Faraday patterns in Bose–Einstein condensates. Amplitude equation for rolls in the parametrically driven, damped Gross–Pitaevskii equation

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The parametrically driven, damped Gross–Pitaevskii equation, which models Bose–Einstein condensates in which the interatomic $s-$wave scattering length is modulated in time, is shown to support spatially modulated states in the form of rolls. A Landau equation with broken phase symmetry is shown to govern the dynamics of the roll amplitude.

I. MODEL

We consider spontaneous pattern formation in dilute Bose–Einstein condensates (BEC) whose interatomic $s-$wave scattering length is periodically varied in time [1]. This modulation can be achieved by different means like, e.g., the use of magnetic [2], electric [3], or light [4] fields. When damping is considered [5] such BEC can be described by the following driven, damped Gross–Pitaevskii (GP) equation,

$$\partial_t \psi(\mathbf{r},t) = (1-i\gamma)\left[-\nabla^2\psi + V(\mathbf{r})\psi + |\psi|^2\psi\right] + \tfrac{a(t)-a_0}{a_0}|\psi|^2\psi. \quad (1)$$

which has been written in appropriate normalized variables. Parameter γ accounts for damping (values on the order $0.01-0.1$ seem to be appropriate for actual BECs [5]), $a(t)$ represents the instantaneous value of the interatomic $s-$wave scattering length and a_0 represents its mean value. We shall assume the simplest case $a(t) = a_0[1+2\alpha\cos(2\omega t)]$.

In the following we shall consider "pancake" (2D) or "cigar" (1D) shaped BECs in which the trapping potential $V(\mathbf{r})$ strongly confines the condensate in one direction or two directions, respectively, whilst along the other direction(s) it extends sufficiently as compared with the typical wavelength of the emerging pattern. If a parabolic trapping potential $V(\mathbf{r}) = -1 + \frac{1}{2}\left(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2\right)$ is used (the arbitrary offset is set to -1 for mathematical convenience) the above situation can be fulfilled whenever $\omega_z \gg \omega \gg \omega_x, \omega_y$ for the 2D case and $\omega_y, \omega_z \gg \omega \gg \omega_x$ for the 1D case. In a first approximation these inequalities allow: (i) to neglect the confined direction(s) in the descripition of the BEC dynamics, and (ii) to approximate the potential by a constant. Both approximations come from the fact that the characteristic wavelength of the selected pattern turns out to be, at the same time, much smaller than the size of the weakly confined direction and much larger than the strongly confined direction. Hence we consider along the rest of this work the following GP equation [1,6]:

$$\partial_t \psi(x,t) = (1-i\gamma)\left[-\partial_x^2\psi - \psi + |\psi|^2\psi\right] + 2\alpha\cos(2\omega t)|\psi|^2\psi. \quad (2)$$

Eq. (2) admits the following homogeneous state

$$\psi = \exp\left[-i(\alpha/\omega)\sin(2\omega t)\right], \quad (3)$$

which, in the absence of modulation ($\alpha = 0$) reduces to the BEC ground state $\psi = 1$ (the chemical potential is null in this representation because of the choice of the offset in the trapping potential).

II. LINEAR STABILITY ANALYSIS: THE PARAMETRIC RESONANCE

We wish to know whether the *spatially homogeneous* external driving is able to induce a spontaneous spatial–symmetry breaking of (3). Fot that we perform next a linear stability analysis of (3) by adding a small perturbation to that solution in the form $\psi = \exp\left[-i(\alpha/\omega)\sin(2\omega t)\right]\left[1 + w(t)\cos(kx)\right]$. Substitution of this expression into Eq. (2) and linearization with respect to w leads to the following coupled equations for $u = \mathrm{Re}\, w$, and $v = \mathrm{Im}\, w$:

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$$u'(t) = -\gamma\left(2+k^2\right)u + k^2 v, \tag{4}$$
$$v'(t) = -\gamma k^2 v - \left[2+k^2+4\alpha\cos(2\omega t)\right]u, \tag{5}$$

which can be combined to yield [1]

$$u''(t) + 2\gamma\left(1+k^2\right)u'(t) + \left[\left(1+\gamma^2\right)\Omega^2(k) + 4k^2\alpha\cos(2\omega t)\right]u(t) = 0, \tag{6}$$

where

$$\Omega(k) = k\sqrt{2+k^2}, \tag{7}$$

is the nonlinear dispersion relation for the perturbations.

