AHARONOV-BOHM EFFECT IN THE QUANTUM HALL REGIME AND FINGERING INSTABILITY

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Abstract The shape of an electronic droplet in the quantum Hall effect is sensitive to gradients of the magnetic field, even if they are placed outside the droplet. Magnetic impurities cause a fingering instability of the edge of the droplet, similar to the Saffman-Taylor fingering instability of an interface between two immiscible phases. We discuss the fingering instability and some algebraic aspects of the electronic states in a strong nonuniform field.

Keywords: Quantum Hall effect, growth problems, fingering, integrable hierarchies

1. Aharonov-Bohm effect and the shape of electronic droplets in a magnetic field

1.1 Introduction

The Aharonov-Bohm effect is a striking manifestation of interference in quantum processes. It has been observed in a number of quantum mechanical and mesoscopic systems and proved to be an important element of our understanding of quantum physics.

In this notes we discuss yet another (so far just theoretical) realization of the Aharonov-Bohm effect, now, in a strong magnetic field. The discussion is based on the recent paper written in collaboration with O. Agam, E. Bettelheim and A. Zabrodin [1].

Electrons confined in a plane in a strong magnetic field form incompressible droplets trapped by an electrostatic potential. The area of

Figure 1. A schematic illustration of the shape of an electronic droplet in a strong magnetic field when some additional magnetic fluxes placed outside of the droplet. Electronic droplet is stratified by semiclassical orbits. The area bounded by each orbit is $\pi N \ell^2$

the droplet is quantized and is equal to $\pi N \ell^2$, where N is a number of electrons in the droplet and ℓ is a magnetic length. If N is large, the droplet is well described in a semiclassical manner. It has a sharp edge distributed over a length ℓ.

If magnetic field is uniform, the shape of the droplet is determined by an equipotential line of the electrostatic landscape. In the case of symmetric potential and a uniform magnetic field the droplet is a disk.

Let us now change magnetic field somewhere away from the droplet in a manner that the magnetic field stays uniform in the area of the droplet. For example we can do this by putting some number of Aharonov-Bohm fluxes or any sort of magnetic impurities. As electrostatic potential, gradients of magnetic field remove the degeneracy of the Landau level and, therefore, affect the shape of the droplet. However, the ways electrostatic and magnetic forces shape the droplet are different.

Electrostatic potential affects the quantum droplet only if it is placed inside the droplet. Its effect decays exponentially with the distance from the droplet. On the contrary, gradients of the magnetic field, even being placed away from the droplet will strongly affect the shape of the droplet. Their effects decay slowly, as a power law in the distance from the droplet.

Moreover, in the situation when potential landscape is negligibly flat, Aharonov-Bohm fluxes placed outside of the droplet cause a fingering instability - an unstable pattern of fingers which grow with increasing the area of the droplet (Fig. 1). A very similar instability is known in non-equilibrium processes driven by diffusion [2].

The effect of magnetic impurities is even more dramatic. Almost any gradient of magnetic field at sufficiently large area of the droplet curves the edge so strongly that segments with the curvature of the order of inverse magnetic length appear inevitably. At these segments the semiclassical description of the droplet and its edge states is no longer valid.

1.2 Electronic droplet in the Quantum Hall regime.

Consider first N spin-polarized electrons on a plane in a uniform, perpendicular magnetic field $B_0 > 0$, in the lowest Landau level:

$$H = \frac{1}{2m}(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A})^2. \quad (1)$$

Degenerate states, written in the symmetric gauge, have the form $f(z)e^{-\frac{|z|^2}{2\ell^2}}$, where $f(z)$ is a holomorphic function. Let us confine electrons in a flat symmetric potential well of large radius R, well exceeding $\ell\sqrt{N}$ ($\ell = \sqrt{2\hbar c/eB_0}$ is a magnetic length). The potential well lifts the degeneracy of the level such that a state with higher angular moment n acquires a higher energy. Near the origin the wave functions are close to the degenerate lowest Landau level wave functions with given orbital momentum. Their orthogonal basis is:

$$\psi^{(0)}_{n+1} = \frac{1}{\sqrt{\pi n!}}\frac{z^n}{\ell^{n+1}}e^{-|z|^2/2\ell^2}. \quad (2)$$

