smaller quantities of the next order. For example

$$H^{-1}\dot\alpha = 2\alpha(\alpha+\beta) \tag{9}$$

is a second order quantity. We use the notation $\simeq$ when the equality is valid only up to the lowest order in the slow-roll approximation.

3 Perturbation

A scalar perturbation in the space-time metric is most generally written as [5]

$$ds^2 = (1+2A)dt^2 - 2a(\partial_i B)dtdx^i - a^2[(1+2\mathcal{R})\delta_{ij} + 2\partial_i\partial_j E]dx^i dx^j\,. \tag{10}$$

Here $\mathcal{R}$ is interpreted as the intrinsic-curvature perturbation in the constant time hypersurface. Let $\delta\phi^a$ be the perturbation in the scalar fields around $\phi^a(t)$. In appendix B, we derived from Eqs.1 and 10 the equation of motion of $\delta\phi^a$ on flat hypersurfaces:

$$\frac{D^2}{dt^2}\delta\phi^a + 3H\frac{D}{dt}\delta\phi^a + R^a{}_{cbd}\dot\phi^c\dot\phi^d\delta\phi^b + \left(\frac{k}{a}\right)^2\delta\phi^a + \delta\phi_b V^{;ab} = \frac{\delta\phi_b}{a^3}\frac{D}{dt}\left[\frac{a^3}{H}\dot\phi^a\dot\phi^b\right] \tag{11}$$

where $R^a{}_{bcd}$ is the Riemannian curvature tensor in the ϕ-space. We work in $\boldsymbol{k}$-space throughout and simply use $\delta\phi^a$ as the Fourier transform of the perturbation. The conformal time η is defined by

$$\eta := \int\frac{dt}{a} = -\frac{1}{aH} + \int\frac{\alpha da}{a^2 H}\,. \tag{12}$$

Since α is slowly varying (see Eq.9), we take α out of the integral and obtain

$$\eta = -(1+\alpha)/(aH)\,. \tag{13}$$

Defining $u^a := a\delta\phi^a$ and working to the first order in α, β, etc., Eq.11 is rewritten as

$$\frac{D^2}{d\eta^2}u^a + k^2 u^a = \frac{1}{\eta^2}(2u^a + 3\epsilon^a_b u^b) \tag{14}$$

where we regard

$$\epsilon_{ab} := \alpha h_{ab} + \left(h_{ac}h_{bd} - \frac{1}{3}R_{acbd}\right)\frac{\dot\phi^c\dot\phi^d}{H^2} - \frac{V_{;ab}}{3H^2} \tag{15}$$

as a first order quantity. In order to solve the differential equation 14, we introduce the orthonormal basis $e^a_{\scriptscriptstyle A}$ (A runs over the number of scalar field components) parallel-transported along the unperturbed trajectory $\phi^a(t)$:

$$\frac{D}{d\eta}e^a_{\scriptscriptstyle A} = 0 \tag{16}$$

so that the symmetric tensor ϵ_{ab} is diagonalized along $\phi^a(t)$ as

$$\epsilon^{ab} \simeq \sum_A \epsilon_{\scriptscriptstyle A}(e^a_{\scriptscriptstyle A} \otimes e^b_{\scriptscriptstyle A})\,. \tag{17}$$

This diagonalization is justified as follows. At some point on $\phi^a(t)$, ϵ_{ab} can be diagonalized exactly as in Eq.17 with the eigen-vectors $e^a_{\scriptscriptstyle A}$. As one moves along $\phi^a(t)$ with $e^a_{\scriptscriptstyle A}$ parallel-transported, $e^a_{\scriptscriptstyle A}$ will not remain the eigen-vectors and off-diagonal components may appear in Eq.17. However, since we are assuming (see Eq.A2) that ϵ_{ab} is covariantly changing slowly along $\phi^a(t)$, the off-diagonal components are second order quantities. Therefore, Eq.17 is valid up to the lowest order in the slow-roll approximation. In short, we treated ϵ_{ab} as a constant in Eq.17, just as we treated α as a constant in Eq.12. From Eqs.16 and 17, the A-component of Eq.14 is written as

$$\frac{d^2 u_{\scriptscriptstyle A}}{d\eta^2} + k^2 u_{\scriptscriptstyle A} = \frac{1}{\eta^2}(2 + 3\epsilon_{\scriptscriptstyle A})u_{\scriptscriptstyle A} \tag{18}$$

where $u_{\scriptscriptstyle A} := u_a e^a_{\scriptscriptstyle A}$. First let us consider microscopic fluctuations, the physical wavelength of which is well-inside the horizon ($-k\eta \to \infty$). When $-k\eta \to \infty$, the right hand side

