

# The hydrodynamical response of a tilted circumbinary disc: linear theory and non-linear numerical simulations

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## Abstract

In this paper we present an analytical and numerical study of the response of a circumbinary disc subject to the tidal-forcing of a binary with a fixed circular orbit. We consider isentropic fluid discs with a range of thicknesses and binaries with a range of mass ratios, orbital separations and inclination angles.

Our numerical simulations are implemented using an SPH code such that we can consider the hydrodynamics fully in three-dimensions. For our unperturbed disc models, we find the numerical shear viscosity to be equivalent to a constant kinematic viscosity and we calibrate its magnitude. Writing a scaling relation for the shear viscosity manifest in our models, we deduce that the disc thickness cannot be varied without affecting the viscosity in these kinds of SPH disc models, as is supported by our numerical results.

It is found that maintenance of an inner cavity owing to the tidal truncation of the disc is effective for non-zero orbital inclinations. Also we show that our model discs may precess approximately like rigid bodies, provided that the disc is able to communicate on a length scale comparable to the inner boundary radius by either sonic or viscous effects, in a sufficiently small fraction of the local precession period. It is found also that the surface density in the disc should not decrease too rapidly with increasing radius, otherwise the disc may separate into disconnected annuli. Furthermore, the disc precession period may tend to infinity if the disc outer edge is allowed to become arbitrarily large, the disc suffering only a modest quasi-steady warp near the inner boundary.

When the disc response is linear, or weakly non-linear, the precession periods and the forms of warping that we measure in our numerical results yield reasonable quantitative agreement with the analytical expressions that we derive from a linear response calculation. For a stronger disc response the results can agree poorly with our linear analysis, although some qualitative features of the response remain intact. We demonstrate that the response of a disc of non-interacting particles is qualitatively different, showing a much larger kinematic

disturbance and an ultimate global thickening of the disc.

This work is of relevance to a number of astrophysical phenomena of current interest in star and planet formation; these include tidal truncation and gap formation in accretion discs and the observational characteristics of some young stellar objects.

## 1 Introduction

Direct imaging of protostellar discs in the Orion nebula (O'Dell et al. 1993, McCaughrean & O'Dell 1996) confirms the long-time hypothesis of their role in low mass star formation. Other studies have shown that binarity in local star-forming regions has a frequency in excess of that normally attributed to solar type main sequence stars in the field. This is found to apply over a wide range of orbital separations (for a review, see Mathieu 1994). Furthermore, it is clear that binary separations in these systems can be of the same order as, or *less* than typical disc sizes. This is suggestive of the existence of circumbinary discs as well as circumstellar discs and raises the question of disc evolution under the influence of bodies other than a central mass alone.

Recent observations have clearly established that protostellar discs can exist in binary and multiple systems. For example, millimetric measurements made by Dutrey et al. (1994) and direct imaging with *adaptive optics* by Roddier et al. (1996), reveal the young binary GG Tauri as occupying a cavity defined by a circumbinary ring of emission, thought to be scattered light from the inner edge of a much larger circumbinary disc. Mathieu et al. (1996, in preparation) have made observations of the remarkable multiple system UZ Tauri. They find that this *quadruple* system consists of a sub-au binary with a circumbinary disc, separated by 500 AU from a 50 AU binary, with at least one of its components carrying a circumstellar disc. Jensen et al. (1996) in a study of 85 systems in Scorpius-Ophiucus and Taurus-Auriga find that the sub-millimeter fluxes from binaries with separations less than about 100 AU is generally less than that for wider or single systems, which they suggest is due to star-disc interactions. Those authors place an upper limit on circumbinary disc masses of  $0.005M_{\odot}$ .

Evidence for non coplanarity of binary orbits and disc planes is provided by the low inclinations of the gas giants in the solar system. Furthermore it is possible to model a small asymmetry in the inner regions of the dusty disc around Beta Pictoris (detected with the *Hubble Space Telescope*, unpublished) as a warp generated by a low mass body on an inclined orbit interior to the disc (Mouillet et al. 1996). Other evidence comes from the third component of the young triple system TY CrA apparently in an orbit at a high inclination to that of the inner binary component (Corporon et al. 1996), and the apparent precession seen in protostellar jets (Bally & Devine 1994). Also Terquem & Bertout (1993, 1996) considered the observational appearance of a thin protostellar disc with warped geometry and were able to synthesize a range of observed spectral energy

distributions (SEDs).

The above is suggestive of non coplanar effects which should be seen in other such binary, and multiple systems. But relative inclinations prove difficult to extract from the observational data available at present and the evidence for non coplanarity to date has been mostly circumstantial in character. Some of the effects we discuss in this paper may become accessible to observation in the near future.

The tidal influence of a companion in a circular orbit upon an accretion disc has been extensively studied analytically in the linear regime and also in the non-linear (as well as the linear) regime by numerical simulations (see Lin & Papaloizou 1993, and references therein). Both circumstellar and circumbinary discs have been considered. Artymowicz & Lubow (1994) modelled the coplanar response of a circumbinary disc to the tide of a binary in an orbit with non-zero eccentricity. However, efforts have focussed for the most part on coplanar configurations.

The tidal disruption of a disc, by encounters with unbound companions at arbitrary inclinations to the disc plane, has been shown to result in a change in the relative inclination of the target disc (Heller 1993, Clarke & Pringle 1993). Papaloizou & Terquem (1995) calculated the linear response of a fluid disc to the tide of a binary companion on an inclined circular orbit exterior to a circumprimary disc. They showed that for a weak enough tidal perturbation, the disc would be expected to precess like a rigid body. Larwood et al. (1996, hereafter LNPT) studied the same problem numerically with a fully non-linear treatment in three spatial dimensions using SPH. They were able to demonstrate disc warping and precession in the tidal field of the companion. The conclusion was that the disc remained only modestly warped, precessing approximately like a rigid body, provided that the sound-crossing time in the disc was a sufficiently small fraction of the precessional period.

In this paper we extend the work presented in LNPT to consider a circumbinary disc with an interior orbiting companion on a fixed circular trajectory at an arbitrary inclination to the disc mid-plane. As in LNPT we study the tidal interaction, disc truncation, warping, and precession as a function of disc thickness, binary mass ratio and orbital inclination.

In Section 2 we review the basic concepts of the theoretical model we employ for our circumbinary disc. In Section 3 we describe and discuss the numerical method we use to perform the simulations. In Section 4 we discuss the linear tidal response of a disc to a binary companion in an inclined orbit. The linear theory we use to describe the form of tidally-induced warps in the thin disc limit is described in Papaloizou & Lin (1995a). Here we perform a linear calculation of the response to the secular terms in the perturbing potential that lead to the warping and precession of the disc. In Section 5 we describe the setting-up of the numerical simulations and some of the diagnostics used. In Section 6 we give an account of our results. Finally, in Section 7 we discuss our findings and give our conclusions.

## 2 Basic Equations

We consider the hydrodynamic equations governing the time evolution of a viscous gaseous medium. Assuming the disc self-gravity to be negligible in comparison to the tidal field of the binary components, the governing equations are the equations of continuity and motion, which may be written

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (1)$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla P - \nabla \Psi + \mathbf{f}_v, \quad (2)$$

where  $\rho$ ,  $\mathbf{v}$  and  $P$  represent the fluid density, velocity and pressure fields respectively. The gravitational potential is  $\Psi$ , and  $\mathbf{f}_v$  is the viscous force per unit mass.