We say that N particles form a droplet, when all first N orbitals, $n = 0, 1\ldots, N-1$ are occupied [3]:

$$\begin{aligned}\Psi^{(0)}(z_1, \cdots, z_N) &= \det \psi^{(0)}_n(z_m)\big|_{n,m<N} \\ &= \frac{1}{\sqrt{N!\tau^{(0)}_N}}\Delta(z)e^{-\frac{1}{2\ell^2}\sum_n |z_n|^2}. \quad (3)\end{aligned}$$

Here $\Delta(z) = \prod_{n<m\leq N}(z_n - z_m) = \det\,(z^n_{m+1})\big|_{0\leq n,m<N}$ is the Vandermonde determinant and the normalization factor $(N!\tau^{(0)}_N)^{-1/2} = \prod_{0\leq n<N} h^{(0)}_n$ is the product of the normalization factors (2) of one-particle states $h^{(0)}_n = (\sqrt{\pi n!}\ell^{n+1})^{-1}$.

In the semiclassical limit $N \gg 1$, this wave function describes a circular shaped droplet of the radius $\ell\sqrt{N}$. In this limit all arguments z_n obey the saddle point equation

$$\sum_{m\neq n}^{N} \frac{2\ell^2}{z_n - z_m} = \bar{z}_n, \quad (4)$$

and are uniformly distributed within a disk of the area $\pi N\ell^2$. The wave function decays exponentially if z_n is found outside the droplet.

Now consider the following arrangement (Fig. 1): the magnetic field remains uniform in the area which includes the droplet (a disk with the radius greater than $\ell\sqrt{N}$). Away from the droplet, the magnetic field is perturbed $B(x,y) = B_0 + \delta B$, in a manner that the nonuniform part does not carry flux $\int \delta B dx dy = 0$. Since $\delta B = 0$ in the area of the droplet, the potential $V(z)$ defined as $\Delta V(z) = -\delta B/2B_0$ is harmonic. The gauge potential is deformed by a harmonic function[1]: $A = A_y - iA_x = \frac{\bar{z}}{\ell^2} - 2\frac{\partial}{\partial z}V(z)$.

Let us set the parameters t_k to be the harmonic moments of the deformed magnetic field:

$$t_k = \frac{1}{\pi k}\int \frac{\delta B(z)}{B_0} z^{-k} d^2 z, \tag{5}$$

For example, in the case of a few thin solenoids with fluxes $\Phi_a < \pi$ added at points ζ_a, the harmonic moments are $t_k = \frac{1}{2B_0}\sum_a \Phi_a \zeta_a^{-k}$. The harmonic potential then is

$$V(z) = Re \sum_{k\geq 1} t_k z^k.$$

A nonuniform part in the magnetic field perturbs the wave function by a "singular gauge transformation"

$$\Psi(z_1, \cdots, z_N) = \frac{1}{\sqrt{N!\tau_N}}\Delta(z) e^{-(\sum_n \frac{1}{2\ell^2}|z_n|^2 - V(z_n))}. \tag{6}$$

The saddle point equation (4) is transformed accordingly

$$\sum_{m\neq n}^{N} \frac{2\ell^2}{z_n - z_m} = \bar{z}_n - 2\ell^2 \frac{\partial}{\partial z} V(z). \tag{7}$$

This result holds in the limit when the radius of the confining potential is very large. In this case the energy splitting of the lowest Landau level due to the confining potential is less than the energy splitting caused by gradients of the magnetic field.

The solution of this equations at large N has been studied in Refs. [4]. The result is as follows: all z_n are uniformly distributed with the density $(\pi\ell^2)^{-1}$in a domain characterized by the following data,

- the area of the domain is $\pi N \ell^2$;

- the harmonic moments of the exterior of the domain

$$t_k = -\frac{1}{\pi k}\int z^{-k} d^2 z, \quad k = 1, 2, \ldots$$

(the integral runs over the exterior of the domain) are equal to the harmonic moments of the nonuniform part of the magnetic field (5).

If the boundary of the domain is smooth and single connected, these data determine the domain.