(RHS) of Eq.18 is negligible compared with the k^2 term, and thus the u_A behave like (real) massless Klein-Gordon fields:

$$u_A(\boldsymbol{k}) = \frac{1}{\sqrt{2k}}\left[a_A(\boldsymbol{k})e^{-ik\eta} + a_A^\dagger(-\boldsymbol{k})e^{ik\eta}\right] \tag{19}$$

where $a_A^\dagger$ and a_A are the creation and annihilation operators of an A-particle:

$$[a_A(\boldsymbol{k}), a_B^\dagger(\boldsymbol{k}')] = \delta_{AB}\delta^3(\boldsymbol{k}-\boldsymbol{k}'), \quad a_A|0\rangle = 0. \tag{20}$$

As $-k\eta$ approaches unity, the RHS of Eq.18 becomes comparable to the k^2 term, and the solution is written in terms of the Hankel functions as

$$u_A(\boldsymbol{k}) = (-k\eta)^{1/2}\left[C_A(\boldsymbol{k})H^{(1)}_{\nu_A}(-k\eta) + C_A^\dagger(-\boldsymbol{k})H^{(2)}_{\nu_A}(-k\eta)\right] \tag{21}$$

where

$$\nu_A := \frac{3}{2} + \epsilon_A\,. \tag{22}$$

Using the asymptotic behavior of the Hankel functions at infinity, the integral constants C_A are determined from Eq.19 as

$$C_A(\boldsymbol{k}) = \sqrt{\frac{\pi}{2}}\exp\left[\frac{i\pi}{4}(2\nu_A+1)\right]\frac{a_A(\boldsymbol{k})}{\sqrt{2k}}\,. \tag{23}$$

Next we go well-outside the horizon, i.e., $-k\eta \to 0$ [but $-\ln(-k\eta)$ is not too large]. Using $H^{(1,2)}_\nu(x) \to \pm\Gamma(\nu)(2/x)^\nu/(i\pi)$ as $x \to 0$, and expanding up to the first order in ϵ_A, one finds

$$u_A(\boldsymbol{k}) \to \frac{i}{\sqrt{2k}}\left(\frac{-1}{k\eta}\right)\frac{\Gamma(\nu_A)}{\Gamma(3/2)}\left(\frac{-2}{k\eta}\right)^{\epsilon_A} b_A(\boldsymbol{k}) \tag{24}$$

$$= \frac{i}{\sqrt{2k}}\left(\frac{-1}{k\eta}\right)\{1+[c-\ln(-k\eta)]\epsilon_A\}\, b_A(\boldsymbol{k}) \tag{25}$$

where

$$b_A(\boldsymbol{k}) := e^{i\pi\epsilon_A/2}a_A(\boldsymbol{k}) - e^{-i\pi\epsilon_A/2}a_A^\dagger(-\boldsymbol{k})\,, \tag{26}$$

and $c := 2 - \ln 2 - \gamma = 0.7296\cdots$ with the Euler number γ. It is clear from Eq.25 that the perturbations become completely classical as we go outside the horizon, because a_A

and $a^\dagger_A$ appear only in the combination of b_A and hence $[u_A, \dot{u}_A] = 0$ follows. Going back to general coordinates, we obtain

$$\delta\phi^a = \frac{iH}{\sqrt{2k^3}} \left\{ (1-\alpha) h^a_b + \left[c + \ln\left(\frac{aH}{k}\right)\right] \epsilon^a_b \right\} b^b \tag{27}$$

with $b^a := \sum_A b_A e^a_A$. For later use, we note that

$$\langle b_a(\boldsymbol{k}) b^\dagger_b(\boldsymbol{k}')\rangle = h_{ab}\delta^3(\boldsymbol{k} - \boldsymbol{k}') \tag{28}$$

where $\langle\cdots\rangle$ reads the vacuum expectation value.