Here we shall ignore the details of heat generation and transport and adopt a simple polytropic equation of state:

$$P = K \rho^{1+1/n}, \quad (3)$$

where  $n$  is the polytropic index and  $K$  is the polytropic constant. For all models considered here we take  $n = 1.5$ . The associated barotropic sound speed,  $c_s$ , is given by

$$c_s^2 = \frac{(n+1)K}{n} \rho^{1/n}.$$

We note that a constant  $K$  specifies isentropy for the system and thus any dissipated energy must be assumed to be radiated away.

### 2.1 Equilibrium Disc Model

In standard notation, the lowest order equation of hydrostatic equilibrium in the vertical direction, for the unperturbed disc is

$$\frac{1}{\rho} \frac{\partial P}{\partial z} = -\Omega^2 z, \quad (4)$$

where  $\Omega$  is the angular velocity.

For a disc satisfying the polytropic equation of state (3), integration of (4) gives

$$\rho = \left( \frac{\Omega^2 H^2}{2K(n+1)} \right)^n (1 - z^2/H^2)^n, \quad (5)$$

where  $H$  is the total vertical semi-thickness which may be a function of  $r$ , and  $\Omega$  is the disc rotation velocity. Hence the surface density  $\Sigma$ :

$$\Sigma = c_n H \left( \frac{\Omega^2 H^2}{2K(n+1)} \right)^n, \quad (6)$$

with  $c_n = \Gamma(1/2)\Gamma(n+1)/\Gamma(n+3/2)$ , where  $\Gamma$  denotes the Gamma function. We also note from equation (5) that

$$\mathcal{M} = \sqrt{2n} \left( \frac{H}{r} \right)^{-1} \quad (7)$$

where  $\mathcal{M} \equiv r\Omega/c_s$  is the Mach number, evaluated at the disc mid-plane. Finally we invoke the standard  $\alpha$ -prescription due to Shakura & Sunyaev (1973) to parameterise an anomalous viscosity. Thus  $\nu = \alpha c_s H$  gives the kinematic viscosity coefficient  $\nu$ .

### 3 Numerical Method

The numerical method we employ to study the dynamics of circumbinary accretion discs is a version of SPH (see Monaghan 1992, and references therein) developed by Nelson & Papaloizou (1993, 1994). The reader is referred to these papers for a description and standard tests of the method. The formulation uses a variable *smoothing length* associated with each particle, which is a function of the particle coordinates, defined in such a way as to ensure accurate energy conservation. Using this method, a particle's smoothing length is defined to be half of the mean distance of the six most distant nearest neighbouring particles chosen from a list of forty-five members at each time-step.

In order to stabilize the calculations in the presence of shocks an artificial viscous pressure term is included, according to the prescription of Monaghan & Gingold (1983) as in LNPT. Although designed to prevent particle penetration while giving positive definite dissipation, the practical implementation of the artificial viscosity introduces a shear viscosity which results in disc spreading much as in the standard theory of accretion discs (Lynden-Bell & Pringle 1974). This effect is quantifiable and utilised in our calculations.

#### 3.1 Viscosity Calibration

In LNPT we described how the shear viscosity present in our model discs was calibrated by comparing SPH calculations of the evolution of an unperturbed accretion disc to solutions obtained by solving the well known diffusion equation for the evolution of the surface density in a thin axisymmetric disc (see Pringle 1981, for a review)

$$\frac{\partial \Sigma}{\partial t} = -\frac{1}{r} \frac{\partial}{\partial r} \left\{ \left[ (r^2 \Omega)' \right]^{-1} \frac{\partial}{\partial r} (\nu \Sigma r^3 \Omega') \right\}, \quad (8)$$

The primes in (8) denote differentiation with respect to  $r$ .

For the standard Monaghan & Gingold (1983) parameters we take (LNPT, see also Artymowicz & Lubow 1994) the numerical viscosity (defined locally) is essentially  $\propto 0.5c_s h$ , where  $h$  is the local smoothing length. Since our smoothing length is spatially adaptive  $h \propto \rho^{-1/3}$  which cancels the density dependence of the sound speed for polytropic models with  $n = 1.5$ . Solving (8) for  $\nu = \text{constant}$  confirms our naive expectation for constant kinematic viscosity to be manifest in these model discs. For constant Mach number discs, the scaling for the effective Shakura–Sunyaev  $\alpha$  is then given by

$$\alpha \equiv \frac{\nu \mathcal{M}^2}{r^2 \Omega} \sim 0.025 (R_o/r)^{1/2} (\mathcal{M}/10)^{2/3}, \quad (9)$$

where  $R_o$  is the radius of the outer boundary of the disc and we have ignored a weak dependence on the total number of particles. So typically we find for  $\mathcal{M} \sim 10$  a Reynolds' number ( $\equiv r^2 \Omega / \nu$ )  $\sim 4000$ , near the outer edge of the disc. We note that this yields the effective Shakura–Sunyaev kinematic viscosity  $\propto \mathcal{M}^{-4/3}$ . It is important to note that in these SPH disc models the disc thickness and viscosity may not be varied independently. Thick discs are inevitably very viscous (see also Artymowicz & Lubow 1994, 1996). Simulation of thick discs with small viscosity has to be performed with other numerical methods.

### 3.2 The Gravitational potential

We assume that the disc circulates around a binary system, the orbit of which is circular with angular frequency  $\omega$ . In addition we suppose that the orbital plane has an initial inclination angle  $\delta$  with respect to the Cartesian  $(x, y)$  plane, which corresponds to the initial mid–plane of the disc. The  $z$  axis is normal to the initial disc mid–plane. We denote the unit vectors in each of the Cartesian coordinate directions by  $\hat{\mathbf{i}}$ ,  $\hat{\mathbf{j}}$  and  $\hat{\mathbf{k}}$  respectively and the associated cylindrical polar coordinates are  $(r, \varphi, z)$ .

For a disc with negligible mass the plane of the orbit is invariable and does not precess. When the mass of the secondary is very much less than the mass of the primary, it is possible to have the origin of the non rotating coordinate system at the the primary. The coordinates and origin of time are such that the line of nodes coincides with, and the secondary is on, the  $x$  axis at  $t = 0$ , the position vector  $\mathbf{D}$  of the secondary is given as a function of time by

$$\mathbf{D} = D \cos \omega t \hat{\mathbf{i}} + D \sin \omega t \cos \delta \hat{\mathbf{j}} + D \sin \omega t \sin \delta \hat{\mathbf{k}}. \quad (10)$$

where  $D = |\mathbf{D}|$ .

The total gravitational potential  $\Psi$  at a point with position vector  $\mathbf{r}$  is given by

$$\Psi = -\frac{GM_p}{|\mathbf{r}|} - \frac{GM_s}{|\mathbf{r} - \mathbf{D}|} + \frac{GM_s \mathbf{r} \cdot \mathbf{D}}{D^3} \quad (11)$$

where  $G$  is the gravitational constant. The first dominant term is due to the primary with mass,  $M_p$ , while the remaining perturbing terms are due to the secondary with mass,  $M_s$ . The last indirect term accounts for the acceleration of the origin of the coordinate system.

