Statistics of finite-time Lyapunov exponents in a random time-dependent potential

H. Schomerus* and M. Titov†
Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Str. 38, 01187 Dresden, Germany
(Dated: April 2002)

The sensitivity of trajectories over finite time intervals t to perturbations of the initial conditions can be associated with a finite-time Lyapunov exponent λ, obtained from the elements M_{ij} of the stability matrix M. For globally chaotic dynamics λ tends to a unique value (the usual Lyapunov exponent λ_∞) as t is sent to infinity, but for finite t it depends on the initial conditions of the trajectory and can be considered as a statistical quantity. We compute for a particle moving in a random time-dependent potential how the distribution function $P(\lambda;t)$ approaches the limiting distribution $P(\lambda;\infty)=\delta(\lambda-\lambda_\infty)$. Our method also applies to the tail of the distribution, which determines the growth rates of positive moments of M_{ij}. The results are also applicable to the problem of wave-function localization in a disordered one-dimensional potential.

PACS numbers: 05.45.-a, 05.40.-a, 42.25.Dd, 72.15.Rn

I. INTRODUCTION

In this work, we give a uniform description of the complete asymptotic statistics of the finite-time Lyapunov exponent for a particle moving in a random time-dependent potential. The Lyapunov exponent λ_∞ characterizes the sensitivity of trajectories to small perturbations of the initial conditions and plays a fundamental role in the characterization of systems which display deterministic chaos [1]. The Lyapunov exponent is defined in the joint limits of vanishing initial perturbation and infinitely large times. In a hyperbolic Hamiltonian system λ_∞ may be obtained from any non-periodic trajectory, because for arbitrarily long times the trajectories uniformly explore the complete phase space.

A widely studied generalization of λ_∞ is the finite-time Lyapunov exponent [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17], which is defined for finite stretches (time interval t) of trajectories (generalizations to finite perturbations also exist [18]). The sensitivity of the dynamics to initial perturbations is given by the stability matrix map M, which is the linearization of the map of initial coordinates to final coordinates. In terms of elements M_{ij} of M, the (complex) finite-time Lyapunov exponent may then be defined as

$$\lambda = \frac{1}{t}\ln M_{ij}. \tag{1}$$

In contrast to λ_∞, λ is not a unique number independent of the initial conditions, but a fluctuating quantity with a distribution function $P(\lambda;t)$ (defined by uniformly sampling all initial conditions in phase space). This distribution function determines, e. g., the generalized entropy and dimension spectra of dynamical systems [1], and more practically the weak-localization correction to the conductance [19] and the shot-noise suppression [20, 21] in mesoscopic systems. Finite-time Lyapunov exponents also determine the wavefront stability of acoustic and electromagnetic wave propagation through a random medium, in the ray-acoustics/ray-optics regime of short wave lengths (for a recent application see Refs. [22, 23]). Moreover, they have shifted into the focus of attention due to recent advances in the understanding of the role of the Lyapunov exponents for quantum-chaotic wave propagation [24, 25, 26, 27, 28, 29, 30]: It has been observed that under certain conditions the Lyapunov exponent can be extracted from the decay of the overlap of two wavefunctions which are propagated by two slightly different Hamiltonians (the so-called Loschmidt echo). Since the overlap is studied as a function of time, the distribution of the finite-time Lyapunov exponent is directly relevant for these investigations. This extends also to related semiclassical time scales, like to the Ehrenfest time $\sim(\log\hbar)/\lambda$, which is a semiclassical estimate of the diffraction time of wave packets due to the chaotic classical dynamics.

In the limit of infinite time t the distribution function $P(\lambda;t)$ in a completely chaotic phase space tends to the limiting form $P(\lambda;\infty)=\delta(\lambda-\lambda_\infty)$. For large but finite t the bulk of the distribution function can be approximated by a Gaussian centered around λ_∞, with the width vanishing $\propto t^{-1/2}$ as $t\to\infty$. However, many of the properties determined by $P(\lambda;t)$ (like the generalized entropy and dimension spectra) cannot be calculated from the Gaussian bulk of the distribution function [1].

In this paper we investigate for a particle moving in a one-dimensional random time-dependent potential how $P(\lambda;t)$ approaches the limiting distribution function $P(\lambda;\infty)=\delta(\lambda-\lambda_\infty)$ for large times. Our approach uniformly applies both to the bulk as well as to the far tail $\lambda\gg\lambda_\infty$ of the distribution function. We find that the cumulant-generating function of $P(\lambda;t)$,

$$\eta(\xi)=\ln\langle\exp(\xi t\lambda)\rangle=\sum_{n=1}^{\infty}\langle\langle\lambda^n\rangle\rangle\frac{(\xi t)^n}{n!}, \tag{2}$$

(where the average $\langle\cdot\rangle$ is over initial conditions and $\langle\langle\cdot\rangle\rangle$ denotes the cumulants), takes the asymptotic form

$$\eta(\xi)=\mu(\xi)t/t_c+\mathcal{O}(t^0), \tag{3}$$

with $\mu(\xi)$ a universal function (within the statistical model) and

$$t_c=\lambda_\infty^{-1}\mu^{(1)} \tag{4}$$

*Electronic address: henning@mpipks-dresden.mpg.de
†Electronic address: titov@mpipks-dresden.mpg.de

a system-specific time-scale which can be determined from the infinite-time-Lyapunov exponent and the constant $\mu^{(1)} = d\mu/d\xi|_{\xi=0}$ (by definition, $d\eta/d\xi|_{\xi=0} = \lambda_\infty t$). The function $\mu(\xi)$ is given by the leading eigenvalue of a second-order differential equation in which ξ appears as a parameter. This eigenvalue can be calculated perturbatively in ξ, which generates the cumulants of λ. The values of μ at integer ξ determine the asymptotic growth rates $(1/t)\ln\langle M_{ij}^{\xi}\rangle = \mu(\xi)/t_c$ of positive moments of elements of the stability matrix. We find that these values are given by the leading eigenvalue of finite-dimensional matrices.

A random time-dependent potential is often considered as a statistical model for the ergodic properties of hyperbolic chaotic motion, in the spirit of the early work of Chirikov [31]. The time dependence of the potential may be considered to mimic the dependence of the potential in the eigentime along the trajectory. In the context of finite-time Lyapunov exponents there have been indications that a statistical description is usually valid for the chaotic background of its distribution [16], while system-specific deviations may exists in some exceptional cases even in the bulk of the distribution function [15]. While the statistical model considered in this work is tailored to a specific class of Hamiltonian systems, it can be modified straightforwardly to other classes of chaotic systems (this is briefly described at the end of this paper).

The problem of finite-time Lyapunov exponents in the random time-dependent potential is equivalent to the problem of wave-localization in a random one-dimensional potential [32, 33, 34, 35, 36], because the equations of motion for the matrix elements M_{ij} are formally equivalent to the corresponding Schrödinger equation [10, 37]. Indeed, the Fokker-Planck equation employed in this work is based on the phase formalism described, e.g., in Ref. [38, 39, 40]. Hence, the asymptotic statistics of the finite-time Lyapunov exponent presented in this work directly is of interest and can be transferred to this field of research. A number of additional areas of application of our method come into scope if one considers the vast arena of problems which can be analyzed by products of random matrices, since the finite-time Lyapunov exponents are a valuable way to characterize the eigenvalues of these products [10].