4 Power spectrum and spectral index of $\mathcal{R}_c$

Sasaki & Stewart [4] showed that the curvature perturbation on a comoving hypersurface $\mathcal{R}_c$ (see Eq.10) during the radiation-dominated phase (after complete reheating) is related to $\delta\phi^a$ as

$$\mathcal{R}_c = N_{,a}\delta\phi^a \,. \tag{29}$$

The RHS is to be evaluated at some time (say t_1) during inflation soon after the scale of the perturbation goes well-outside the horizon (but does not depend on the exact value of t_1, as shown below). It is also assumed in Eq.29 that the space-time is foliated on a flat hypersurface at t_1, in accord with Eq.11. Here

$$N(\phi) := \int_{t_1(\phi)}^{t_2} H dt \tag{30}$$

is the number of e-folds in the background universe, and t_2 is the time corresponding to some fixed energy density during the radiation-dominated phase. In general N can depend on both $\phi^a(t_1)$ and $\dot{\phi}^a(t_1)$. However, as we are assuming that slow-roll has been achieved, the $\dot{\phi}$-dependence should be eliminated using the slow-roll trajectory which is given in Eq.C12 up to second order. The power spectrum $P(k)$ of $\mathcal{R}_c$ is defined by

$$\langle \mathcal{R}_c(\boldsymbol{k}) \mathcal{R}^\dagger_c(\boldsymbol{k}')\rangle = 2\pi^2 k^{-3} P(k) \delta^3(\boldsymbol{k} - \boldsymbol{k}') \,. \tag{31}$$

From Eqs.27, 28 and 29, one finds

$$P(k) = N^{,c} N_{,c} \left(\frac{H}{2\pi}\right)^2 \left\{ 1 - 2\alpha + 2\left[c + \ln\left(\frac{aH}{k}\right)\right] \epsilon_{ab} M^{ab} \right\} \tag{32}$$

where $M_{ab} := N_{,a}N_{,b}/N^{,c}N_{,c}$. Thus the spectral index is

$$n := 1 + \frac{d\ln P}{d\ln k} = 1 - 2\epsilon_{ab}M^{ab} \tag{33}$$

which is the identical result with Ref.[4]. In order to calculate n up to the second order, we rewrite Eq.32 to show more explicitly that $P(k)$ does not depend on time. We expand the prefactor of Eq.32 as

$$N^{,c}N_{,c}H^2 = (N^{,c}N_{,c}H^2)_{aH=k}\left[1 + \left.\frac{d\ln(N^{,c}N_{,c}H^2)}{d\ln a}\right|_{aH=k}\ln\left(\frac{aH}{k}\right)\right]. \tag{34}$$

Substituting Eq.34 into 32, and using Eq.C11, we see that the $\ln(aH/k)$ terms cancel and obtain

$$P(k) = N^{,c}N_{,c}\left(\frac{H}{2\pi}\right)^2(1 - 2\alpha + 2c\epsilon_{ab}M^{ab})\bigg|_{aH=k}. \tag{35}$$

In this expression for $P(k)$, the k-dependence of the LHS is such that the RHS (which is a function only of time) is evaluated at the horizon-crossing time $aH = k$. Thus $P(k)$ does not depend on time, as noted above. To avoid any confusion, let us define $Q(a)$ to be the RHS of Eq.35 so that

$$P(k) = Q(a)|_{aH=k}. \tag{36}$$

Then, using Eqs.9, C10 and C14, n is calculated up to second order as

$$n = 1 + \frac{d\ln Q}{d\ln aH} = 1 + (1+\alpha)\frac{d\ln Q}{d\ln a} \tag{37}$$

$$= 1 - 2\alpha + 2\lambda_{ab}M^{ab} - 2(3-2c)\alpha^2 - 4(1-c)\alpha\beta + \frac{8}{3}\alpha\lambda_{ab}M^{ab} + 4c(\lambda_{ab}M^{ab})^2$$
$$-\frac{2}{3}(6c-1)M_{ab}\lambda_a^c\lambda_{bc} - \frac{4}{3}\frac{N^{,a}}{N^{,c}N_{,c}}\frac{\dot\phi^b}{H}\frac{V_{;ab}}{3H^2} - \frac{2\alpha}{N^{,a}N_{,a}} - \frac{2}{3}(3c+1)\frac{M^{ab}}{H}\frac{D}{dt}\lambda_{ab} \tag{38}$$

where

$$\lambda_{ab} := \left(\frac{1}{3}R_{acbd} - h_{ac}h_{bd}\right)\frac{\dot\phi^c\dot\phi^d}{H^2} + \frac{V_{;ab}}{3H^2} \tag{39}$$
$$= \alpha h_{ab} - \epsilon_{ab}. \tag{40}$$