For secondary masses comparable to the primary mass it is convenient to refer the description of an external disc to a coordinate system based on the centre of mass of the binary. Then the gravitational potential takes the form

$$\Psi = -\frac{GM_p}{|\mathbf{r} - \mathbf{D}_p|} - \frac{GM_s}{|\mathbf{r} - \mathbf{D}_s|}, \quad (12)$$

where  $\mathbf{D}_s = M_p \mathbf{D}/(M_p + M_s)$  and  $\mathbf{D}_p = -M_s \mathbf{D}/(M_p + M_s)$ . Note that in this case because the origin is not accelerating there is no indirect term.

These potentials are used in the SPH calculations with the minor modification that the potential due to a point mass is softened. That is the standard Newtonian potential due to a point mass  $M$ , namely  $-GM/r$ , is replaced by  $-GM/\sqrt{r^2 + b^2}$ , where  $b$  is the softening parameter, chosen to be small compared with the binary separation.

## 4 Viscous Evolution of the Disc Under Tidal Forcing

### 4.1 The Coplanar Case

The evolution of binaries with coplanar circumbinary discs has been studied previously using particle simulation methods (see Lin and Papaloizou 1979, Artymowicz et al. 1991, Artymowicz and Lubow 1994) and also finite difference methods (Lin and Papaloizou 1993).

The natural tendency in an isolated disc is for material to slowly move radially inwards while angular momentum is transported outwards by the action of viscosity.

When the central region of the disc is occupied by a binary in a coplanar circular orbit, there is a tidal interaction between the disc and binary, the latter not being a point mass. Because the binary rotates faster than the circumbinary disc material, angular momentum is transported from the orbit to the disc. If the viscosity is not too large, inward radial migration of disc material is prevented and a cavity or ‘gap’ is maintained (Lin & Papaloizou 1979).

The basic evolutionary effects on the disc may be found from (8) with the addition of an angular momentum source term to account for the angular momentum injection resulting from the tidal effects of the secondary (Lin & Papaloizou 1986b, Pringle 1991). Pringle (1991) found that the disc surface density profile became steadily shallower as the outer boundary moved outwards to increasingly large radii on the viscous timescale. For sufficiently strong injections of angular momentum, disc material never penetrated interior to the inner radius

where the source was located. Whether or not the inner cavity is maintained depends upon the rate of angular momentum input due to tidal effects in comparison to its transport due to viscous effects (see Goldreich & Tremaine 1978, 1982, Papaloizou & Lin 1984, Lin & Papaloizou 1993). For the case of small  $M_s/M_p$ , Lin & Papaloizou (1986a, 1993) give the condition for maintenance of the gap against viscous diffusion as

$$\frac{M_s}{M_p} > \frac{40\nu}{D^2\Omega}. \quad (13)$$

Those authors found this expression to be valid for  $M_s/M_p < 10^{-2}$ . It was not found to be applicable to binary systems with a large mass ratio, where it would over-estimate the strength of tidal torques in comparison to viscous diffusion. In addition, for a gap to form in a disc with Mach number  $\mathcal{M}$ , we expect, for a low mass ratio, that the disc thickness should not be more than the Roche lobe size associated with the secondary. This gives the thermal condition

$$\left(\frac{M_s}{M_p}\right)^{1/3} > \frac{1}{\mathcal{M}}. \quad (14)$$

## 4.2 The Non-Coplanar Case

From the study presented in LNPT, as well as very general considerations, we expect tidal effects to work to maintain an inner cavity as in the coplanar case but at a somewhat reduced level when the orbital and disc planes are inclined. However, we also expect the phenomena of warping and twisting of the disc as well as disc precession to occur in this case.

An initially large inner boundary radius of a circumbinary disc shrinks because of viscous evolution until tidal effects become important enough to halt it.

During this process the disc is also expected to develop a twisted and warped structure. The timescale for setting this up is expected to be on the order of the time the disc takes to communicate with itself over a scale length comparable to the inner boundary radius either through sonic or viscous effects. This is not greater than the viscous diffusion timescale on which the disc contracts inwards. In addition, as a result of the net torque acting on it because of the inclined binary orbit, precession of the disc is expected to be induced.

In order to carry out an independent investigation of the warping and precession of a circumbinary disc, we consider the linear response of a disc to the component of the time averaged perturbing potential which is mainly responsible for producing these effects. We remark that because the SPH circumbinary discs are inevitably very viscous the inner edge moves close to the binary orbit where it becomes in general strongly perturbed. The quantitative applicability of a linear response calculation is accordingly restricted to low mass ratios and small orbital inclinations. However, we find that qualitative trends can be reproduced in other cases.



### 4.3 The Linear Secular Response due to Binary with an Inclined Orbit

For the problem of an external disc we can consider an expansion of (12) in powers of  $D/r$ , taking the part with odd symmetry with respect to reflection in the  $z$ -plane as the important component contributing to a secular non-coplanar response (Papaloizou & Terquem 1995). So keeping terms up to second order in  $D/r$  and assuming a thin disc, ie.  $z/r \ll 1$ , we find the zero-frequency component of the tidal potential for azimuthal mode number  $m = 1$  to be the real part of  $\Psi' \exp(i\varphi)$ , where

$$\Psi' = i \frac{3}{4} \frac{\Omega^2}{r} \frac{M_s M_p}{(M_p + M_s)^2} D^2 z \sin(2\delta). \quad (15)$$

We consider the linear response of a thin disc to the perturbing potential  $\Psi'$ . Papaloizou and Lin (1995a) derived the equation needed to describe the secular warped structure taken on by a thin fluid disc, which may have a *small* kinematic viscosity, when perturbed by a non-coplanar binary companion in the form:

$$\frac{d}{dr} \left\{ \frac{\mu \Omega^2}{[\Omega^2(1 - i\alpha)^2 - \kappa^2]} \frac{dg}{dr} \right\} = \frac{i\mathcal{I}}{r}, \quad (16)$$

where

$$\mathcal{I} = \int_{-\infty}^{+\infty} \frac{\rho z}{c_s^2} (\Psi' + 2i\omega_p r z \Omega \sin \delta) dz. \quad (17)$$

$$\mu = \int_{+\infty}^{-\infty} \rho z^2 dz,$$

$$\kappa^2 = \frac{2\Omega}{r} \frac{d}{dr} (r^2 \Omega),$$

and we have allowed for a Shakura–Sunyaev  $\alpha$  viscosity to act on the vertical dependence of horizontal motions. Note that the effect of viscosity on the radial dependence of the vertical displacement has been neglected. This might be important in cases showing a strong non-linear response. It is likely that this leads to the weaker response we find in our model discs than is indicated by the linear theory described here. The complex function  $g$  gives the ratio of the vertical to rotational velocity  $g \exp(i\varphi)$ . In terms of the vertical displacement  $\zeta(r) \exp(i\varphi)$ ,  $g = i\zeta/r$ .