The plan of this paper is as follows: In Sec. II we formulate the problem of finite-time Lyapunov exponents in the one-dimensional random time-dependent potential. In Sec. III we show how the cumulant-generating function can be related to the parameterized eigenvalue of a second-order differential equation, and that the cumulants can be calculated systematically. Positive moments of M_{ij} are calculated in Sec. IV. We close the paper with discussion and conclusions in Sec. V.

II. FORMULATION OF THE PROBLEM

A. Statistical model

Let us consider a time-dependent Hamiltonian system with one degree of freedom (canonically conjugated coordinates x, p), and the Hamiltonian given by

$$H = \frac{p^2}{2m} + V(x,t) + \frac{V_2}{2}x^2. \tag{5}$$

Here $V(x,t)$ is a time-dependent potential and m is a mass. We also allow for an additional static potential with curvature V_2 acting in the background of the random potential (this potential is repulsive for $V_2 < 0$ and attractive for $V_2 > 0$).

We introduce the map $\mathcal{F}_t$ which propagates initial conditions (x_i, p_i) over a time interval t to the final coordinates $(x_f, p_f) = \mathcal{F}_t(x_i, p_i)$. The stability matrix M is the linearization of the map $\mathcal{F}_t$ and describes the sensitivity of the final coordinates to a small perturbation of the initial conditions,

$$M = \frac{\partial(x_f, p_f)}{\partial(x_i, p_i)} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}. \tag{6}$$

Area preservation of the dynamics in phase space entails the property $\det M = 1$ of the stability matrix.

We are interested in the evolution of the stability matrix with given initial conditions and increasing time interval t. According to Hamilton's equations of motion the stability matrix fulfills the differential equation

$$\frac{dM}{dt} = KM, \quad K = \begin{pmatrix} 0 & m^{-1} \\ v & 0 \end{pmatrix}, \tag{7}$$

where the function $v(t)$ in the matrix K is given by

$$v = -V_2 - \left.\frac{d^2V}{dx^2}\right|_{(x,p)=(x_f,p_f)}. \tag{8}$$

This differential equation is supplemented by the initial conditions

$$M(0) = \mathrm{diag}(1,1), \tag{9}$$

corresponding to the identification of the initial and final coordinate systems for $t = 0$.

In order to study the statistical behavior of the stability matrix we now assume that $v(t)$ is a randomly fluctuation function equivalent to Gaussian random δ-correlated noise,

$$\langle v(t)\rangle = -V_2, \qquad \langle v(t_1)v(t_2)\rangle = 2D\delta(t_1 - t_2). \tag{10}$$

The condition of a vanishing mean of the time-dependent part of v corresponds to the observation that the incidence of positive and negative curvature of the potential landscape along a typical chaotic trajectory should be identical. The δ-function correlations are valid if the correlation time of the fluctuations is smaller than the mean free transport time in the random potential. The constant D (similar to a diffusion constant, but not identical with conventional diffusion constants of motion in phase space) can be related to the strength of the temporal fluctuations of the potential $V(x,t)$. However, both D as well as the mass m can be eliminated from the subsequent analysis by rescaling quantities in the following way:

$$\begin{aligned} &t = t_c t', \qquad v = (D/m)t_c v', \qquad V_2 = (D/m)t_c V_2' \\ &M_{12} = (t_c/m)M_{12}', \qquad M_{21} = (m/t_c)M_{21}', \\ &M_{11} = M_{11}', \qquad M_{22} = M_{22}'. \end{aligned} \tag{11}$$

Here we defined the characteristic time scale

$$t_c = m^{2/3} D^{-1/3}. \tag{12}$$

[In the course of our analysis we will see that this time scale also can be found from Eq. (4).] The rescaled (primed) quantities fulfill Eqs. (7), (9), (10) with $D = m = 1$. Also note that the rescaling leaves the property $\det M = 1$ invariant.

B. Relation to one-dimensional localization

The set of linear first-order differential equations (7) can be decoupled by converting them into second-order differential equations. It is useful to note (as mentioned in the introduction) that the equations for the elements M_{11} and M_{12} are equivalent to the Schrödinger equation, at energy $E = V_2/m$, of a particle of mass $\hbar^2/2$ in a one-dimensional random potential $(v + V_2)/m$ (of vanishing mean), with t playing the role of the spatial coordinate,

$$\frac{d^2 M_{11}}{dt^2} = \frac{v}{m} M_{11}, \quad \frac{d^2 M_{12}}{dt^2} = \frac{v}{m} M_{12}, \tag{13}$$

while the other matrix elements are directly related to them by

$$M_{21} = m\frac{dM_{11}}{dt}, \quad M_{22} = m\frac{dM_{12}}{dt}. \tag{14}$$

The problem of finite-time Lyapunov exponents hence is closely related to the problem of one-dimensional localization in a random potential, in which the Lyapunov exponent corresponds to the inverse decay length of the wave function.

III. CUMULANTS OF THE FINITE-TIME LYAPUNOV EXPONENT

We now solve the problem of finding the probability distribution function of matrix elements M_{ij} within the statistical model of chaotic dynamics, defined by the evolution equation (7) for M, with initial condition (9), and the statistical properties (10) of the random function v. For the sake of definiteness we will consider in this section the statistics of the upper diagonal element M_{11}. The results directly carry over to the other elements of M, as is discussed in Sec. IV C.

A. Cumulant-generating function as an eigenvalue

We introduce the quantities

$$u = \ln M'_{11}, \quad z = \frac{M'_{21}}{M'_{11}}, \tag{15}$$

where the relation $u = \lambda t$ to the finite-time Lyapunov exponent λ is established by Eq. (1) [note that $M_{11} = M'_{11}$ in the rescaling Eq. (11)]. According to Eqs. (7) and (11), u and z fulfill the differential equations

$$\frac{du}{dt'} = z, \qquad \frac{dz}{dt'} = v' - z^2. \tag{16}$$

Note that the evolution equation of z decouples from u and can be interpreted as a Langevin equation. Hence the distribution $P(z;t')$ can be calculated from a Fokker-Planck equation, which was considered before in the context of wavefunction localization [39, 40],

$$\partial_{t'} P(z;t') \;=\; \mathcal{L}_z P(z;t'), \tag{17a}$$

$$\mathcal{L}_z \,\cdot \;=\; \partial_z (z^2 + V_2' + \partial_z) \;\cdot\, . \tag{17b}$$

For large t' the distribution function $P(z;t')$ approaches the stationary solution [38, 39, 40]

$$P_{\text{stat}}(z) \;=\; \tilde{N} \int_{-\infty}^{z} dy\, K(y,z), \tag{18a}$$

$$K(y,z) \;=\; e^{(y^3 - z^3)/3 + V_2'(y-z)}, \tag{18b}$$

$$\tilde{N} \;=\; \pi^{-2} [\mathrm{Ai}^2(-V_2') + \mathrm{Bi}^2(-V_2')]^{-1}. \tag{18c}$$

Here Ai and Bi are Airy functions. The normalization constant is directly related to the integrated density of states in the localization problem [38, 39, 40]. For $V_2' = 0$, $\tilde{N} = 3^{5/6} 2^{-1/3} \pi^{-1/2} / [\Gamma(1/6)]$. Because $du/dt = z/t_c$ it is clear [40] that the infinite-time Lyapunov exponent can be obtained from $\lambda_\infty = \langle z \rangle / t_c$; this relation will be demonstrated explicitely in Sec. III C.