We see that gradients of magnetic field (say, Aharonov-Bohm fluxes) placed away of the semiclassical orbits affect the shape of the droplet. The effect of the gradients dies slow with the distance between the droplet and the position of the gradients. Indeed, if L is a typical distance between the droplet and the gradients, then $t_k \sim L^{2-k}$ decay slowly with L. In particular, the quadrupole moment t_2 of the magnetic field is transferred to a droplet from an arbitrary distance. The third moment severely disturbs the shape of the droplet. Its effect decays with the distance as $1/L$.

In the next paragraph we argue that the distortion of the droplet caused by a generic gradient of the magnetic field (magnetic impurities) not only strong, but unstable. The magnetic impurities cause a fingering instability. Afterwards, we discuss the origin of the Eq. (7).

1.3 Laplacian growth problem.

Consider a process where the area of the droplet $\pi t = \pi N \ell^2$ grows, while the gradients of the magnetic field δB remains intact. This can be achieved by increasing the number of electrons N (by changing the gate voltage, for example), or by decreasing the uniform part of the magnetic field. In this process the moments t_k are fixed. This leads to the following geometrical problem:

- find the dynamics of a domain while its area increases while harmonic moments t_k remain fixed.

This problem has been discussed in the context of pattern formations in non-equilibrium processes when a front between two immiscible phases advances with the normal velocity proportional to the gradient of a harmonic field - a mechanism often referred as Laplacian growth (for a review see, e.g., [2]).

Viscous or Saffman-Taylor fingering is one of the most studied instabilities of this type. It occurs at the interface between two incompressible fluids with different viscosities when a less viscous fluid is injected into a more viscous one in a 2D geometry (typically, the fluids are confined in the Hele-Shaw cell – a thin gap between two parallel plates - or in porous media [5]).

In a thin cell, the local velocity of a viscous fluid is proportional to the gradient of pressure: $\vec{v} = -\vec{\nabla} p$. Incompressibility implies that the pressure $p(z)$ is a harmonic function of $z = x + iy$ with a sink at infinity:

$$\nabla^2 p(z) = 0, \quad p(z) \to -\frac{1}{2} \log |z|, \quad |z| \to \infty \tag{8}$$

If the difference between viscosities is large, the pressure is constant in the less viscous fluid and, if the surface tension is ignored, it is also constant (set to zero) on the interface. Thus on the interface

$$p(z) = 0, \quad v_n = -\partial_n p(z). \tag{9}$$

If less viscous liquid is supplied through the origin with a constant rate, the area πt of the less viscous fluid grows linearly with time t.

A simple consequence of the growth process defined by these equations (8, 9) is that harmonic moments of the viscous fluid domain,

$$t_k = -\frac{1}{\pi k} \int z^{-k} d^2 z, \quad k = 1, 2, \ldots$$

where the integral runs outside of the droplet, do not change in time [11]. They are initial data of evolution. Indeed,

$$\frac{d}{dt} t_k = \frac{1}{\pi k} \oint_{\text{interface}} z^{-k} \partial_n p(z) |dz| = 0$$

since the pressure is a harmonic function and is a constant on the interface. Conservation of the harmonic moments is an equivalent formulation of Laplacian growth, where surface tension is ignored (9).

We conclude that the growth of the semiclassical electronic droplet in a strong magnetic field is equivalent to the propagation of a "water" drop (less viscous liquid) in "oil" (more viscous fluid). This result is not surprising: both dynamics are determined by the condition of incompressibility.

1.4 Fingering instability, finite-time singularities and destruction of edge states.

The Saffman-Taylor problem has been intensively studied experimentally and analytically. It has been found that a small (almost arbitrary) deviation from a circular form of the initial shape of the droplet is unstable. The droplet forms a pattern of growing fingers whose shapes become complex as the area of the droplet increases [2]. Infact, the situation is even more dramatic. It is known that some fingers develop cusp-like singularities within a finite time of growth [6], i.e., when the area of the droplet is finite. In other words, fingers growing with $N\ell^2$, become thinner, and reach the atomic/molecular scale at the finite area of the droplet.

Similarly, one can cause a fingering instability of the quantum Hall droplet by changing the gradients of the magnetic field at fixed area. Fingers will be driven to a cusp-like singularity by adiabatically changing any of the harmonic moments t_k. When this occurs, the Laplacian growth equations (8,9) are no longer valid. At this point corrections obtained from the Navier-Stokes equations must be taken into account. They introduce a microscale in the form of the surface tension, that stops the curvature of the interface. Another mechanism to cure the singularities is the discretization of the liquid. In this case one assumes that the "water" domain consists of small particles of non-vanishing size [7].