[Note that $1/(N^{,a}N_{,a})$ is a first order quantity.] The k-dependence of n is understood in the same way as in Eq.35. Rewriting Eq.38 in terms of V, we find

$$
\begin{aligned}
n = 1 &- U^{,a}U_{,a} + 2M^{ab}W_{ab} - \frac{1}{2}(U^{,a}U_{,a})^2 - \frac{2}{3}(3c-2)U^{;ab}U_{,a}U_{,b} + M^{ab}U_{;ab}U^{,c}U_{,c} \\
&+4c(M^{ab}W_{ab})^2 - \frac{2}{3}(6c-1)M^{ab}W_{ac}W^{c}_{b} + \frac{2}{3}(3c+1)M^{ab}[U_{;abc}U^{,c} + \frac{1}{3}R_{acbd;e}U^{,c}U^{,d}U^{,e}] \\
&+\frac{4}{9}M^{ab}R_{acbd}U^{,c}U_{,e}[U^{,e}U^{,d} + (3c+2)U^{;ed}] \qquad (41)
\end{aligned}
$$

where

$$U := \ln V \,, \qquad (42)$$

$$W_{ab} := U_{;ab} + \frac{1}{3}R_{acbd}U^{,c}U^{,d} \simeq \lambda_{ab} \,. \qquad (43)$$

($U_{,a}U_{,b}$ and $U_{;ab}$ are first order quantities.)

5 Summary

We have derived general analytic formulae for the power spectrum P (Eq.35) and spectral index n (Eq.38 or 41) of the curvature perturbation $\mathcal{R}_{\rm c}$ produced during inflation driven by a multi-component inflaton field, up to the second order in the slow-roll approximation. Once one specifies a model of inflation and calculates the number of e-folds $N(\phi)$ (Eq.30) in the background universe, then the substitution of them into our general formulae immediately yields the power spectrum and its index with accuracy. We anticipate that, to lowest order, $N_{,a}$ can be calculated from the inflationary phase only. However, to the next order, as considered in this paper, the contribution to $N_{,a}$ from the reheating and radiation-dominated phases should be significant.

The magnitude of the first order terms is of order $N_{\rm end}^{-1}$ in many inflation models ($N_{\rm end}$ is the number of e-folds from the horizon-crossing time to the end of inflation), and if thermal inflation [6] occurs after ordinary inflation, we have $N_{\rm end} \sim 30$–40. In this case, the correction terms in Eq.35 should be observable, while those in Eqs.38 and 41 may be marginally observable. Thus, our formulae may be useful in testing models of inflation when P and n are observed accurately by presently-planned experiments such as the *Microwave Anisotropy Probe*. At the same time, the theories and models of inflation need to progress to make more precise predictions.

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Appendices

A Slow-roll conditions

Here we summarize all the conditions assumed in our calculation, without derivation (One can derive them by differentiating Eq.3 several times and using the resultant equations recursively). We assumed that i) the potential has sufficiently gentle slope, ii) slow roll has been achieved, and iii) the curvature of ϕ-space is not too large and is slowly varying [i) is a necessary condition for ii)]. To write the conditions quantitatively, let us define the "norm"

$$\|X^a\| := (X^a X_a)^{1/2}, \quad \|Y_{ab}\| := (Y_{ab}Y^{ab})^{1/2} \tag{A1}$$

of a vector X^a and a symmetric second-rank tensor $Y_{ab} = Y_{ba}$, and introduce a small quantity $\varepsilon \ll 1$. Then, in the calculation of P, we assumed [conditions i) and iii)]

$$\begin{aligned} &\|U_{,a}U_{,b}\| < \varepsilon, \quad \|U_{;ab}\| < \varepsilon, \quad \|R_{acbd}U^{,c}U^{,d}\| < \varepsilon, \\ &\|U_{;abc}U^{,c}\| < \varepsilon^2, \quad \|(R_{acbd}U^{,c}U^{,d})_{;e}U^{,e}\| < \varepsilon^2 \end{aligned} \tag{A2}$$

so that Eq.35 is valid up to order ε. Also we need the extra but rather weak condition:

$$\left\| \frac{1}{H^4}\frac{D^3}{dt^3}\dot{\phi}^a \right\| < \varepsilon^{5/2} \tag{A3}$$

because of ii). The second line in Eq.A2 is necessary to make sure that the first order quantities are slowly varying. Similarly, in order that Eqs.38 and 41 are valid up to order ε^2, the calculation of n assumes

$$\|U_{;abcd}U^{,c}U^{,d}\| < \varepsilon^3, \quad \|U_{;abc}U^{;cd}U_{,d}\| < \varepsilon^3, \quad \|[(R_{acbd}U^{,c}U^{,d})_{;e}U^{,e}]_{;f}U^{,f}\| < \varepsilon^3 \tag{A4}$$

in addition to Eq.A2, and

$$\left\| \frac{1}{H^5}\frac{D^4}{dt^4}\dot{\phi}^a \right\| < \varepsilon^{7/2}, \tag{A5}$$

instead of Eq.A3.