We suppose our coordinate system, defined as above with respect to the disc at some initial epoch, and hence the disc rotation axis, precesses about the orbital rotation axis with angular velocity  $\omega_p$ . This is necessary (at least in

the case of a finite disc) because if the unperturbed disc density vanishes at an inner boundary with  $r = R_i$  and an outer boundary with  $r = R_o$ , there is an integrability condition associated with (16) which is obtained by integrating (16) over the disc. This is

$$\int_{R_i}^{R_o} \int_{-\infty}^{+\infty} \frac{\mathcal{I}}{r} dz dr = \int_{R_i}^{R_o} \int_{-\infty}^{+\infty} \frac{\rho z (\Psi' + 2i\omega_p r z \Omega \sin \delta)}{c_s^2 r} dz dr = 0. \quad (18)$$

The precession frequency  $\omega_p$  is chosen so that the integrability condition (18) is satisfied. This gives, assuming a linear dependence for  $\Psi'$ ,

$$\omega_p = - \int_{R_i}^{R_o} \frac{\Sigma}{r \Omega^2} \left( \frac{\partial \Psi'}{\partial z} \right)_{z=0} dr \bigg/ \int_{R_i}^{R_o} \frac{2ir \Sigma \sin(\delta)}{\Omega r} dr \equiv - \frac{3 GM_s M_p D^2}{4 (M_s + M_p)} \cos \delta \frac{\int \Sigma r^{-2} dr}{\int \Omega \Sigma r^3 dr}. \quad (19)$$

Here we have used

$$\int_{+\infty}^{-\infty} \frac{\rho z^2}{c_s^2} dz = \frac{\Sigma}{\Omega^2},$$

an identity which follows by multiplying (4) by  $\rho z/c_s^2$  and integrating through the disc.

For a Keplerian disc, (19) gives the precession frequency as the total torque acting on the disc divided by the total disc angular momentum. For a finite disc this is always non zero. However, for an infinite disc with infinite angular momentum content the precession frequency is clearly zero. In such cases it is possible that the inner regions adjust into a warped structure while exhibiting no precession, even though there is a net torque on the disc (see below). After having determined  $\omega_p$  by the above procedures, equation (16) may be integrated to give

$$g = i \int_{R_i}^r \frac{[\Omega^2(1 - i\alpha)^2 - \kappa^2]}{\mu \Omega^2} \left( \int_{R_i}^r \frac{\mathcal{I}}{r} dr \right) dr \quad (20)$$

Here, we have assumed that  $g = 0$  for  $r = R_i$  but note that an arbitrary constant may be added to  $g$ . This corresponds to a small arbitrary rigid tilt (Papaloizou & Lin 1995a) which we assume to be taken care of by the choice of coordinate system. In order for the disc to approximately precess as a rigid body  $g$  calculated from (20) must be small.

As stated above, when the disc has infinite extent and angular momentum content,  $\omega_p$  is identically zero, although there is a net torque on the disc. None the less (20) may give a convergent expression for  $g$ . Physically this corresponds to the situation when the disc is distorted in the inner regions without undergoing any global precession.

#### 4.4 The response of a disc of infinite extent

We illustrate the possibility of solutions of this type by considering polytropic models of the kind corresponding to our simulations ( $n = 1.5$ ). We also make the replacement  $\Omega^2 = \kappa^2$ , and neglect  $\alpha^2$  in comparison to  $\alpha$ . This procedure is expected to be valid in the bulk of the disc if  $\alpha \geq (H/r)^2$  and also if  $\alpha\Omega$  exceeds the precession period of a particle orbit locally (see for example Papaloizou & Pringle 1983, Papaloizou & Lin 1995a). These conditions are expected to hold for our simulations for the smallest perturbing mass presented below. The response equation behaves then like a diffusion equation. In this case (20) gives for  $\omega_p = 0$ , and  $\alpha$  assumed to be constant

$$g = \frac{3i(2n+3)\alpha}{2} \sin(2\delta) \frac{M_s M_p}{(M_p + M_s)^2} D^2 \int_{R_i}^r \frac{1}{\Sigma H^2} \left( \int_{R_i}^r \frac{\Sigma}{r^2} dr \right) dr. \quad (21)$$

Equation (21) gives a convergent expression for  $g$  as  $r \rightarrow \infty$ , provided  $1/(\Sigma H^2)$  vanishes sufficiently rapidly. For illustrative purposes, we consider the case when  $H/r = \text{constant}$  and  $\Sigma \propto r^{-1/2}$  for  $r > R_i$ , otherwise  $\Sigma = 0$ . But note that taking other power laws for  $\Sigma$  yields very similar results as long as the integrals are convergent. In considering such models, we neglect the structure of the inner disc edge, but as there is no divergence of the integrals there, this should make little difference. For this simple model (21) gives

$$g = 12i\alpha \sin(2\delta) \frac{M_s M_p}{(M_p + M_s)^2} \left( \frac{D}{R_i} \right)^2 \left( \frac{r}{H} \right)^2 \left[ \frac{3}{4} - \left( \frac{R_i}{r} \right)^{1/2} + \left( \frac{R_i}{2r} \right)^2 \right]. \quad (22)$$

The dependence on azimuth is such that the relative displacement,  $\zeta/r$ , is maximised on the  $x$  axis, being an increasing function of  $x$  for both  $x > 0$  and  $x < 0$ . This behaviour is similarly found to occur for the zero-frequency response of a circumprimary disc (Papaloizou et al. 1995, LNPT).

We define the total range of  $\zeta/r$  as

$$\Delta(\zeta/r) = 9\alpha \sin(2\delta) \frac{M_s M_p}{(M_p + M_s)^2} \left( \frac{D}{R_i} \right)^2 \left( \frac{r}{H} \right)^2. \quad (23)$$

The condition for  $\Delta(\zeta/r)$  to be small, and hence for the linear analysis to have some validity, is, for  $\alpha \sim H/r$ , roughly that the sound crossing time across a scale  $\sim R_o$  be less than the inverse precession frequency. For a very viscous disc, the criterion should be that the viscous diffusion time be less than the inverse precession frequency (Papaloizou & Pringle 1983). But note that because the simulations are for finite discs, they exhibit a non zero (albeit small) pseudo rigid body precession rate. This makes essentially no difference to the above discussion if the model is a cut off version of an infinite one with convergent

$\Delta(\zeta/r)$ . However, it is in general possible for some models that  $\Delta(\zeta/r)$  is divergent. In this case the disc may split into disconnected pieces each of which has good enough internal communication to maintain approximate rigid body precession. This kind of behaviour was found to occur in very thin circumprimary disc models in LNPT (see also model 13, below).

## 5 Numerical Simulations

We have performed simulations for binary mass ratios,  $q = M_s/M_p$  equal to 0.01, 0.1, and 1. The range of Mach numbers considered is 10 – 30, and we take the inclination between binary orbit and initial disc plane to be in the range 0 – 45 degrees.

For  $q = 0.01$  we adopted a coordinate system with the origin located at the primary (Case I). For the other mass ratios, we adopted a coordinate system with the origin located at the centre of mass of the binary (Case II). In each of these cases the  $(x, y)$  plane of the fixed Cartesian coordinate system used to describe the disc was taken to coincide with the initial disc mid-plane.

### 5.1 Initial Conditions

A disc containing 27000 identical particles was initially set in orbital motion about the origin of the coordinate system. Each particle was then put into a circular orbit, obtained assuming a unit mass, acting as a softened point mass, centred at the origin (see above). The rotation law adopted for the disc was thus of the modified-Keplerian form:

$$\Omega = \frac{1}{(r^2 + b^2)^{3/4}}. \quad (24)$$

where  $b$  is the softening length.