The Fokker-Planck equation for the joint distribution function $P(u,z;t')$ is given by

$$\partial_{t'} P = -z \partial_u P + \mathcal{L}_z P. \tag{19}$$

This Fokker-Planck equation with $V_2' = 0$ has been derived in Ref. [19] for the autonomous chaotic scattering of a particle from a dilute collection of scatterers (with more than one degree of freedom).

The joint distribution function $P(u,z;t')$ does not approach a stationary limit because u runs away to infinitely large values. In order to analyze the behavior of the distribution function $P(u,z;t')$ for large times we convert the Fokker-Planck equation (19) into an eigenvalue problem which discriminates between the different time scales involved in this evolution. For this purpose, we introduce into Eq. (19) the ansatz

$$P(u,z;t') = \int_{-i\infty}^{+i\infty} \frac{d\xi}{2\pi i} \sum_{n=0}^{\infty} \exp(\mu_n t' - \xi u) f_n(\xi, z). \tag{20}$$

(The integration contour along the imaginary axis corresponds to an inverse Laplace transformation.) It follows that the functions f_n fulfill the differential equation

$$\mu_n f_n(\xi,z) = (\xi z + \mathcal{L}_z) f_n(\xi,z), \tag{21a}$$

in which μ_n and ξ appear as parameters. However, in order to obtain a meaningful probability distribution function (20) we have to impose boundary conditions on $f_n(\xi,z)$ at $z \to \pm\infty$. It is convenient to express these boundary conditions by the requirement

$$\mathcal{P} \int_{-\infty}^{\infty} dz\, f_n(\xi,z) z < \infty. \tag{21b}$$

Here $\mathcal{P}$ denotes the principal value with respect to the integration boundaries at $\pm\infty$. Condition (21b) follows from the behavior $z \approx (t' - t'_\infty)^{-1}$ of the solution of the differential equation (16) close to times $t' \approx t'_\infty$ where $|z| \to \infty$ (and hence v' can be ignored). In practical terms, the condition (21b) guarantees that the drift of u remains finite for all times.

Eqs. (21) form an eigenvalue problem, since condition (21b) only can be fulfilled for a discrete set of numbers μ_n—note that these eigenvalues depend on the parameter ξ. In the limit of large t' only the largest eigenvalue $\mu_0(\xi) \equiv \mu(\xi)$ is relevant, because the other eigenvalues give rise to exponentially smaller contributions. This eigenvalue vanishes as $\xi \to 0$, *i. e.*, $\mu(0) = 0$, because the stationary distribution of z, Eq. (18), must be recovered for large times from Eq. (20) by integrating out u.

The moments of u are given by

$$\begin{aligned} \langle u^n \rangle &= \int_{-i\infty}^{+i\infty} \frac{d\xi}{2\pi i} \int_{-\infty}^{\infty} du\, \exp(\mu t' - \xi u) f(\xi) u^n \\ &= \lim_{\xi \to 0} \partial_\xi^n \exp(\mu(\xi) t') f(\xi), \end{aligned} \tag{22}$$

where the coefficients $f(\xi) = \int_{-\infty}^{\infty} dz f_0(\xi, z)$ are determined, in principle, by the initial condition for $P(u, z; t')$ at $t' = 0$. From Eq. (22) we obtain the moment-generating function

$$\chi(\xi) = \langle \exp(\xi u) \rangle = \exp(\mu(\xi) t / t_c) f(\xi), \tag{23}$$

where we re-introduced the original time variable $t = t_c t'$ by Eq. (11). The cumulant-generating function (2) hence takes the form of Eq. (3), including the corrections of order t^0,

$$\eta(\xi) = \ln \chi(\xi) = \mu(\xi) t / t_c + \ln f(\xi). \tag{24}$$

The cumulants $\langle\langle \lambda^n \rangle\rangle$ of the finite-time Lyapunov exponent are obtained by expanding the generating function η in powers of ξ, see Eq. (2). In terms of the coefficients of the Taylor expansion

$$\mu = \sum_{n=1}^{\infty} \xi^n \mu^{(n)} \tag{25}$$

[which starts with the linear term in ξ because $\mu(0) = 0$], according to Eqs. (2) and (24) the nth cumulant of λ is then given by

$$\langle\langle \lambda^n \rangle\rangle = n! \mu^{(n)} t_c^{-1} t^{1-n} + \mathcal{O}(t^{-n}). \tag{26}$$

This equation means that within the statistical model the cumulants are universal quantities in the leading order in t, in the sense that the initial conditions $P(z, u; 0)$ only enter the next-order corrections. The only system-specific parameters which enter the cumulants are the time scale t_c and the (rescaled) strength V_2' of the static potential. Note that ratios of cumulants are even independent of the time scale t_c (and hence of the parameters D and m of the statistical model).

The form (4) of t_c follows from Eq. (26) when t_c is expressed in terms of the infinite-time Lyapunov exponent λ_∞ with help of the definition

$$\lambda_\infty = \lim_{t \to \infty} \langle \lambda \rangle = \mu^{(1)} / t_c. \tag{27}$$

In terms of the bare quantities of the statistical model,

$$\lambda_\infty = \mu^{(1)} D^{1/3} m^{-2/3}. \tag{28}$$

In the next two sections we obtain general expressions for the expansion coefficients $\mu^{(n)}$ and calculate explicitely the proportionality factor $\mu^{(1)} = d\mu/d\xi|_{\xi=0}$ in (28), as well as the first few coefficients $\mu^{(2)}$, $\mu^{(3)}$, ..., which determine, respectively, the variance and the leading non-Gaussian corrections (higher cumulants) of the fluctuations of the finite-time Lyapunov exponent around its limiting value λ_∞.

B. Recursion relations for the cumulants

We now show how the cumulants can be calculated from Eq. (26) by recursively solving a hierarchy of equations for coefficients $\mu^{(n)}$ in the Taylor expansion of $\mu(\xi)$, Eq. (25).

In analogy to Eq. (25) let us also expand the function $f_0(\xi, z)$ in powers of ξ,

$$f_0(\xi, z) = \sum_{n=0}^{\infty} \xi^n f_0^{(n)}(z). \tag{29}$$

With Eqs. (25) and (29) the eigenvalue problem (21) can now be written order by order in powers of ξ^n. For $n = 0$ we recover the stationary variant (17) of the Fokker-Planck equation (19),

$$\mathcal{L}_z f_0^{(0)}(z) = 0, \tag{30}$$

which is solved by the stationary solution $f_0^{(0)}(z) = P_{\rm stat}(z)$, Eq. (18). For $n > 1$ the differential equations are of the form

$$\mathcal{L}_z f_0^{(n)}(z) = -z f_0^{(n-1)}(z) + \sum_{l=1}^{n} \mu^{(l)} f_0^{(n-l)}(z). \tag{31}$$