B Derivation of Eq.11

From Eq.1, the stress tensor is given by

$$T^{\mu}_{\nu} = h_{ab} g^{\mu\rho} \partial_\rho \phi^a \partial_\nu \phi^b - \left(\frac{1}{2} h_{ab} g^{\rho\sigma} \partial_\rho \phi^a \partial_\sigma \phi^b - V \right) g^{\mu}_{\nu} \,, \tag{B1}$$

and the Euler-Lagrange equation is

$$\left[\frac{D}{dx^\mu} + (\partial_\mu \ln\sqrt{-g}\,) \right] h_{ab} g^{\mu\nu} \partial_\nu \phi^b + V_{,a} = 0 \tag{B2}$$

which yields Eq.3. Choosing the gauge $\mathcal{R} = B = 0$ in Eq.10, the relevant components of the metric perturbation are (we work in $\boldsymbol{k}$-space)

$$\delta g_{00} = -\delta g^{00} = 2A \,, \quad \delta g_{0i} = \delta g^{0i} = 0 \,, \quad \delta \ln\sqrt{-g} = A - k^2 E \,. \tag{B3}$$

We perturb T^{μ}_{ν} covariantly with respect to ϕ^a and define the covariant perturbation operator $\delta := \delta\phi^a \nabla_a$; for example, $\delta h_{ab} = 0$. Since $\delta\phi^a$ is Lie-transported along $\phi^a(t)$, i.e., $[\delta\phi, \dot{\phi}]^a = 0$, it follows that

$$\delta(\dot{\phi}^a) = \frac{D}{dt} \delta\phi^a. \tag{B4}$$

Using Eq.D.7 in Ref.[5] to calculate the perturbation in the Einstein tensor, $\delta G^0_0 = \delta T^0_0$ and $\delta G^0_i = \delta T^0_i$ give

$$-6H^2 A - 2k^2 H \dot{E} = -A \dot{\phi}^a \dot{\phi}_a + \dot{\phi}^a \frac{D}{dt} \delta\phi_a + \delta\phi^a V_{,a} \,, \tag{B5}$$

$$2HA = \dot{\phi}_a \delta\phi^a, \tag{B6}$$

respectively. Using Eq.3, one finds

$$\dot{A} + k^2 \dot{E} = \delta\phi^a \frac{D}{dt} \left(\frac{\dot{\phi}_a}{H} \right) \,. \tag{B7}$$

Perturbing Eq.B2 covariantly gives

$$\left(\frac{D}{dt} + 3H \right) \left(\frac{D}{dt} \delta\phi_a - 2A \dot{\phi}_a \right) + \left(\delta \frac{D}{dt} - \frac{D}{dt} \delta \right) \dot{\phi}_a$$

$$+(\dot{A}-k^2\dot{E})\dot{\phi}_a+\left(\frac{k}{a}\right)^2\delta\phi_a+\delta\phi^b V_{;ab}=0. \tag{B8}$$

Here the second term is calculated as

$$\left(\delta\frac{D}{dt}-\frac{D}{dt}\delta\right)\dot{\phi}_a = (\delta\phi^b\nabla_b\dot{\phi}^c\nabla_c-\dot{\phi}^c\nabla_c\delta\phi^b\nabla_b)\dot{\phi}_a \tag{B9}$$

$$= 2\delta\phi^b\dot{\phi}^c\dot{\phi}_{a;[cb]}+\delta\phi^b\dot{\phi}^c{}_{;b}\dot{\phi}_{a;c}-\dot{\phi}^c\delta\phi^b{}_{;c}\dot{\phi}_{a;b} \tag{B10}$$

$$= R_{acbd}\dot{\phi}^c\dot{\phi}^d\delta\phi^b. \tag{B11}$$

on account of Eq.B4. Therefore

$$\frac{D^2}{dt^2}\delta\phi_a+3H\frac{D}{dt}\delta\phi_a+R_{acbd}\dot{\phi}^c\dot{\phi}^d\delta\phi^b+\left(\frac{k}{a}\right)^2\delta\phi_a+\delta\phi^b V_{;ab}$$
$$= (\dot{A}+k^2\dot{E})\dot{\phi}_a+2A(\ddot{\phi}_a+3H\dot{\phi}_a)\,. \tag{B12}$$

From Eqs.B6 and B7, it is easy to show that the RHS of Eq.B12 is equivalent to that of Eq.11.