The quantum Hall effect may be considered as a quantum version of the Laplacian growth problem [1]. It also provides an attractive mechanism of regularizing cusp-like singularities on the scale of the magnetic length, as it will become apparent in the following.

For the electronic droplet, a singularity means that, by increasing the number of particles at fixed gradients of magnetic field, the curvature of some segments of the droplet becomes so large that the semiclassical description is no longer valid. The electronic states of a sharp segment of the edge of the droplet are not separated from the bulk. They enjoy universal conformal properties that are very different from the conformal properties of edge states on a smooth part of the edge.

It is important that a singularity occurs inevitably.

1.5 Quantization of a singularity.

Summing up, at some point on a quantum Hall plateau, the edge of the droplet becomes very sharp and does not obey the standard semiclassical description. It cannot be described by a conformal field theory. Edge states at the cusp-like singularity of the classical edge seem important in tunneling processes and in the transitions between plateaus.

Analysis of the singularity is equally important for Laplacian growth problem. Quantum Hall effect provides a "quantized" version of the Laplacian growth where no singularity is possible on a scale less than magnetic length. Quantization may be seen as yet another regularization of singularity. The study, which we do not present here, shows that the states at the singularity enjoy universal scaling features, depending only on the qualitative character of the singularity. The analysis of the scaling behavior at the singularity is technically involved. Its algebraic aspects are similar to the universal scaling behavior of random surfaces, intensively studied in the context of 2D quantum gravity at $c < 1$, the so-called double scaling limit (for a review, see [8]). Physics of the states on a sharp edge is a subject of current studies of the author.

In the rest of these notes, we review some algebraic aspects of the dynamics of the electronic droplet in a nonuniform magnetic field and its relation with the Laplacian growth.

2. Algebraic aspects of electronic states in the quantum Hall regime and Laplacian growth.

2.1 Laplacian growth as an evolution of conformal maps.

The Laplacian growth can be conveniently reformulated as a problem of evolutions of conformal maps.

Let $w(z,t)$ is a conformal map of the exterior of the droplet to the exterior of the unit disk $|w| \geq 1$ in such a manner that the source at $z = \infty$ is mapped to infinity. In terms of the conformal map the pressure is $p = -\frac{1}{2}\log|w(z,t)|$ and the complex velocity in the viscous fluid is $v(z) = v_x - iv_y = \frac{1}{2}\partial_z \log w(z)$. On the interface, it is proportional to the harmonic measure:

$$v_n(z,t) = \frac{1}{2}|w'(z,t)|. \tag{10}$$

The complex velocity is conveniently written using the Schwarz function, $S(z)$: this is an analytic function in the domain containing the contour such that $S(z) = \bar{z}$ on the boundary [10]. The complex velocity is expressed in terms of this function by $\partial_t S(z)$. The equation (identity)

$$\partial_t S(z) = \partial_z \log w(z).$$

describes the evolution of the droplet under the condition that all parameters t_k are kept fixed.

One may be interested in the evolution of the droplet under a change of some particular t_k if the area and all other moments are kept fixed. This has been studied in Ref.[4]. For references we list the result here. The evolution reads:

$$\partial_{t_k} S(z) = \partial_z H^{(k)}(z), \quad k = 1, 2, \ldots \tag{11}$$

were the k-th Hamiltonian is a nonnegative part of the k-th power ofthe inverse conformal map $z(w)$. They are $H_k = \Big(z^k(w)\Big)_+ + \frac{1}{2}\Big(z^k(w)\Big)_0$. The symbols $(f(w))_\pm$ mean the truncated Laurent series where only terms with positive (resp. negative) powers of w are kept, while $(f(w))_0$ is the constant term (w^0) of the series. The derivatives in the last equations are taken at fixed z.