Let us assume that we have solved the hierarchy of equations up to order $n-1$. In the next order n both the unknown quantities $f_0^{(n)}$ as well as $\mu^{(n)}$ appear. The unknowns can be separated by integrating the differential equation (31) over z from $-\infty$ to ∞: The integrated left-hand side vanishes because of condition (21b) of the eigenvalue problem. The integrated right-hand side can be rearranged to give $\mu^{(n)}$,

$$\mu^{(n)} = \int_{-\infty}^{\infty} dz\, [z f_0^{(n-1)}(z) - \sum_{l=1}^{n-1} \mu^{(l)} f_0^{(n-l)}(z)], \tag{32a}$$

which only involves quantities up to order $n-1$. Subsequently, $\mu^{(n)}$ can be inserted into Eq. (31). The function

$$\begin{aligned} f_0^{(n)}(z) &= \int_{-\infty}^{z} dy \int_{-\infty}^{y} dx\, K(y, z) \\ &\quad \times [-x f_0^{(n-1)}(x) + \sum_{l=1}^{n} \mu^{(l)} f_0^{(n-l)}(x)] \end{aligned} \tag{32b}$$

[with the kernel $K(y, z)$ defined in Eq. (18)] is then obtained by solving the resulting inhomogeneous differential equation with help of the partial solution $f_0^{(0)}(z)$ of its homogeneous counterpart, Eq. (30). This inhomogeneous part of the functions $f_0^{(n)}(z)$ is fixed by the requirement that $f_0^{(0)}(z)$ is normalized to 1. Adding the homogeneous solution to $f_0^{(n)}(z)$ in any order gives rise to additional terms in all higher orders, but these combine in such a way that they drop out of the calculation of the coefficients $\mu^{(n)}$, which hence are uniquely determined by Eq. (32a).

The recursion relations (32) can be iterated to calculate successively all cumulants of λ.

C. Explicit expressions and numerical values

According to Eq. (26), the two numbers $\mu^{(1)}$ and $\mu^{(2)}$ determine mean and variance of the distribution function of λ, which then is approximated by a Gaussian. The coefficient $\mu^{(1)}$ has been obtained in Ref. [40] from the Fokker-Planck equation (17) for arbitrary V_2. For the special case $V_2 = 0$, the two coefficients $\mu^{(1)}$ and $\mu^{(2)}$ have been obtained in Ref. [19] from the Fokker-Planck equation (19). However, the deviations from the Gaussian distribution function are not at all negligible for many chaotic systems, which is most clearly displayed in their generalized dimension and entropy spectra [1]. As we have seen in the previous subsection III B, our approach of reduction to the eigenvalue problem (21) allows to analyze the non-Gaussian deviations by the higher cumulants of λ. [In next section IV, we show that one can even obtain from our analysis the positive moments of M_{11}, which are determined by the far tail $\lambda \gg \lambda_\infty$ of $P(\lambda;t)$, while the bulk of the distribution is essentially irrelevant for these moments.]

Explicit expressions for the first few coefficients $\mu^{(1)}$, $\mu^{(2)}$, $\mu^{(3)}$, and $\mu^{(4)}$ result from Eq. (32a),

$$\mu^{(1)} = \int_{-\infty}^{\infty} dz\, z f_0^{(0)}(z), \tag{33a}$$

$$\mu^{(2)} = \int_{-\infty}^{\infty} dz\, (z-\mu^{(1)}) f_0^{(1)}(z), \tag{33b}$$

$$\mu^{(3)} = \int_{-\infty}^{\infty} dz\, [(z-\mu^{(1)}) f_0^{(2)}(z) - \mu^{(2)} f_0^{(1)}(z)], \tag{33c}$$

$$\mu^{(4)} = \int_{-\infty}^{\infty} dz\, [(z-\mu^{(1)}) f_0^{(3)}(z) - \mu^{(2)} f_0^{(2)}(z) - \mu^{(3)} f_0^{(1)}(z)], \tag{33d}$$

where $f_0^{(0)}(z) = P_{\text{stat}}(z)$ is given by the stationary distribution function of z, Eq. (18), while the other functions follow

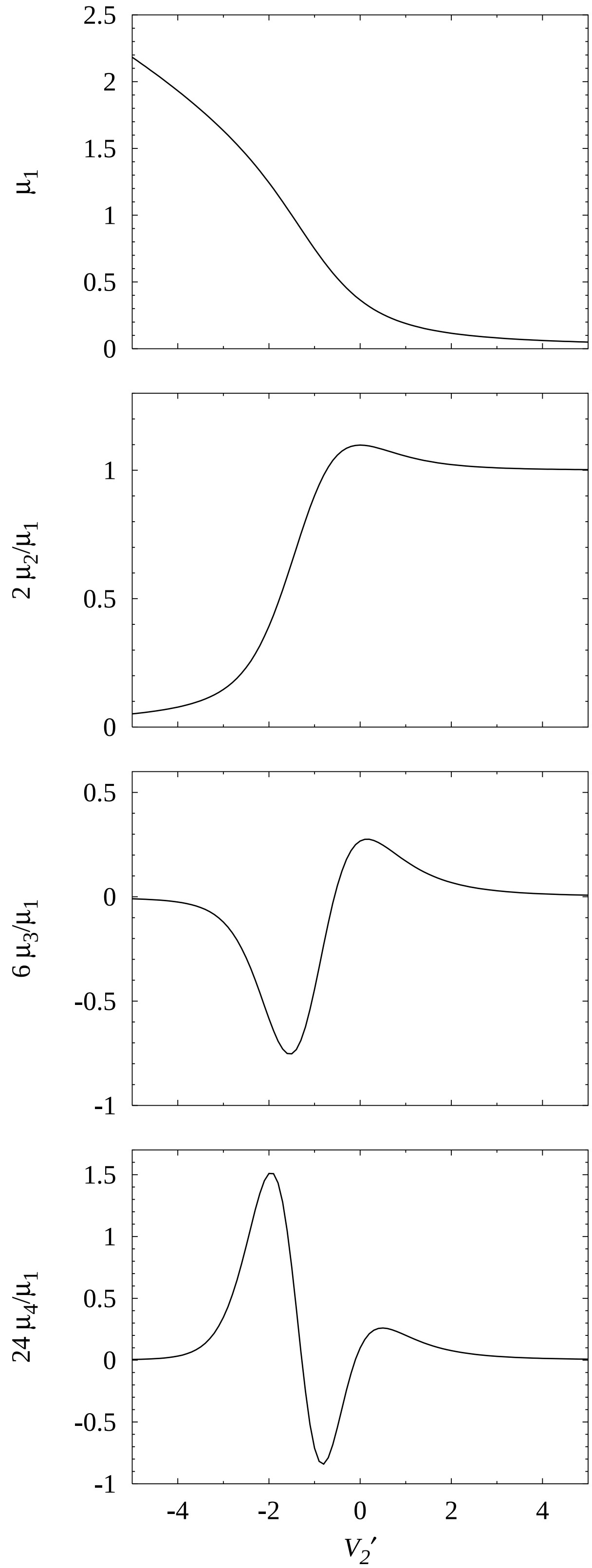

FIG. 1: Coefficient $\mu^{(1)}$ of the first cumulant, and the ratios $n!\mu^{(n)}/\mu^{(1)}$ for the coefficients of the second, third, and fourth cumulant [cf. Eq. (26)], as a function of the strength V_2' of the static background potential.

n	1	2	3	4
$n!\mu^{(n)}$	0.365	0.401	0.0975	0.0361
n	5	6	7	8
$n!\mu^{(n)}$	-0.266	-0.628	-0.554	3.71

TABLE I: First eight coefficients $n!\mu^{(n)}$ of the cumulants of finite-time Lyapunov exponents [cf. Eq. (26)], in absence of the static background potential ($V_2'=0$).