We adopt units of mass such that for Case I, the primary mass  $M_p = 1$ , and for Case II the total binary mass  $M_s + M_p = 1$ . In both cases the gravitational constant  $G = 1$ , and the unit of length was chosen such that the initial outer disc boundary was at  $r = R_o = 5$ . The softening length for the primary was taken to  $b = 0.2$  in these units. We then adopt the natural time unit, being the inverse Keplerian frequency for unit central mass at  $r = 1$ . When  $q = 1$  the softening length for the secondary is taken to have the same value as for the primary, when  $q \neq 1$  we use  $b = 0.001$  for the secondary softening.

The initial inner boundary of the disc was taken to be at  $r = R_i = 1.5$  in all cases with  $q \neq 1$ . For  $q = 1$ , we took  $R_i = 2$ , the larger value being necessary because of the strong tidal effects that occur when  $q = 1$ .

The particle positions were initialised by dividing the disc into 100 annuli of equal width and placing in each one an equal number of identical particles so as to obtain on average  $\Sigma \propto 1/r$ . Initially 30 particles were placed at random

in the mid-plane of each annulus giving a disc plane containing a total of 3000 particles. Eight other identical planes were then placed symmetrically about the mid-plane with equal vertical separation, such that the total initial disc semi-thickness was  $\bar{H}$ . The constant value of  $\bar{H}$  was chosen such that  $R_o/\bar{H} \equiv \bar{\mathcal{M}}$ , the required Mach number (we shall subsequently use  $\bar{\mathcal{M}}$  when we refer to the single Mach number associated with a particular model). The polytropic constant  $K$  was adjusted so that the Mach number calculated in the mid-plane at  $r = R_o$  from the equation of state was equal to the required value.

We remark that after relaxing to vertical hydrostatic equilibrium, noting that the mid-plane sound speed is almost unchanged, the total vertical semi-thickness  $H = \sqrt{3}\bar{H}$  at the outer boundary (see equation (5)). The mid-plane Mach number is then found to be approximately independent of radius so that we can effectively characterize each initial disc model by just one value. We note that (6) implies that in the mid-plane  $\mathcal{M} \propto r^{1/8}$ , being a very weak radial dependence.

For all of the calculations described here the smoothing length turned out to be less than the disc semi-thickness, this indicates that the latter was determined by genuine pressure effects and not kernel support, which is necessary in order to model the hydrodynamics in three dimensions.

For  $q = 0.01$  and  $q = 1$ , the disc models were allowed to relax for approximately  $2\Omega(R_o)^{-1}$ , evaluated at the initial outer boundary, under the gravity of a central unit mass. During the initial relaxation period the outside edge of the disc expanded by about 20 percent. This effect was mostly due to a pressure-driven expansion into the surrounding vacuum. After relaxation, the Mach number was found to be approximately constant through the disc. Subsequently the full gravitational potential corresponding to a binary in a circular orbit with separation  $D = 1$  was introduced. However, we note that the relaxation time for vertical hydrostatic equilibrium is short compared to the disc evolution timescales of interest. Accordingly, after long times the results were found to be robust to variation in the details of the initial conditions. For the results presented with  $q = 0.1$ , the full gravitational potential was introduced immediately without prior relaxation. The orbital separation was taken to be  $D = 0.7$  for these models. With this choice of  $D$  for  $q = 0.1$  our different mass ratio cases divide into three groups according to the relative strength of the linear secular tidal potential.

The potential expansion we carry out allows us to make a parameterisation for the ‘tidal strength’  $\varsigma \equiv \frac{M_s M_p}{(M_p + M_s)^2} D^2$ . This parameter gives a crude relative measure of the effectiveness of the tidal torques in truncating a circumbinary disc, for fixed orbital inclinations. We also consider the relative strength of the ‘warping tide’,  $\varsigma \sin(2\delta)$ . With this choice of parameter the cases for mass ratios  $q = 0.01, 0.1$  and  $1.0$  correspond to  $\varsigma = 0.01, 0.05$  and  $0.25$  respectively, so that an increment in the  $q$ -value of the model modifies the tidal strength by a constant multiplicative factor. We test our models for various values of  $\bar{\mathcal{M}}, \varsigma$

and  $\delta$  but for clarity we refer to  $q$  in our discussion of the results, noting that the relative strengths of the tidal torques at work for the respective models is related in the way described above.

## 5.2 Measuring Warping and Precession

In order to measure the degree of warping and amount of precession manifest in our calculations we introduce an angle  $\iota$ , being the angle between the angular momentum vector of the disc material contained within a specified cylindrical annulus, and the total angular momentum vector of the disc. We define  $\iota$  through

$$\cos \iota = \frac{\mathbf{J}_A \cdot \mathbf{J}_D}{|\mathbf{J}_A||\mathbf{J}_D|}.$$

The disc angular momentum is  $\mathbf{J}_D$ , calculated as the sum over all disc particle angular momenta. The angular momentum within an annulus is  $\mathbf{J}_A$ , calculated as the sum of all disc particle angular momenta within the annulus. If  $\iota$  is small its total range is equivalent to the total range of  $\zeta/r$ , defined above.

The angle  $\Pi$  measures the amount of precession of the disc angular momentum vector,  $\mathbf{J}_D$ , about the binary orbital angular momentum vector,  $\mathbf{J}_O$ . We define  $\Pi$  through

$$\cos \Pi = \frac{(\mathbf{J}_O \times \mathbf{J}_D) \cdot \mathbf{u}}{|\mathbf{J}_O \times \mathbf{J}_D||\mathbf{u}|}.$$

The reference vector  $\mathbf{u}$  may be taken to be any arbitrary vector in the orbital plane. For convenience we choose  $\mathbf{u}$  such that  $\Pi$  takes the initial value of  $\pi/2$  radians in all models. For retrograde precession of  $\mathbf{J}_D$  about  $\mathbf{J}_O$  the angle  $\Pi$  should decrease linearly with time as is found in practice.

## 6 Numerical Results

Before describing the results of our simulations in detail, we give a brief summary of our findings. In all cases the initial location of the disc inner boundary was too great for angular momentum input from tidal torques to balance inward viscous diffusion. All the coplanar models except those with  $q = 0.01$  and  $\bar{M} = 10$  reached a quasi-equilibrium in which a central cavity containing the binary was essentially maintained, with further disc contraction through viscosity being halted because of the angular momentum input by tidal torques. The gap-clearing efficiency of tidal torques appeared to become reduced as the inclination of the disc plane to that of the orbit increased. This was apparent due to the reduced size of the cavity at higher inclinations of the companion's orbit compared with the greater scale of the cleared cavity present for lower

inclinations. When  $q = 0.01$  and  $\bar{\mathcal{M}} = 10$ , the viscous condition for tidal truncation (13) gives  $\alpha/\mathcal{M}^2 < 2.5 \times 10^{-4}$ , which is not satisfied near the inner disc boundary. Accordingly a breakdown of tidal truncation is observed in these cases.

The tidally truncated inclined discs also exhibit warping. For mass ratios  $q = 0.01$  and  $q = 0.1$ , we found that the magnitude of the elevation of the disc mid-plane above its initial level behaved qualitatively as predicted in Section (4) above. In addition, except when  $q = 1$ , very slow quasi-rigid body precession was observed. These cases also implied a very slow evolution (possibly on the global viscous timescale) of the relative mean disc plane inclination to the orbital plane (cf. Papaloizou & Terquem 1995).