Eq. (6) is a Mathieu equation with damping, analogous to that describing parametrically driven, damped pendula, and ubiquitous in the description of parametric forcing [7]. Its solutions are well known which, according to Floquet's Theorem, can be written as

$$u(t) = \operatorname{Re} f(t)\, e^{\mu t}, \tag{8}$$

where f is a periodic complex function of period π/ω and μ is the so called Floquet exponent (in the case of the Mathieu equation μ/i is known as Mathieu characteristic exponent). The BEC ground state (3) will be unstable against perturbations with wavenumber k whenever $\operatorname{Re}\mu > 0$. A general property of Eq. (6) is that $\operatorname{Re}\mu > 0$ within a series of resonance "tongues" very much like the parametric resonances observed in liquids vibrated vertically [7]. For small γ and α (the cases we consider here) these tongues are located, as a function of Ω, around $\Omega(k_n) = n\omega$, $n = 1, 2, 3, \ldots$ Within these tongues the BEC ground state undergoes a spontaneous spatial-symmetry breaking and perturbations with wavenumber $k_n = \sqrt{\sqrt{1+n^2\omega^2}-1}$ amplify. The Floquet exponents can be numerically determined by using standard mathematical methods [8]. Useful analytical information can be obtained in the limit of weak damping $\gamma \ll 1$. Taking into account that for $\gamma = \alpha = 0$ the solution to (6) is of the form $u(t) = \operatorname{Re} U e^{i\Omega t}$, if we consider $\alpha \sim |\omega - \Omega| \sim \gamma$, a perturbative expression for u can be obtained by allowing U to be a slowly varying function of time with the result

$$u(t) = \operatorname{Re}\left[U_+\exp(\lambda_+ t) + U_-\exp(\lambda_+ t)\right]e^{i\Omega t}, \tag{9}$$
$$\lambda_\pm = i(\omega-\Omega) - \gamma\left(1+k^2\right) \pm \sqrt{\left(\frac{\alpha k^2}{\Omega}\right)^2 - (\omega-\Omega)^2}. \tag{10}$$

Note that (9) can be written as (8) with $f(t) = \exp(i2\omega t)$, and

$$\mu = -i\omega - \gamma\left(1+k^2\right) + \sqrt{\left(\frac{\alpha k^2}{\Omega}\right)^2 - (\omega-\Omega)^2}. \tag{11}$$

Finally the condition $\operatorname{Re}\mu \geq 0$ reads

$$\alpha \geq \frac{\Omega}{k^2}\sqrt{\gamma^2\left(1+k^2\right)^2 + (\omega-\Omega)^2}, \tag{12}$$

where the equality defines the boundary (neutral stability line) of the first resonance tongue. Note that (12) indicates that, for fixed ω, the threshold for pattern formation is minimum at $\Omega = \omega$, i.e., for wavenumbers $k = k_1 = \sqrt{\sqrt{1+\omega^2}-1}$. The minimum parametric driving is hence predicted to be $\alpha_{\min}^{n=1}(\omega) = \gamma\omega\sqrt{1+\omega^2}/\left(\sqrt{1+\omega^2}-1\right)$. Following a similar analysis for the second resonance tongue ($\Omega \simeq 2\omega$) in the limit $\alpha \sim |\omega - \Omega| \sim \gamma^{1/2}$, it follows that $u(t)$ can be written as (9) with

$$\lambda_\pm = i(2\omega-\Omega) - \gamma\left(1+k^2\right) \pm \frac{1}{3}\sqrt{20\left(\frac{\alpha^2 k^4}{\Omega^3}\right)^2 - 24\frac{\alpha^2 k^4}{\Omega^3}(2\omega-\Omega) - 9(2\omega-\Omega)^2}, \tag{13}$$

from which the second tongue ($\operatorname{Re}\lambda_+ \geq 0$) runs:

$$\alpha \geq \frac{\Omega^{3/2}}{k^2}\sqrt{\frac{3}{10}}\sqrt{\sqrt{5\gamma^2\left(1+k^2\right)^2 + 9(2\omega-\Omega)^2} + 2(2\omega-\Omega)}. \tag{14}$$

The minimum driving amplitude needed in this case, $\alpha_{\min}^{n=2}$, occurs at $\Omega = 2\omega$. In this case $\alpha_{\min}^{n=2} \sim \sqrt{\gamma}$ hence the second resonance tongue is excited at larger driving amplitudes than the first one, for which $\alpha_{\min}^{n=1} \sim \gamma$. Note in both cases that for $\gamma = 0$ (no damping) the threshold for both (and in fact any) resonance tongues is $\alpha = 0$ hence all are simultaneously excited at vanishingly small values of driving amplitude. Damping hence breaks this degeneracy and selects the first resonance tongue at low drivings.