ξ	1	2	3	4
μ	0	$2^{2/3}$	$24^{1/3}$	$84^{1/3}$
ξ		5	6	7
μ		$2(14+3\sqrt{19})^{1/3}$	$(252+24\sqrt{79})^{1/3}$	$2(63+15\sqrt{10})^{1/3}$

TABLE II: Exponential growth rates $\mu(\xi)$ of the first few moments $\langle M_{11}^{\xi}\rangle$ [cf. Eq. (38)], in absence of the static background potential ($V_2'=0$).

from Eq. (32b),

$$f_0^{(1)}(z) = \int_{z>y>x} dy\,dx\,K(y,z)(\mu^{(1)}-x)f_0^{(0)}(x), \quad (34a)$$

$$f_0^{(2)}(z) = \int_{z>y>x} dy\,dx\,K(y,z)[(\mu^{(1)}-x)f_0^{(1)}(x) - \mu^{(2)}f_0^{(0)}(x)], \quad (34b)$$

$$f_0^{(3)}(z) = \int_{z>y>x} dy\,dx\,K(y,z)[(\mu^{(1)}-x)f_0^{(2)}(x) - \mu^{(2)}f_0^{(1)}(x) - \mu^{(3)}f_0^{(0)}(x)]. \quad (34c)$$

The coefficient $\mu^{(1)}$ is then given by [40]

$$\mu^{(1)} = \frac{1}{2}\frac{d}{dV_2'}\log\tilde{N}, \quad (35)$$

where $\tilde{N}$ is given in Eq. (18), while the cumulants for $n\geq 2$ can be obtained quickly by numerical integration of $2n$-fold integrals. The effort of integration can be greatly reduced down to the expense equivalent to a single integral, because the integrand factorizes. An efficient recursive scheme is described in the Appendix. In Fig. 1 we plot the coefficient $\mu^{(1)}$ and the ratios $n!\mu^{(n)}/\mu^{(1)}$ for $n=2,3,4$ as a function of V_2'. The non-Gaussian corrections are largest around $V_2'=0$, while they become irrelevant for large negative or positive values of V_2'.

For strong confinement, $V_2' \gg 1$, the coefficients $n!\mu^{(n)}/\mu^{(1)} \to \delta_{1n}+\delta_{2n}$, with δ_{mn} the Kronecker symbol, and the Gaussian approximation

$$\mu_{\text{Gaussian}}(\xi) = \mu^{(1)}\left(\xi+\frac{1}{2}\xi^2\right) \quad (36)$$

becomes valid. [In the context of wave localization, this corresponds to the well-known limit of a large Fermi energy $E\sim V_2'$ (cf. Sec. II B).]

Analytical results can be found in the case $V_2'=0$ for the first two coefficients,

$$\mu^{(1)} = \frac{(3/2)^{1/3}\sqrt{\pi}}{\Gamma(1/6)}, \quad (37a)$$

$$\mu^{(2)} = \frac{5\pi^2}{18}\tilde{N} - \frac{\pi}{2\sqrt{3}}\tilde{N}\,{}_3F_2\left(1,1,\frac{7}{6};\frac{3}{2},\frac{3}{2};\frac{3}{4}\right), \quad (37b)$$

where $\tilde{N}(V_2'=0) = 3^{5/6}2^{-1/3}\pi^{-1/2}/[\Gamma(1/6)]$, while ${}_3F_2$ is a generalized hypergeometric function. Incidentally, the numerical value given for $\mu^{(2)}$ in Ref. [19] is wrong, but the analytic expression given in that paper is equivalent to Eqs. (33b) and (37b). In Tab. I we tabulate the numerical values of the first eight coefficients $n!\mu^{(n)}$ for $V_2'=0$.

IV. POSITIVE MOMENTS

A. Formally exact expressions

In view of Eqs. (15) and (23) we find that the exponential growth rates of the positive moments of M_{11} are given by the eigenvalue $\mu(\xi)$ of Eq. (21) at integer values of ξ:

$$\frac{d\ln\langle M_{11}^{\xi}\rangle}{dt} = \frac{\mu(\xi)}{t_c}. \quad (38)$$

As we will now show, for integer values of ξ the eigenvalue problem (21) can be reduced to a matrix eigenvalue problem of finite dimension. For the first few moments the leading eigenvalue can be calculated explicitely, while for larger values it is formally given by the largest root of the corresponding characteristic polynomial.

In order to obtain a solution of the differential equation (21a), we write

$$f_0(\xi,z) = \int_{-\infty}^{z} dy\,K(y,z)\frac{g(z)}{g(y)^2} \quad (39)$$

[with the kernel $K(y,z)$ defined in Eq. (18)], and obtain for g the differential equation

$$(\mu-\xi z)g = -(z^2+V_2')\partial_z g + \partial_z^2 g \quad (40)$$

(a triconfluent Heun's equation with singularity at $1/z=0$). We introduce into this equation the polynomial ansatz

$$g(z) = \sum_{n=0}^{\xi} c_n z^n. \quad (41)$$

Power matching results in the following recursion relation

$$(\xi-n)c_n = \mu c_{n+1} + (n+2)[V_2' c_{n+2} - (n+3)c_{n+3}] \quad (42a)$$

for the coefficients c_n, with initial conditions

$$c_\xi = 1, \qquad c_{\xi-1} = \mu, \qquad c_{\xi-2} = \mu^2/2. \quad (42b)$$

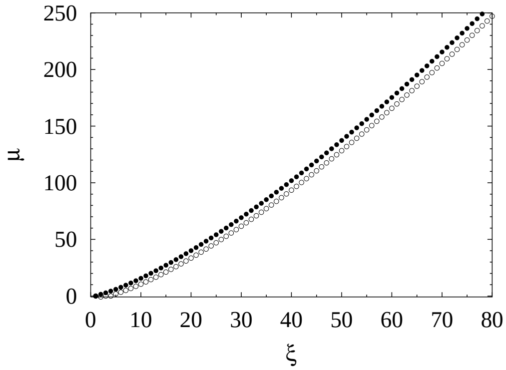

FIG. 2: Growth rates $\mu(\xi)$ of the moments $\langle M_{11}^m \rangle$ [cf. Eq. (38)] in absence of the static background potential ($V_2' = 0$), obtained as the largest eigenvalue of the matrix (44) (full circles). Also shown is the real part of the subleading eigenvalue of this matrix (open circles).

For integer ξ this recursion relation terminates. We obtain functions $c_0(\mu)$, $c_1(\mu)$, and $c_2(\mu)$ and an additional condition from the term in Eq. (40) which is constant in z,

$$p_\xi(\mu) = \mu c_0 + V_2' c_1 - 2c_2 = 0, \tag{43}$$

where $p_\xi(\mu)$ is a polynomial of degree $\xi + 1$.

The polynomial $p_\xi(\mu)$ can also be interpreted as the characteristic polynomial of the $(\xi + 1) \times (\xi + 1)$-dimensional matrix

$$\begin{pmatrix} 0 & -V_2' & 1\cdot 2 & 0 & 0 & \cdots & \cdots & \cdots \\ \xi & 0 & -2V_2' & 2\cdot 3 & 0 & \cdots & \cdots & \cdots \\ 0 & \xi-1 & 0 & -3V_2' & 3\cdot 4 & \cdots & \cdots & \cdots \\ 0 & 0 & \xi-2 & 0 & -4V_2' & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \xi-3 & 0 & \ddots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & (1-\xi)V_2' & (\xi-1)\xi \\ \vdots & \vdots & \vdots & \vdots & \vdots & 2 & 0 & -\xi V_2' \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 1 & 0 \end{pmatrix}, \tag{44}$$

which is the matrix representation of the eigenvalue problem (21) in the space of the monomial expansion of $g(z)$.