In refs.[4], the set of equations (11) was identified with the dispersionless limit of the integrable Toda lattice hierarchy. The compatibility of these equations give a set of nonlinear equations which describe the evolution of conformal maps under a deformation of the domain. For example, the first equation of the hierarchy is written for the conformal radius $r = \frac{1}{2\pi i}\oint \frac{dz}{w(z)}$ and $u = -\frac{1}{2\pi i r}\oint w(z)dz$. They are dispersionless limit of Toda equation and Kadomtzev-Petviashvili (KP) equations:

$$\partial_{t_1}\partial_{\bar{t_1}} \log r^2(t) = \partial_t^2 r^2(t) \tag{12}$$

$$3\partial_{t_2}^2 u_n + \partial_{t_1}(-4\partial_{t_3}u_n + 12u_n\partial_{t_1}u_n) = 0. \tag{13}$$

We will not develop this aspect further. See refs. [4] for the details.

2.2 The wave function in a nonuniform magnetic field.

We now return to the problem of the electronic droplet in a nonuniform magnetic field. Since the magnetic field is uniform inside the droplet, the one-particle wave functions in this area are obtained by linear combinations of the wave functions (2) times the gauge factor $e^{V(z)}$. They have the form

$$\psi_{n+1}(z) = P_n(z)e^{-\frac{|z|^2}{2\ell^2}+V(z)}, \tag{14}$$

where P_n is a holomorphic polynomial of the degree n.

There are two equivalent ways for finding the polynomials. One uses the fact that the deformed wave functions are still orthogonal. Therefore the holomorphic polynomials are bi-orthogonal with the measure $e^{-\frac{|z|^2}{\ell^2}+2V(z)}$. This condition uniquely determines the polynomials. Their explicit form is known [13]. It is given by a multiple integral

$$P_n(z) = \kappa_n^{-1}\int \Delta(\xi)\prod_{i\le n}(z-\xi_i)e^{-\frac{|z_i|^2}{2\ell^2}+V(z_i)}d^2\xi_i \tag{15}$$

where the normalization factor $\kappa_n^2 = n!(n+1)!\tau_n\tau_{n+1}$ and τ_n is the tau-function:

$$\tau_N = \frac{1}{N!}\int |\Delta(\xi)|^2\prod_n e^{-\frac{|\xi_i|^2}{\ell^2}+2V(\xi_i)}d^2\xi_n\,. \tag{16}$$

In the case of a uniform magnetic field the integrals are computed exactly: $P_n^{(0)}(z) = \frac{1}{\sqrt{\pi n!}} \frac{z^n}{\ell^{n+1}}$.

Computing the Slater determinant $\det(\psi_{n+1}(z_m))$, we obtain the multiparticle wave function (6) (we used the fact that $\det P_n(z_m) = \frac{1}{\sqrt{\tau_N}}\Delta(z)$, where $1/\sqrt{\tau_N}$ is the product of the coefficients of the highest monomials of $P_n(z)$). The formula of the tau- function (16) follows from the normalization condition for the wave-function.

Another way is to obtaine the orthogonal set of one-particle states as an overlap between $N+1$- and N-particle states (6):

$$\psi_{N+1}(z) = \int \Psi(z, \xi_1, \ldots, \xi_N)\overline{\Psi(\xi_1, \ldots, \xi_N)} \prod_{n \le N} d^2\xi_n. \tag{17}$$

This prompts the Eqs.(14,15).

2.3 Semiclassical states.

At large N, one may treat the formulas (6,16) in the semiclassical approximation. This immediately yields Eq.(4), and to the shape of the droplet described after this equation. It is interesting to go one step further to find a semiclassical form of the wave function in a nonuniform magnetic field characterized by the harmonic moments t_k (5). The result is sketched below (Ref.[1]).

A semiclassical state is characterized by the orbit - a smooth, closed and single connected loop with the area $\pi n \ell^2$ and a given harmonic moments t_k. We recall that they are the moments of the nonunoform part of the magnetic field (5). The semiclassical form of the wave function of this orbit (14) is found to be

$$\psi_N(z) \simeq \left(\frac{w'(z)}{2\pi\ell\sqrt{\pi}}\right)^{1/2} e^{-\frac{1}{\ell^2}\mathcal{A}(z,\bar{z})} e^{i\Phi(z)}$$

Here $w(z)$ is a conformal map of the exterior of the orbit to th exterior of the unit disk, a geometrical phase $2\pi\Phi(z)/\pi\ell^2$ is the area of a sector bounded by a ray $\arg z$ and some reference axis. The action $\mathcal{A}(z,\bar{z}) = \frac{1}{2}|z|^2 - Re\,\Omega(z)$, where $\Omega(z)$ is defined such that $\partial_z\Omega(z) = S(z)$ is the Schwarz function of the domain. The action is positive in the vicinity of the contour and everywhere in the exterior domain. Its variation normal to the orbit reads

$$\mathcal{A}(z + \delta_n z) = |\delta_n z|^2 - \frac{1}{3}\kappa(z)(\delta_n z)^3 + \ldots\,,$$

where $\delta_n z$ is a normal deviation from a point z of the orbit and $\kappa(z)$ is the curvature of the orbit.