When  $q = 1$  the tidally truncated inclined discs were severely warped in comparison to the other models. However, a quasi-rigid body precession was still observed for small inclinations such that the disc maintained itself as a unit. For high inclinations the disc distortion became so severe that there were indications that it could evolve into disconnected annuli. Parameters for all the models we consider are presented in Table (1).

## 6.1 Coplanar Models

We begin with a discussion of our results obtained when the disc and orbital planes were coincident. These cases, which may be related to previous work, provide reference points to which the non coplanar cases may be compared (see Lin & Papaloizou 1993, and references therein, and also Artymowicz & Lubow 1994).

### 6.1.1 Models with $q=0.01$

#### *Model 1*

This model had  $\bar{\mathcal{M}} = 20$ . Figure (1) shows a particle projection plot onto the  $(x, y)$  plane after relaxation was complete, but without a binary companion ( $M_s = 0$ ). This is to be compared to Figure (2) which shows a similar projection plot of the disc after the companion had been included for an additional time of approximately 150 units. This shows a cavity to have been cleared out to as far as a radius  $\simeq 1.4$ . A faint density wave with  $m = 1$  is clearly seen, as well as a ‘wake-like’ feature trailing the companion, causing the shape of the inner boundary to deviate from axisymmetry.

At this stage the inner boundary region of the disc appears to have reached a quasi-steady state with tidal interaction providing a continuous supply of angular momentum to balance viscous effects, which otherwise would cause material to flow inwards. This is consistent with the viscous condition for tidal truncation (13) which, being marginally satisfied at the inner disc boundary, implies that tidal truncation should just be possible.

However, a few particles were found to move inside the cavity as a material stream reached toward the companion from the inner boundary. This leakage is possibly due to poor statistics combined with the effects of large smoothing lengths near the inner boundary. Due to these effects, small numbers of particles could be propelled into unstable coorbital trajectories. The numbers of particles involved is too small to make any inference about consequent local fluid properties. We comment that, for this to be possible, a more thorough understanding of the behaviour of SPH particles in low density regions and near edges than appears to be currently available is required.

As the ‘wake–stream’ of particles extends from the inner boundary to the companion (see Fig. (3)), its inner edge becomes increasingly poorly defined. It is not clear what role numerical effects play in its existence. Although, this feature of our numerical results can be understood in terms of a simple impulse model for tidal interaction (Lin & Papaloizou 1985). It is plausible that, if the companion mass were to be reduced, the wake–stream may ultimately tend to the streamline configuration seen in the vicinity of an accreting protoplanet fully embedded in a thin disc (Korycansky & Papaloizou 1996).

#### *Model 6*

We describe this model here as it had the same parameters as model 1 except that the Mach number was halved by comparison. This had the consequence that the effective shear viscosity was approximately two and a half times larger (see equation (9)). In this case the condition (13) implies a failure of tidal truncation near the inner disc boundary. In model 6 the inner disc is found to approach the companion’s orbit more closely than in model 1. This is consistent with the disc material seeking a stronger tidal torque to balance the increased viscous transport.

Indeed the inner boundary for this model contracted to a mean radius of approximately 1.2, which is about a companion Roche lobe radius away from the coorbital radius. Thus the inner boundary had closed 0.2 radius units, which is more than one would expect from thermal effects alone. As expected the large viscosity in these low mass ratio models determines the size of the gap.

When the disc inner edge had approached so close to the coorbital radius that non–linear tidal disruption of the inner boundary occurred, many particles were found to move interior to the binary orbit in an apparently continuous transfer of material. The mean rate of flow of particles into the inner region was found to be comparable to the rate of delivery of material expected through unimpeded viscous transport. However, the number was still too small for detailed fluid properties to be represented beyond saying that the inflow did occur.

Figure (4) shows a projection plot onto the initial disc mid–plane for model 6 after a time of about 125 units after the introduction of the secondary. Any particles contained within a radius of unity of the computational origin were extracted at a time of about 75 units. Therefore, in this plot, all the particles interior to the coorbital radius ( $\sim 100$  of them) found their way there over a



period of about 50 time units. We note that the character of the gap breakdown in this marginal case is to maintain the cavity, allowing leakage through the vicinity of the companion’s position without it ‘accreting’ any material.

The gap opening criteria (13) & (14) are found to be consistent with our results, confirming our calibration of the disc viscosity. It is a consequence of the scaling for the shear viscosity (9) that the disc thickness and viscosity are not independently variable as model parameters.

### 6.1.2 Model with $q=1$

By contrast with the marginal case of model 6 above, the unit mass ratio model 11 with  $\bar{M} = 20$  displayed strong tidal torques with no indication of any particle leakage into the interior or the development of wake-streams. Results were found to match expectation (Lin & Papaloizou 1979, Artymowicz & Lubow 1994). A cavity was cleared out to a mean distance of about 1.9 units, measured from the binary centre of mass.

## 6.2 Models with non-zero inclination

As the inclinations were increased in all our models, the incidence of wake-streams and gap leakage increased also. This is clearly due to the weakened tidal torques at higher orbital inclinations allowing the closer approach of the inner boundary to the binary components. This highlights that the formation of wake-streams depends on the relative strengths of the tidal and viscous torques, so that such features are expected to occur only when viscosity predominates. Furthermore, the gap edge in low inclination, low Mach number (i.e. large viscosity) models showed similar behaviour to that for higher Mach number (i.e. smaller viscosity), higher inclination cases.

We now proceed to discuss the response of our models with inclination different from zero.

### 6.2.1 Models with $q=0.01$

Taking the final state of model 6 as the set of initial conditions (see Figure (2)), we introduced an inclination of the secondary’s orbit to the mid-plane of the disc, namely the  $(x, y)$  plane. The inclination,  $\delta$ , was increased in steps of 10 degrees, each change being initiated after an additional relaxation period of approximately 150 time units. For each value of  $\delta$ , the disc was evolved for several hundred time units enabling the disc edge region and warped structure to achieve a quasi-steady state.

In general the degree of warping was mild and much less than would be expected from a collisionless particle simulation (see below). Rough agreement was obtained with the simple linear theory described in Section 4, at least when the degree of warping was small, as in model 2. In addition we observed in such

cases that the disc behaved in a manner consistent with approximate rigid body precession. However, the precession frequencies were small so that total angles of precession were small. Thus this effect was not dynamically very significant and as we have indicated above would probably disappear if the disc were to have arbitrarily large extent.

As discussed above, the introduction of a non-zero inclination results in a reduction in the effectiveness of the tidal torques in maintaining a cleared gap. This results in the disc inner edge moving inwards in comparison to the coplanar case for all models. Consequently tidal truncation could break down at a finite inclination.

#### *Models 2–5*

Figure (5) shows a projection plot of the model 4 data at a time of 265 units after the introduction of the companion on an inclined orbit at 30 degrees to the initial disc mid-plane. The gap generally remains cleared, suffering only a small contraction of the disc inner edge. Also a wake-stream seems to have become a continuous feature in this case, as compared with its episodic growth and decay in the coplanar case of model 1. The rate of inflow of particles inside of the projected orbital path is still very slight though, being much less than that found in the coplanar model 6. We find that the gap is maintained for a finite orbital inclination of the companion for all of the models 2–5, with indications that the gap would breakdown if the inclination were much higher.