The exponential growth rate $\mu(\xi)$ of the ξth moment is given by the largest root of $p_\xi(\mu)$, or equivalently by the largest eigenvalue of the matrix (44). In subsection IV B we will see for the examples $\xi = 1, 2$ that the other roots show up in the transient behavior of the moments.

First we present results in absence of the static background potential, $V_2' = 0$. The values for the first few moments are given in Tab. II. Figure 2 shows the growth rates and the real part of the subleading eigenvalue for values of ξ up to 80. A log-normal statistics of M_{11} (corresponding to a Gaussian statistics of the finite-time Lyapunov exponents) would result in the quadratic dependence Eq. (36) of $\mu(\xi)$ on ξ, while the plot shows a weaker (approximately linear) dependence for large ξ. This results from the influence of the terms $\mu^{(n)}\xi^n$ for $n \geq 3$ in the complete Taylor expansion of μ, Eq. (25). Further note that the subleading eigenvalue stays at a finite distance to the leading eigenvalue (indeed, their distance increases with increasing ξ), as we have assumed before in restricting the attention to the leading eigenvalue μ_0 of the eigenvalue problem (21).

For finite V_2', the growth rate of the first moment

$$\mathrm{Re}\,\mu(1) = |\,\mathrm{Im}\,\sqrt{V_2'}| \tag{45}$$

vanishes in the case of confinement, $V_2' > 0$. This will be confirmed by the direct computation in Sec. IV B. The growth rate of the second moment is given by

$$\begin{aligned} \mu(2) \;&=\; 2^{1/3}\left(1+\sqrt{1+16V_2'^3/27}\right)^{1/3} \\ &\quad + 2^{1/3}\left(1-\sqrt{1+16V_2'^3/27}\right)^{1/3} \end{aligned} \tag{46}$$

[with the roots taken such that $\mu(2)$ is real]. We plotted the real parts of the leading and subleading growth rates [eigenvalues of matrix (44)] for the first four moments in Fig. 3.

B. Direct computation of the first and second moment

In order to illustrate the results for the growth rates of the moments $\langle M_{11}^\xi \rangle$ we compare the results for $\xi = 1$ and $\xi = 2$ to the exact results for all times (including the transient behavior). A formal solution of the differential equation (13) in terms of a series in the disorder potential is obtained by integrating Eq. (13) twice, under observation of the initial conditions $M_{11} = 1$, $dM_{11}/dt = 0$ for $t = 0$, and iterating the resulting integral relation

$$\begin{aligned} M_{11}(t) \;&=\; 1 + \int_0^t dt_1 \int_0^{t_1} ds_1 \frac{v(s_1)}{m} M_{11}(s_1) \\ &=\; 1 + \int_0^t dt_1 (t-t_1) \frac{v(t_1)}{m} M_{11}(t_1). \end{aligned} \tag{47}$$

The formal solution is of the form

$$M_{11}(t_0) = 1 + \sum_{n=1}^{\infty} \prod_{k=1}^{n} \int_0^{t_{k-1}} dt_k\, (t_{k-1} - t_k) \frac{v(t_k)}{m}, \tag{48}$$

where we introduced $t_0 = t$ for notational convenience.

For the first moment we can average Eq. (48) directly. Because of the factors $(t_{k-1} - t_k)$ and the time ordering, the random function v never appears instantaneously in second or higher order in any of the integrals. Hence we can replace v by its average, given in Eq. (10). Consequently, the first moment is given by

$$\langle M_{11} \rangle = \cos[(t/t_c)\sqrt{V_2'}] = \frac{1}{2} e^{(t/t_c)\sqrt{-V_2'}} + \frac{1}{2} e^{-(t/t_c)\sqrt{-V_2'}}. \tag{49}$$

For $V_2' = 0$ the first moment is constant and given by its initial value, $\langle M_{11}\rangle = 1$. This means that negative deviations $M_{11} \ll 0$, corresponding to inverse hyperbolic motion, cancel precisely the positive deviations $M_{11} \gg 0$ of hyperbolic motion. For negative V_2' the first moment grows, while for positive V_2' it oscillates and stays of order unity. In the decomposition of the cosine into the two exponentials, we identify in the exponents the two roots $\pm\sqrt{-V_2'}$ of the characteristic polynomial $p_{\xi=1}(\mu) = \mu^2 + V_2'$ of the matrix (44) with $\xi = 1$. For negative V_2', the subleading exponent hence governs the transient behavior of the first moment.

For the second moment let us restrict for simplicity to the case $V_2' = 0$. We group the functions v in the two factors of M_{11} in pairs and then invoke the delta-correlations of Eq. (10). Performing the time-ordered integrals we obtain

$$\langle M_{11}^2\rangle = 1 + \sum_{n=1}^{\infty}(2/t_c^3)^n \prod_{k=1}^{n}\int_0^{t_{k-1}} dt_k\,(t_{k-1}-t_k)^2$$
$$= \frac{1}{3}[e^{\mu(2)t/t_c} + e^{-(-1)^{1/3}\mu(2)t/t_c} + e^{-(-1)^{-1/3}\mu(2)t/t_c}]. \quad (50)$$

The asymptotic growth rate of the second moment is given by the leading root $\mu(2) = 2^{2/3}$ of the characteristic polynomial $p_2(\mu) = \frac{1}{2}\mu^3 - 2$, which is in accordance to Tab. II. The second and third exponent are the other two roots of this polynomial.

C. Equivalence of matrix elements

So far we mainly studied the statistics of the upper diagonal element M_{11} of the stability matrix M. At this point now we can discuss how the results for the cumulant-generating function and the positive moments can be transferred to the other elements of M.

The differential equation (14) for M_{22} can be integrated similarly as the one for M_{11}, from which we obtain analogously to Eq. (48) the formal solution

$$M_{22}(t_0) = 1 + \sum_{n=1}^{\infty}\prod_{k=1}^{n}\int_0^{t_{k-1}} dt_k\,(t_k - t_{k+1})\frac{v(t_k)}{m}. \quad (51)$$

Here we defined in each term of order n that $t_{n+1} = 0$. It follows by direct computation that the first two moments of M_{22} are identical to those of M_{11},

$$\langle M_{22}\rangle = \langle M_{11}\rangle, \quad \langle M_{22}^2\rangle = \langle M_{11}^2\rangle. \quad (52)$$

These explicit results already suggest that the statistics of the two diagonal matrix elements is the same. Indeed, the transformation $t_k = t - \tilde{t}_{n+1-k}$, $v(t-\tilde{t}) = \tilde{v}(\tilde{t})$ brings Eq. (51) into the form of Eq. (48) and leaves the properties of the Gaussian noise (10) invariant. Hence even the transient behavior of the diagonal elements is completely identical, for arbitrary values of V_2'.