III. THE ROLL PATTERN: AMPLITUDE EQUATION

In the following we study the simplest pattern supported by the GP equation (2) under the previously described parametric instability. We consider a roll pattern in the form

$$\psi(x,t) = e^{-i(\alpha/\omega)\sin(2\omega t)}\left[1+\varepsilon w_1(t)\cos(kx)+\varepsilon^2 w_2(x,t)+\varepsilon^3 w_3(x,t)+O\left(\varepsilon^4\right)\right], \tag{15}$$

where $0<\varepsilon\ll 1$ is an auxiliary small parameter. In order to deal with a small roll component (of order ε) we assume that the amplitude of the parametric driving is also small, say $\alpha = O\left(\varepsilon^2\right)$ (that this is the proper scaling for α is justified a posteriori by the consistency of the final result). On the other hand for the parametric excitation to be effective Eq. (12) must be fulfilled. If we want to take into account both the resonance condition and the effect of damping we must impose $\gamma = O\left(\varepsilon^2\right)$ and $(\omega-\Omega) = O\left(\varepsilon^2\right)$. Summarizing we consider in the following the scalings

$$\omega = \Omega+\varepsilon^2\omega_2, \quad \alpha = \varepsilon^2\alpha_2, \quad \gamma = \varepsilon^2\gamma_2. \tag{16}$$

Our goal is to find an equation for the roll complex amplitude w_1. This will be done using a standard multiple timescale technique [7,9]. For this purpose we introduce a slow time

$$\tau = \varepsilon^2 t, \tag{17}$$

and allow all coefficients of the expansion to depend formally both on t and τ:

$$w_1(t) = u_{11}(t,\tau)+iv_{11}(t,\tau), \tag{18}$$

$$w_j(x,t) = u_j(x,t,\tau)+iv_j(x,t,\tau). \tag{19}$$

Finally Eqs. (15), (16), (18) and (19) are introduced into the GP equation (2). After using the chain rule for differentiation $\partial_t \to \partial_t+\varepsilon^2\partial_\tau$ and equating equal powers in ε an infinite hierarchy of differential equations is obtained.

A. Order ε^1

This is the first nontrivial order and reads

$$\partial_t u_{11}(t,\tau)-k^2 v_{11}(t,\tau) = 0, \tag{20}$$

$$\partial_t v_{11}(t,\tau)+\left(2+k^2\right)u_{11}(t,\tau) = 0, \tag{21}$$

whose solution can be written as

$$u_{11}(t,\tau) = \left[r(\tau)e^{i\Omega t}+\overline{r}(\tau)e^{-i\Omega t}\right], \tag{22}$$

$$v_{11}(t,\tau) = i\frac{\Omega}{k^2}\left[r(\tau)e^{i\Omega t}-\overline{r}(\tau)e^{-i\Omega t}\right], \tag{23}$$

where $r(\tau)$ stands for a yet arbitrary complex function of the slow time, and the overbar denotes complex conjugation.

B. Order ε^2

At this order we find

$$\partial_t u_2(x,t,\tau)+\partial_x^2 v_2(x,t,\tau) = \left[1+\cos(2kx)\right]u_{11}(t,\tau)v_{11}(t,\tau), \tag{24}$$

$$\partial_t v_2(x,t,\tau)+\left(2-\partial_x^2\right)u_2(x,t,\tau) = -\tfrac{1}{2}\left[1+\cos(2kx)\right]\left[3u_{11}^2(t,\tau)+v_{11}^2(t,\tau)\right], \tag{25}$$

whose solution can be written as

$$u_2(x,t,\tau) = u_{20}(t,\tau)+u_{22}(t,\tau)\cos(2kx), \tag{26}$$

$$v_2(x,t,\tau) = v_{20}(t,\tau)+v_{22}(t,\tau)\cos(2kx), \tag{27}$$

where each of the coefficients verifies

$$\partial_t u_{20}(t,\tau) = i\frac{\Omega}{k^2}\left[r^2(\tau)e^{i2\Omega t} - \overline{r}^2(\tau)e^{-i2\Omega t}\right], \tag{28}$$

$$\partial_t v_{20}(t,\tau) = -2u_{20}(t,\tau) - 2\left(2+k^{-2}\right)|r(\tau)|^2 + \left(k^{-2}-1\right)\left[r^2(\tau)e^{i2\Omega t}\overline{r}^2(\tau)e^{-i2\Omega t}\right], \tag{29}$$

$$\partial_t u_{22}(t,\tau) = 4k^2 v_{22}(t,\tau) + i\frac{\Omega}{k^2}\left[r^2(\tau)e^{i2\Omega t} - \overline{r}^2(\tau)e^{-i2\Omega t}\right], \tag{30}$$

$$\partial_t v_{22}(t,\tau) = -\left(2+4k^2\right)u_{22}(t,\tau) - 2\left(2+k^{-2}\right)|r(\tau)|^2 + \left(k^{-2}-1\right)\left[r^2(\tau)e^{i2\Omega t}\overline{r}^2(\tau)e^{-i2\Omega t}\right], \tag{31}$$

which solved yield

$$u_{20}(t,\tau) = -2\left(1+k^{-2}\right)|r(\tau)|^2 + \frac{1}{2k^2}\left[r(\tau)e^{i\Omega t} + \overline{r}(\tau)e^{-i\Omega t}\right]^2, \tag{32}$$