The precession angle,  $\Pi$ , evolution for all of these models is plotted in Figure (6) and the precessional timescales (or inverse precession frequencies) that we infer,  $\langle\omega_p^{-1}\rangle$ , are given in the second column of Table (2), for each value of  $\delta$ . Typical values are in broad agreement with equation (19); e.g. assuming  $R_i/R_o = 1.2/6$ ,  $\Sigma \propto 1/r$ , Keplerian rotation and  $\delta = 20$  degrees, we find  $\omega_p^{-1} \simeq 3000$ . None of these models showed precession through more than about 10 degrees. As we shall see, when the strength of the tidal force increases with the larger mass ratio cases, the disc precession can be of much greater significance.

Figure (7) plots  $\iota$  as a function of radius for model 5 with  $\delta = 40$  degrees, the inclination profile appears to reach a state where it changes very slowly with time. We remark that some residual evolution would be expected on a viscous timescale as the disc density decreases in the vicinity of the binary orbit. The data implies a range of  $\zeta/r \simeq 0.14$  in this case, for model 2 with  $\delta = 10$  we infer  $\Delta(\zeta/r) \simeq 0.06$ . This is in reasonable agreement with what we obtain using (23); for the typical values  $M_s/M_p = 0.01$ ,  $\alpha = 0.04$ ,  $\mathcal{M} = 20$ ,  $R_i/D = 1.5$ , we find  $\Delta(\zeta/r) \simeq 0.21 \sin(2\delta)$ . Then for  $\delta = 10$  and 40 degrees,  $\Delta(\zeta/r) \simeq 0.07$  and 0.21 respectively (note that we do not make any notational distinction between inferred and predicted values). Model 2 gives the better agreement, as might be expected considering the weaker warping tide in this case. In view of the uncertainties involved in the details of the edge region and the  $\Sigma$  profile to be used we consider the general qualitative agreement that we find for these models to be satisfactory.

### *Models 7–10*

Performing similar calculations for a lower Mach number such as in models 7–10, results in a lower set of values for  $\Delta(\zeta/r)$  in all cases (as we expect from considering a disc with a larger sound speed). For  $\delta = 10$  and 40 degrees we expect  $\Delta(\zeta/r) \simeq 0.013$  and 0.039 respectively. We infer from the data the values 0.011 and 0.018 which shows once again that the best agreement is for the weakest warping tide (i.e. smallest value of  $\delta$ ) and is poorest when the warping tide is strongest. The agreement is generally poorer in this case though, probably due to non-linearity and excessive gap leakage occurring near the inner edge. The precession frequencies we infer, however, are hardly altered by comparison with those for models 2–5. This shows that the global secular response is determined by communication over the whole disc and the inner warp is set up locally.

Agreement with our analytical expressions matches well in these cases as it did for models 2–5. The only significant difference is the extent of the gap leakage, which is larger at low inclinations than in the higher Mach number models, becoming increasingly important at larger inclinations of the companion’s orbit.

### **6.2.2 Comparison with a disc of non-interacting particles**

We now highlight the striking difference in the response of a fluid disc with hydrodynamic forces and a disc of non-interacting particles. The response of such a ballistic model to the tide of a companion on an inclined orbit is to warp the disc by the propagation of a kinematic bending wave from the innermost parts to the outermost parts of the disc. This reaches a radius where the precessional timescale for non-interacting orbital planes is roughly equal to the time after initiation (Mouillet et al. 1996). For small inclinations the scale of the warp is typically on the order of the maximum vertical displacement of the companion and the disc does not precess as a single entity.

Figure (8) shows a comparison between a ballistic model and model 5 at similar times. The ballistic model was setup by randomising particle positions within a fixed opening angle to obtain the appropriate aspect ratio for direct comparison with the hydrodynamic model. The SPH disc in model 5 shows a damping of the large warp present in the ballistic disc at short times. In this case the warp amplitude is affected by communication with the outer parts through pressure waves on a short timescale and viscous effects on a much longer timescale. The vertical scale of the inner warp is clearly much smaller than in the ballistic case. Further, the scale of the warp is found to approach a quasi-steady configuration with this small magnitude, as can be seen in Figure (7).

At long times the ballistic disc became generally thickened on the scale of the original inner warp. The ballistic disc was not found to show global rigid-body precession on any timescale.

### 6.2.3 Models with $q=0.1$

Models 14 & 15 were initialised with  $\bar{\mathcal{M}} = 20$ , and model 16 with  $\bar{\mathcal{M}} = 30$ . In each of these cases the binary separation was taken to be  $D = 0.7$ . In Model 14 the orbit was given an initial inclination to the disc of  $\delta = 10$  degrees, and in models 15 and 16 this was taken to be  $\delta = 45$  degrees. For these models the companion was introduced directly without prior disc relaxation. The sudden introduction of the binary resulted in transient axisymmetric waves which propagated and damped on the sound-crossing timescale. The disc inner boundary spread inwards until it became truncated at  $r \simeq 1$  with an inner cavity being maintained for the duration of the runs.

In spite of non-linearity,  $\zeta/r$  has a similar form to that in the lower mass ratio cases, with the maximum values occurring along the  $x$  axis. We show this in Figure (9) by comparing a ‘sectional-plot’ (in which we only consider particles such that  $-0.5 < x, y < 0.5$ ) of the disc as seen projected onto the  $(x, z)$  and  $(y, z)$  planes for model 14 after a time of about 310 units.

The longest run for the case of  $q = 0.1$  was for model 16, with a run time of 410 units. Note that because the binary separation is  $D = 0.7$ , this corresponds to 700 inverse binary frequency time units. After this time only slow evolution of the warp was observed as in the  $q = 0.01$  case. In order to illustrate the behaviour of the warp as a function of Mach number, we give  $\iota$  versus radius plots for models 15 and 16 at the same time of 310 units in Figure (10). This demonstrates that the warp amplitude is less for the lower Mach number case. For cases with  $q = 0.1$ , we infer an extremely long quasi-rigid body precessional timescale  $\sim 10^4$  time units. This is probably because relatively little of the disc matter finds its way to small radii as a result of the large tidal angular momentum input, yielding a small net precessional torque. In all of these cases, the larger part of the disc essentially maintained its original plane over the period of the simulations.

### 6.2.4 Models with $q=1.0$

These unit mass ratio cases show the strongest warping perturbations and are therefore instructive in understanding the way that a circumbinary disc responds to warping perturbations. Figure (11) shows sectional-plots for the model 12 disc at times of about 250 and 450 units after the introduction of the companion on a path inclined at  $\delta = 10$  degrees (after the coplanar relaxation of model 11). In the first frame we see that the warping has its largest affect in the inner regions of the disc, with an annulus of material lifting out of the original plane. Consistent with our choice of surface density profile and equation (21), the annulus does not separate from the outer regions of the disc but, as shown in the second frame, settles together with the outer parts to give a stable global warp.

Figure (12) shows the precession angle evolution for the duration of this run.