The results for the cumulant-generating function $\eta(\xi)$ (hence also the growth rates of the moments, but not the tran-

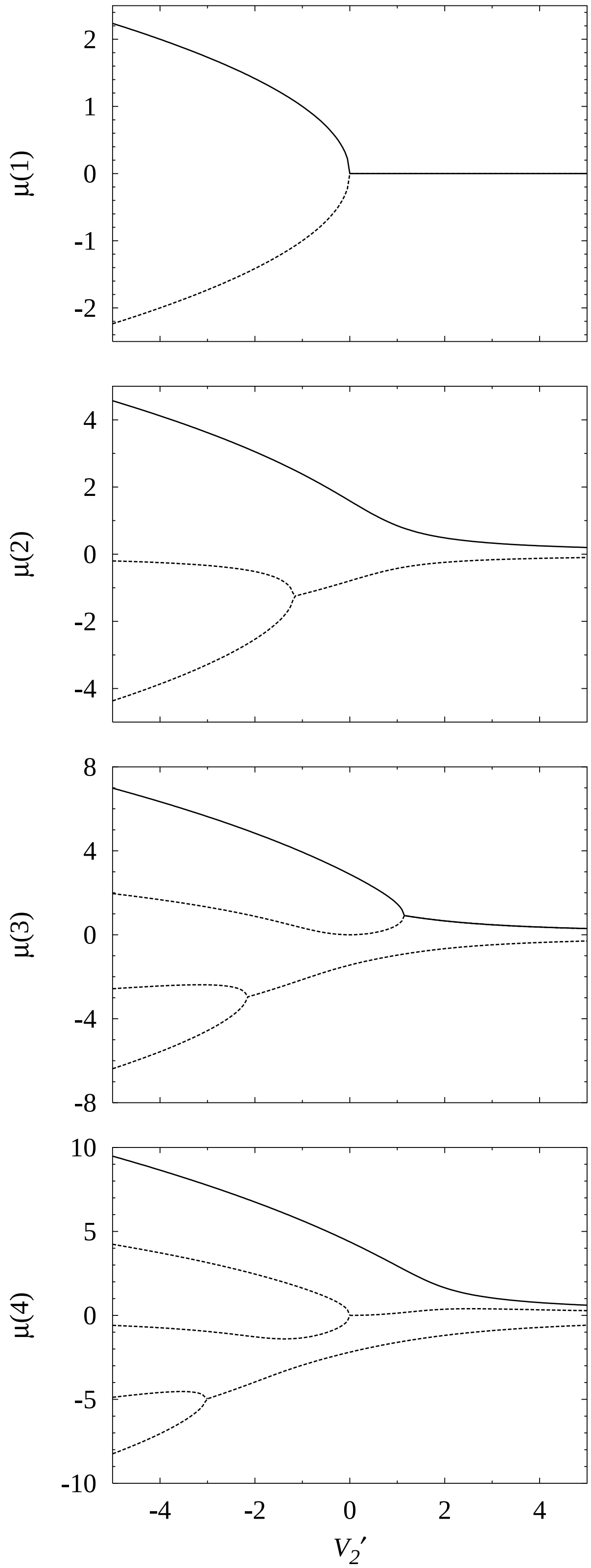

FIG. 3: Growth rates $\mu(\xi)$ the moments $\langle M_{11}^m\rangle$ [cf. Eq. (38)], for $m = 1, 2, 3, 4$, as a function of the strength V_2' of the static background potential. Also shown (dashed lines) is the real part of the subleading growth rates [subleading eigenvalues of matrix (44)].

sient behavior) can also be transferred to the offdiagonal matrix elements of M: The element M_{12} fulfills the same differential equation as M_{11}, see Eq. (13), while M_{21} fulfills the same differential equation as M_{22}. The initial conditions of the offdiagonal matrix elements differ from those of the diagonal elements. However, according to Eq. (24) this only affects the function $f(\xi)$ in the subleading corrections of the cumulant-generating function [which, for the example of the second moment, results in factors in front of the exponential functions which are different than in Eq. (50)].

Let us add that from Eqs. (48) and (51) we find for $V_2' = 0$ the cross-correlator

$$\langle M_{11}M_{22}\rangle = \frac{1}{2} + \frac{1}{2}\langle M_{11}^2\rangle. \tag{53}$$

As a consequence, for $V_2' = 0$ the trace $\mathrm{tr}\, M = M_{11} + M_{22}$ of the stability matrix has the following first two moments

$$\langle \mathrm{tr}\, M\rangle \;=\; 2, \tag{54a}$$
$$\langle (\mathrm{tr}\, M)^2\rangle \;=\; 1 + e^{\mu(2)t/t_c} + 2\;\mathrm{Re}\, e^{-(-1)^{1/3}\mu(2)t/t_c}. \tag{54b}$$

V. DISCUSSION

In this work we presented a uniform approach to the asymptotic statistics of finite-time Lyapunov exponents, for the model (described in Sec. II) of a particle moving in a random time-dependent potential. The cumulant-generating function $\eta(\xi)$ was found to be directly proportional to the eigenvalue $\mu(\xi)$ of a parameterized differential equation, defined by Eqs. (21). This facilitated an effective analysis of the statistics, including the non-Gaussian deviations of the distribution function. These deviations are especially important for the positive moments of the elements of the stability matrix, since their growth rate *cannot* be predicted by the Gaussian approximation Eq. (36).

We limited our attention to the case of time-dependent Hamiltonian systems with a single degree of freedom and a Hamiltonian (5) which is of the special type of kinetic energy plus potential energy, with time-dependence only in the potential energy. This case is of particular interest because of its direct applicability to specific dynamical systems as in the random wave-propagation problem of Refs. [22, 23], and because of its applicability to one-dimensional wave localization. For the Hamiltonian (5) the matrix K in the differential equation (7) is purely off-diagonal, with fluctuations only in the lower-left element. For Hamiltonians which do not separate into kinetic and potential energy, the differential equation (5) for M involves the matrix K in the more general form

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 H}{\partial x \partial p} & \frac{\partial^2 H}{\partial p^2} \\ -\frac{\partial^2 H}{\partial x^2} & -\frac{\partial^2 H}{\partial x \partial p} \end{pmatrix}. \tag{55}$$

A generalized statistical model now arises by introducing noise into all of the matrix elements of K. (One may also allow for correlations between the different matrix elements or for finite correlation times by introducing auxiliary variables for the noise in the standard way.)

Let us point out two particular cases for which a statistical description promises to result in direct applications to physical situations of interest. One case is more relevant to wave-function localization while the other is more relevant for chaotic dynamics.

a) The diagonal elements $K_{11} = -K_{22} = 0$ still vanish identically, but both off-diagonal elements K_{12} and K_{21} fluctuate with a vanishing mean. This situation appears to be related to the band-center case of one-dimensional localization in the Anderson model [41, 42] (where space is discretized on the lattice), since at the band-center the effective mass of the particle diverges (and hence the mean of K_{12} vanishes).

b) Chaotic dynamics with an isotropic phase space may be modeled by independent fluctuations of all four matrix elements K_{ij} with identical amplitude and vanishing mean. Hamiltonian dynamics gives rise to the further constraint $K_{11} = -K_{22}$. Isotropic dynamics arises in typical chaotic maps (some maps, like the Baker map or the cat map, however, are not isotropic—the directions of stable and unstable manifolds are known by construction). Good candidates are the Poincaré surface of section of autonomous systems with two degrees of freedom, in which the motion in four-dimensional phase space is restricted to three-dimensional manifolds of constant energy and the coordinate along the flow field is taken as a time.