A semiclassical state is localized at the minimum of $\mathcal{A}(z)$, where the amplitude has a sharp maximum. All orbits have the same harmonic moments t_k and are differed by the area. The holonomy of the state is $2\pi N$.

This result is easy to understand. The wave function (17,2.3) is a matrix element of the vertex operator at the edge of the quantum Hall state. The edge states are conformal invariant. Therefore the vertex operators on the edge of a circular droplet (in a uniform magnetic field) and a perturbed droplet (in a nonuniform magnetic field) differ by the conformal transformation: $\psi(z) \to (w'(z))^h \psi(z)$ where h is a dimension of the vertex operator. In the integer Hall effect the edge excitations are free fermions: $h = 1/2$. Similar calculations for the semiclassical limit of Laughlin's FQHE-$1/(m)$ states are expected to give the prefactor in (2.3) equal $(w'(z))^{1/2m}$.

In the classical approximation the amplitude of the wave function reads

$$|\psi_N|^2 \simeq \frac{1}{2\pi} |w'(z)| \delta(z), \tag{18}$$

where the δ-function is localized on the orbit.

2.4 Integrable structure of QHE states.

An integrable structure for the dynamics of the semiclassical droplet (evolution of conformal maps) suggests that the electronic states in quantum Hall regime may also obey an integrable nonlinear equations. This is, indeed, true.

Let us vary magnetic field and follow an evolution of the matrix elements of electronic operators. They evolve according to the Toda lattice hierarchy. A precursor of the integrability has been found in Refs.[12]. There, the operator content of QHE was identified with the $W_{+\infty}$ algebra. We will address this issue in details elsewhere (see also [4, 14] and references therein). Below we will write the major formulas.

The polynomials (15) represent the coherent states of the operator of magnetic translations $Z = \ell^2(-2\partial_{\bar z} + \bar A)$ in the arrangements where nonuniform field is located outside of the droplet, i.e., when $A - \ell^{-2}\bar z = -2\frac{\partial}{\partial z} V(z)$ is a holomorphic function. This operator annihilates all wave functions (14) of the first Landau level $Z\psi_n(z) = 0$ and acts as a multiplicator on the polynomials $ZP_n(z) = zP_n(z)$. The Hermitian conjugated operator $\bar Z = \ell^2(2\partial_z + A)$ differentiates the polynomials $\bar Z \psi_n = e^{-\frac{1}{\ell^2}|z|^2 + V(z)} \ell^2 \partial_z P_n(z)$. In terms of these operators the

Hamiltonian (1) is $H = \frac{1}{2m\ell^4}(Z\bar{Z} + \bar{Z}Z)$. Obviously

$$[\bar{Z}, Z] = 2\ell^2. \tag{19}$$

As on operator acting on polynomials, Z is a lower triangular matrix with the upper adjacent diagonal. The operator $\bar{Z}$ is a lower diagonal matrix

$$Z_{nm}P_m(z) = zP_n(z), \quad Z_{nm} = 0 \text{ at } \ m > n+1. \tag{20}$$

$$\bar{Z}_{nm}P_m(z) = 2\ell^2\partial_z P_n(z), \quad \bar{Z}_{nm} = 0 \text{ at } \ m \geq n. \tag{21}$$

Their matrix elements depend on the magnetic field and are parametrized by t_k.