C Some useful formulae

To evaluate Eqs.34 and 37, we calculate

$$\frac{d\ln(N^{,a}N_{,a}H^2)}{d\ln a}=2\frac{\dot{H}}{H^2}+2\frac{N^{,a}\dot{N}_{,a}}{N^{,c}N_{,c}H} \tag{C1}$$

where $\dot{N}_{,a}$ is calculated as

$$\dot{N}_{,a} = \dot{\phi}^b\nabla_b\nabla_a N \tag{C2}$$

$$= \nabla_a(\dot{\phi}^b\nabla_b N)-(\nabla_a\dot{\phi}^b)(\nabla_b N) \tag{C3}$$

$$= -H_{,a}-N^{,b}\dot{\phi}_{b;a}\,. \tag{C4}$$

Taking the gradient of Eqs.3 and 4, one obtains

$$H_{,a}=\frac{1}{6H}(V_{,a}+\dot{\phi}^b\dot{\phi}_{b;a}) \tag{C5}$$

$$\dot{\phi}_{b;a}=-\frac{1}{3H}[3\dot{\phi}_b H_{,a}+R_{acbd}\dot{\phi}^c\dot{\phi}^d+V_{;ab}+\dot{\phi}^c{}_{;a}\dot{\phi}_{b;c}+(\dot{\phi}_{b;a})^{\cdot}]\,. \tag{C6}$$

The second term in Eq.C6 arises when one commutes the covariant derivatives of $\dot\phi^c\nabla_a\nabla_c\dot\phi_b$. From these equations, $\dot\phi_{a;b} \simeq \dot\phi_{b;a}$ holds to the lowest order and thus the second term in Eq.C5 becomes $\ddot\phi_a$. Using Eq.3 again, one finds

$$H_{,a} \cong -\frac{1}{2}\dot\phi_a\,. \tag{C7}$$

We use $\cong$ when the equation is valid up to the next order. Defining the first order quantity

$$\gamma_{ab} := \left(\frac{1}{2}h_{ac}h_{bd} - \frac{1}{3}R_{acbd}\right)\frac{\dot\phi^c\dot\phi^d}{H^2} - \frac{V_{;ab}}{3H^2}\,, \tag{C8}$$

Eq.C6 can be written iteratively as

$$\dot\phi_{b;a} \cong H\gamma_{ab} - \frac{1}{3}H\gamma_a^c\gamma_{bc} - \frac{(H\gamma_{ab})\dot{}}{3H}\,. \tag{C9}$$

From Eqs.C1, C4, C7, C9 and 39, one obtains

$$\frac{d\ln(N^{,a}N_{,a}H^2)}{d\ln a} \cong -2\epsilon_{ab}M^{ab} + \frac{2}{3}M^{ab}\left(\lambda_a^c\lambda_{bc} + \alpha\lambda_{ab} - \frac{\dot\lambda_{ab}}{H}\right) - \frac{2\alpha}{N^{,c}N_{,c}} - \frac{4}{3}\frac{N^{,a}}{N^{,c}N_{,c}}\frac{\dot\phi^b}{H}\frac{V_{;ab}}{3H^2} \tag{C10}$$

$$\simeq -2\epsilon_{ab}M^{ab}\,. \tag{C11}$$

Also it is easy to show that

$$\frac{\dot\phi^a}{H} \cong -(U^{,a} + \frac{1}{3}U^{;ab}U_{,b}) \simeq -U^{,a}\,, \tag{C12}$$

$$H^{-1}\dot N_{,a} \simeq \lambda_{ab}N^{,b}\,, \tag{C13}$$

$$H^{-1}\dot M_{ab} \simeq 2(\lambda^c{}_{(a}M_{b)c} - M_{ab}M_{cd}\lambda^{cd})\,. \tag{C14}$$

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