The disc exhibits uniform precession through an angle of approximately 180 degrees and we infer  $\langle \omega_p^{-1} \rangle \simeq 300$  time units, consistent to within a factor of two with equation (19), which yields  $\omega_p^{-1} \simeq 180$  (assuming  $\Sigma \propto 1/r$ ,  $R_i/R_o = 1.9/6$  and Keplerian rotation).

Taking the relaxed configuration of model 12, we incremented the inclination to  $\delta = 30$  degrees. This resulted in an extreme non-linear response of the disc such that the disc became severely warped and a dense ring of material gathered at the inner boundary. This redistribution of the surface density enabled the ring to partially disconnect from the outer disc. The disconnection results in two separate inclined discs which try to precess separately (see also LNPT) but which have a strong interaction region where material from the two discs grazes. It appears that non-linear dissipation has occurred in a shocked region where material from the almost disconnected discs interact. This interaction attempts to bring the discs back to synchronous precession. The simulations we have carried out indicate that structures like this can be long lived. In a real system this effect might produce an observational signature as a ring of emission. Figure (13) displays the position data resulting from this simulation in a particle projection onto the  $(x, y)$  plane. Figure (14) takes a projection at a viewing angle such that the outer disc is seen almost edge-on. In this figure, an inner ring is clearly seen at a small inclination to the outer disc.

## 7 Discussion

### 7.1 Tidal Truncation

As was found for circumprimary discs in LNPT, provided the viscosity is small enough, circumbinary discs are truncated at the radius where viscous and tidal torques balance. As far as it can be investigated with our SPH method, gap clearing in a disc by an embedded companion on a fixed circular orbit is effective in discs with vertical structure and consistent with theoretical expectation, both in the coplanar and inclined orbit cases (Lin & Papaloizou 1993, LNPT). The effect of the tidal torques that clear a gap in the disc becomes weaker as the disc plane and binary orbit become more inclined, with the consequence that the disc inner boundary moves inwards and that the gap may breakdown at a finite inclination.

However, it should be emphasized that for the SPH models considered here, the viscosity is inevitably large corresponding to Reynolds' numbers typically  $< 10^4$ . In addition the disc thickness and viscosity are not independently variable as model parameters. The effective shear viscosity increases with disc thickness so that the viscous torques increase in their effectiveness, in addition to the improved sonic communication. Thick discs with very small viscosity or weakly viscous discs truncated by low mass companions cannot be studied by this method.

When the viscosity is too large for its effects to be balanced by tidal torques, an approximate inner boundary of the disc may be maintained while particles spill into the gap at a significant rate through material streams. This could be potentially important with regard to planetary formation and accretion in the early solar nebula, but note that the  $\alpha \sim 0.03$  viscosity parameter, for which we mostly observe such effects, is larger than those inferred for protostellar discs from estimates of disc lifetimes (Beckwith et al. 1990), or obtained so far from magnetohydrodynamical simulations (e.g. Brandenburg et al. 1996). But note values of  $\alpha \sim 0.03$  have been found for unstable self-gravitating discs (see Papaloizou & Lin 1995b, and references therein).

## 7.2 The Affect of Non-Coplanarity on Circumbinary Discs

Provided sonic or viscous communication over a scale length comparable to the inner boundary radius takes less than the local inverse precession frequency, and the surface density is not a very steep power law of radius, the inclined disc responds to the secular torques by precessing approximately as a rigid body. The precession frequencies were seen to be small in the  $q = 0.01$  and  $q = 0.1$  cases. It is likely, as demonstrated in a linear response calculation, that were the disc to extend to an arbitrarily large radius, the precession frequency would be zero. In addition to precession the disc takes on a warped structure in the vicinity of the disc inner boundary, which in general has the same qualitative form as that given by our linear response calculation. For the  $q = 1$  case and high inclination, the disc apparently could not maintain itself against the strong differential precession, there being evidence of separation into two separate sonically connected annuli. In this state the disc gave the appearance of a highly warped state.

Finally we comment that the response of discs with pressure and viscosity is qualitatively and significantly different from that seen for non-interacting particle discs.

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Table 1: Model parameters.

Model	$\mathcal{M}$	$q$	$D$	$\delta$	$\varsigma \sin(2\delta)$
1	20	0.01	1.0	0	0.000
2	20	0.01	1.0	10	0.003
3	20	0.01	1.0	20	0.006
4	20	0.01	1.0	30	0.009
5	20	0.01	1.0	40	0.010
6	10	0.01	1.0	0	0.000
7	10	0.01	1.0	10	0.003
8	10	0.01	1.0	20	0.006
9	10	0.01	1.0	30	0.009
10	10	0.01	1.0	40	0.010
11	20	1.0	1.0	0	0.000
12	20	1.0	1.0	10	0.086
13	20	1.0	1.0	30	0.217
14	20	0.1	0.7	10	0.017
15	20	0.1	0.7	45	0.049
16	30	0.1	0.7	45	0.049

Table 2: Inferred precessional timescales for models 2–5.

$\delta$	$\langle \omega_p^{-1} \rangle$
10	2300
20	2700
30	3400
40	4400

Figure 1: The relaxed disc without a companion. Particle positions are projected onto the  $(x, y)$  plane. The sense of positive rotation is anti-clockwise.

Figure 2: The inner regions of the disc, relaxed with a companion included for  $\simeq 150$  time units. The primary is shown as an asterisk, the companion as a circle and its projected path as a dashed line. For clarity, this plot shows only the inner regions of the disc.

Figure 3: The inner disc with a 'wake-stream' extending from the disc inner edge to the direction of the companion after a further binary orbital period.

Figure 4: The thicker more viscous disc of model 6 showing gap leakage at a time of approximately 125 units after the introduction of the companion. We note also that the particles inside the cavity have not suffered large vertical displacements and are therefore representative of gap leakage.

Figure 5: The thin disc of model 4 with the companion at an inclination of 30 degrees. Data is shown at a time of approximately 265 units after incrementing the companion's inclination. The cavity was cleared of particles at a time of 140 units. We note also that the particles inside the cavity have not suffered large vertical displacements and are therefore representative of gap leakage.

Figure 6: The precession angle versus time for models 2–5, shown for a representative subset of the SPH data.

Figure 7: The time evolution of disc inclination versus radius for model 5. Note that the relative inclination angle plotted is essentially equivalent to  $|\zeta/r|$ . For clarity we only plot data at intervals of approximately 200 time units.

Figure 8: We project particle positions onto the  $(x, z)$  plane. The lower frame, for the ballistic data, is to be compared with the SPH data in the upper frame; each is taken at a time of approximately 300 units. Notice that the fluid disc has taken on a mild global warp, the ballistic disc is only warped locally, at this time, and by a much greater amount.

Figure 9: Sectional projection plots for model 14 at a time of approximately 300 units show that the warp is largest along the  $x$ -axis, 90 degrees out of phase with the maximum of the warping potential.

Figure 10: Disc inclination versus radius plots for models 15 and 16, each at a time of approximately 310 units.

Figure 11: Sectional projection plots for model 12 at times of 250 (upper frame) and 450 (lower frame) units are shown.

Figure 12: The precession angle evolution for model 12. We plot the magnitude of negative values.

Figure 13: Projection onto the  $(x, y)$  plane for model 13 at a time of 730 units.

Figure 14: Projection of model 13 data at a time of 730 units, taken at a viewing angle so that the outer disc appears approximately edge-on.