The integrable hierarchy describes an evolution of the wave functions, or, equivalently, the matrix elements of the operators Z and $\bar{Z}$ as functions of parameters t_k:

$$\partial_{t_k}\psi_n(z) = H^{(k)}_{nm}(z)\psi_m(z) \tag{22}$$

The commutative set of Hamiltonians $H^{(k)}_{nm}(z)$ are proved to be a set matrices with zeros in the lower triangular part. They are

$$H^{(k)} = \left(Z^k\right)_+ + \frac{1}{2}\left(Z^k\right)_0 \tag{23}$$

where $\left(Z^k\right)_+$ and $\left(Z^k\right)_0$ are upper triangular and diagonal parts of the k-th power of the matrix Z_{nm}. In terms of operator Z the evolution equations read

$$\frac{\partial Z}{\partial t_k} = [H^{(k)}, Z], \quad \frac{\partial \bar{Z}}{\partial t_k} = [H^{(k)}, \bar{Z}] \tag{24}$$

These equations and the vanishing commutators among the Hamiltonians give a set of nonlinear equations for the matrix elements and coefficients of the polynomials. For example, the equation for the "quantum conformal radius" $r_n = Z_{n,n+1}$ and $u_n = \frac{Z_{n,n+2}}{Z_{n,n+1}}$ are the celebrated Toda and KP equations:

$$\ell^2\partial^2_{t_1\bar{t}_1}\log r_n^2 = r_{n+1}^2 - 2r_n^2 + r_{n-1}^2, \tag{25}$$

$$3\partial^2_{t_2}u_n + \partial_{t_1}(\ell^{-2}\partial^3_{t_1}u_n - 4\partial_{t_3}u_n + 12u_n\partial_{t_1}u_n) = 0. \tag{26}$$

In the terminology of integrable hierarchies, the operators Z and $\bar{Z}$ are a pair of Lax operators; the wave function $\psi_n(z)$ is the Baker-Akhiezer function; Eq. (19) is the string equation; Eqs. (22-24)) are the Lax-Sato equation. Finally, (16) represents the tau-function of the hierarchy.

The connection with the semiclassical description is transparent. The classical limits ($\ell^2 \to 0$) of the operators Z and $\bar{Z}$ are the coordinate of the droplet and its Schwarz function. The classical (dispersionless) limit of the Lax-Sato equations describe the evolution of conformal maps (11). The Toda equation (25) is reduced to the dispersionless Toda equation (12) for the conformal radius in the limit $n \to \infty$ while $t = n\ell^2$ is kept fixed.

2.5 Random matrix representations.

The wave functions of the Quantum Hall effect are naturally related to random matrices. The square of the amplitude of the multiparticle wave function (6) can be obtained as a result of the integration of $e^{-\frac{1}{\ell^2}\mathrm{tr}MM^\dagger + \mathrm{tr}V(M,M^\dagger)}$ over certain ensembles of complex matrices. Here $2V(M, M^\dagger) = \sum_k (t_k M^k + \bar{t}_k (M^k)^\dagger)$.

One ensemble is $N \times N$ normal matrices with a given set of complex distinct eigenvalues $z_1, \dots, z_N$ [14, 4]. We recall that the normal matrices are the complex matrices with a relation $[M,\, M^\dagger] = 0$. Integration over these matrices recovers (6) up to a factor.

Another ensemble has been pointed to the author by M. Hastings. This is an ensemble of arbitrary complex matrices [15]. We recall this relation briefly. Any complex matrix with distinct eigenvalues $z_1, \dots, z_N$ can be decmposed into as $M = U^\dagger(\mathrm{diag}(z_1, \dots, z_N) + R)U$, where U is a unitary matrix and R is an upper triangular complex matrix. Potential $\mathrm{tr}V(M, M^\dagger) = \sum_n V(z_n, \bar{z}_n)$ depends only on eigenvalues, while the measure of the integral $D[M] = D[U]D[R]\,|\Delta(z)|^2$, and $\mathrm{tr}MM^\dagger = \sum_n |z_n|^2 + \sum_{i>j} |R_{ij}|^2$ are factorized. The volume of the unitary group $\int D[U]$ and the gaussian integration over matrix elements R_{ij} of the matrix R contributes just numerical factors. As a result $|\Psi(z_1, \dots, z_N)|^2 \sim \int DMDM^\dagger e^{-\frac{1}{\ell^2}\mathrm{tr}MM^\dagger + \mathrm{tr}V(M,M^\dagger)}$.

Appearance of integrable hierarchies and random matrices ties the Laplacian growth and the dynamics of quantum Hall edge states to a number of important problems of theoretical and mathematical physics.

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Notes

1. We set $\hbar = e = c = 1$ hereafter.

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