It would be interesting to compare the outcome of an analysis of model b) with the findings in the literature [15, 16] which indicate a certain degree of robustness (if not universality) of the distribution of finite-time Lyapunov exponents for typical chaotic systems.

Acknowledgments

We gratefully acknowledge useful discussions with Philippe Jacquod and Holger Kantz, and especially with Steven Tomsovic who motivated us to study this problem.

APPENDIX: INTEGRALS FOR THE HIGHER CUMULANTS

The cumulants of order n result from the recursion relations Eq. (32) in the form of $2n$-fold integrals. Usually, the numerical evaluation of such integrals for large n is very time-consuming, since the number of points on a grid covering the integration domain with lattice constant $(1/N)$, $N \gg 1$, grows rapidly with n as N^{2n}. However, presently the integrand factorizes and the expense of the integration can be reduced from exponential to algebraic n-dependence $\sim nN$. The principle can be demonstrated for the example of the two-fold integral

$$I^{(1)} = \int_{-z_0}^{z_1} dz\, I^{(2)}(z), \qquad I^{(2)}(z) = g(z)\int_{-z_0}^{z} dy\, I^{(3)}(y), \tag{56}$$

where g is an arbitrary function and $I^{(3)}$ may itself be a multi-dimensional integral.

We introduce an index m which denotes that the argument of a function is taken at the mth lattice point on the appropriate axis of the grid. The initial values of $I_m^{(n)}$ at $m=0$ (the lower integration boundary) are zero. We now can write recursively, by incrementally increasing the integration variables,

$$I_{m+1}^{(2)} = \frac{g_{m+1}}{g_m} I_m^{(2)} + \frac{1}{N} g_m I_m^{(3)}, \tag{57a}$$

$$I_{m+1}^{(1)} = I_m^{(1)} + \frac{1}{N} I_{m+1}^{(2)}. \tag{57b}$$

Moreover, when $I^{(3)}$ itself is a multi-dimensional integral of type $I^{(1)}$, its current value can be obtained recursively in the same way as the value of $I^{(1)}$. Since each additional integral will give rise to only one additional equation [similar either to Eq. (57a) or to Eq. (57b)], the number of operations grows linearly with n, as advertised above. [The recursion relations (57) have the additional advantage for the present problem that they avoid over- and underflow in the evaluation of the kernel $K(y,z) = \exp(y^3/3 + V_2' y - z^3/3 - V_2' z)$.]

[1] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, 1993).
[2] H. Fujisaka, Prog. Theor. Phys. **70**, 1264 (1983).
[3] P. Grassberger and I. Procaccia, Physica D **13**, 34 (1984).
[4] R. Badii and A. Politi, Phys. Rev. A **35**, 1288 (1987).
[5] T. Tél, Phys. Rev. A **36**, 2507 (1987).
[6] P. Grassberger, R. Badii, and A. Politi, J. Stat. Phys. **51**, 135 (1988).
[7] M. A. Sepúlveda, R. Badii, and E. Pollak, Phys. Rev. Lett. **63**, 1226 (1989).
[8] S. Vaienti, J. Stat. Phys. **56**, 403 (1989).
[9] F. J. Romeiras, C. Grebogi, and E. Ott, Phys. Rev. A **41**, 784 (1990).
[10] A. Crisanti, G. Paladin, and A. Vulpiani, *Products of Random Matrices* (Springer, Berlin, 1993);
[11] B. Eckhardt and D. Yao, Physica D **65**, 100 (1993).
[12] C. Amitrano and R. S. Berry, Phys. Rev. E **47**, 3158 (1993).
[13] M. H. Ernst, J. R. Dorfman, R. Nix, and D. Jacobs, Phys. Rev. Lett. **74**, 4416 (1995).
[14] A. Adrover and M. Giona, Physica A **253**, 143 (1998).
[15] A. Prasad and R. Ramaswamy, Phys. Rev. E **60**, 2761 (1999).
[16] F. K. Diakonos, D. Pingel, and P. Schmelcher, Phys. Rev. E **62**, 4413 (2000).
[17] H. Yamada and T. Okabe, Phys. Rev. E **63**, 026203 (2001).
[18] E. Aurell, G. Boffetta, A. Crisanti, G. Paladin, and A. Vulpiani, J. Phys. A **30**, 1 (1997).
[19] I. L. Aleiner and A. I. Larkin, Phys. Rev. B **54**, 14423 (1996); Phys. Rev. E **55**, R1243 (1997).
[20] O. Agam, I. Aleiner, and A. Larkin, Phys. Rev. Lett. **85**, 3153 (2000).
[21] S. Oberholzer, E. V. Sukhorukov, and C. Schönenberger, Nature **415** (6873), 765 (2002).
[22] M. A. Wolfson and S. Tomsovic, J. Acoust. Soc. Am. **109**, 2693 (2001).
[23] M. G. Brown, J. A. Colosi, S. Tomsovic, A. Virovlyansky, M. A. Wolfson, and G. M. Zaslavsky, preprint nlin.CD/0109027 (2001).
[24] R. A. Jalabert and H. M. Pastawski, Phys. Rev. Lett. **86**, 2490 (2001).
[25] Ph. Jacquod, P. G. Silvestrov, and C. W. J. Beenakker, Phys. Rev. E **64**, 055203 (2001).
[26] F. M. Cucchietti, H. M. Pastawski, and D. A. Wisniacki, Phys. Rev. E **65**, 045206(R) (2002).
[27] N. R. Cerruti and S. Tomsovic, Phys. Rev. Lett. **88**, 054103 (2002).
[28] T. Prosen, preprint quant-ph/0106149 (2001).
[29] Ph. Jacquod, I. Adagideli, and C. W. J. Beenakker, preprint nlin.CD/0203052 (2002).
[30] F. M. Cucchietti, C. H. Lewenkopf, E. R. Mucciolo, H. M. Pastawski, and R. O. Vallejos, preprint nlin.CD/0112015 (2001).
[31] B. V. Chirikov, Phys. Rep. **52**, 263 (1979).
[32] P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
[33] R. E. Borland, Proc. R. Soc. London A **274**, 529 (1963).
[34] R. Landauer, Philos. Mag. **21**, 863 (1970).
[35] N. F. Mott and W. D. Twose, Adv. Phys. **10**, 107 (1961).
[36] For reviews on localization see, e.g., P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985); B. Kramer and A. MacKinnon, Rep. Prog. Phys. **56**, 1469 (1993); C. W. J. Beenakker, Rev. Mod. Phys. **69**, 731 (1997); A. D. Mirlin, Phys. Rep. **326**, 259 (2000).
[37] A. Crisanti, G. Paladin, and A. Vulpiani, Phys. Rev. A **39**, 6491 (1989).
[38] H. L. Frisch and S. P. Lloyd, Phys. Rev. **120**, 1175 (1960).
[39] B. I. Halperin, Phys. Rev. **139**, A104 (1965).
[40] I. M. Lifshitz, S. A. Gredeskul, and L. A. Pastur, *Introduction to the Theory of Disordered Systems* (Wiley, New York, 1988).
[41] M. Kappus and F. Wegner, Z. Phys. B **45**, 15 (1981).
[42] I. Goldhirsch, S. H. Noskowicz, and Z. Schuss, Phys. Rev. B **49**, 14504 (1994).