$$u_{22}(t,\tau) = -\frac{1}{2k^2}\left[r(\tau)e^{i\Omega t} + \overline{r}(\tau)e^{-i\Omega t}\right]^2, \tag{33}$$

$$v_{20}(t,\tau) = v_{200}(\tau) + \frac{i}{2\Omega}\left[r^2(\tau)e^{i2\Omega t} - \overline{r}^2(\tau)e^{-i2\Omega t}\right], \tag{34}$$

$$v_{22}(t,\tau) = -\frac{i\Omega}{2k^4}\left[r^2(\tau)e^{i2\Omega t} - \overline{r}^2(\tau)e^{-i2\Omega t}\right], \tag{35}$$

where $v_{200}(\tau)$ is a yet undetermined function of the slow time, which is not fixed by the present analysis. This information should be obtainable by extending the calculation up to higher orders of the expansion. Anyway, as will be seen below, the knowledge of $v_{200}(\tau)$ is not relevant for our purposes.

C. Order ε^3. The amplitude equation

This is the last order we will consider. It reads

$$\partial_t u_3(x,t,\tau) + \partial_x^2 v_3(x,t,\tau) = f_u(t,\tau)\cos(kx) + g_u(t,\tau)\cos(3kx), \tag{36}$$

$$\partial_t v_3(x,t,\tau) + \left(2-\partial_x^2\right)u_3(x,t,\tau) = f_v(t,\tau)\cos(kx) + g_v(t,\tau)\cos(3kx), \tag{37}$$

where

$$f_u = -\partial_\tau u_{11} - \gamma_2\left(2+k^2\right)u_{11} + \left(2v_{20}+v_{22}\right)u_{11} + \left(2u_{20}+u_{22}\right)v_{11} + \tfrac{3}{4}\left(u_{11}^2+v_{11}^2\right)v_{11}, \tag{38}$$

$$f_v = -\partial_\tau v_{11} - \gamma_2 k^2 v_{11} - 4\alpha_2\cos\left(2\Omega t + 2\omega_2\tau\right)u_{11} - 3\left(2u_{20}+u_{22}\right)u_{11} - \left(2v_{20}+v_{22}\right)v_{11} - \tfrac{3}{4}\left(u_{11}^2+v_{11}^2\right)u_{11} \tag{39}$$

$$g_u = u_{11}v_{22} + u_{22}v_{11} + \tfrac{1}{4}\left(u_{11}^2+v_{11}^2\right)v_{11}, \tag{40}$$

$$g_v = -3u_{11}u_{22} - v_{11}v_{22} - \tfrac{1}{4}\left(u_{11}^2+v_{11}^2\right)u_{11}. \tag{41}$$

The solution to Eqs. (36) and (37) can be written as

$$u_3(x,t,\tau) = u_{31}(t,\tau)\cos(kx) + u_{33}(t,\tau)\cos(3kx), \tag{42}$$

$$v_3(x,t,\tau) = v_{31}(t,\tau)\cos(kx) + v_{33}(t,\tau)\cos(3kx), \tag{43}$$

where each of the coefficients verifies

$$\partial_t u_{31}(t,\tau) - k^2 v_{31}(t,\tau) = f_u(t,\tau), \tag{44}$$

$$\partial_t v_{31}(t,\tau) + \left(2+k^2\right)u_{31}(t,\tau) = f_v(t,\tau), \tag{45}$$

$$\partial_t u_{33}(t,\tau) - 9k^2 v_{33}(t,\tau) = g_u(t,\tau), \tag{46}$$

$$\partial_t v_{33}(t,\tau) + \left(2+9k^2\right)u_{33}(t,\tau) = g_v(t,\tau). \tag{47}$$

The last two equations do not give us relevant information for our purposes. On the contrary Eqs. (44) and (45) determine the evolution equation for the complex amplitude of the roll r. This comes from the fact that these equations contain resonant terms which yield divergent solutions unless a solvability condition is imposed. This is clearly seen by writing Eqs. (44) and (45) in vector form:

$$\partial_t\begin{bmatrix} u_{31}(t,\tau) \\ v_{31}(t,\tau)\end{bmatrix} = \begin{bmatrix} 0 & k^2 \\ -\left(2+k^2\right) & 0\end{bmatrix}\begin{bmatrix} u_{31}(t,\tau) \\ v_{31}(t,\tau)\end{bmatrix} + \begin{bmatrix} f_u(t,\tau) \\ f_v(t,\tau)\end{bmatrix}. \tag{48}$$

Upon diagonalizing this equation we obtain

$$\partial_t \chi (t, \tau) = -i\Omega \chi (t, \tau) + f (t, \tau) , \tag{49}$$

(and its complex conjugate) where

$$\chi (t, \tau) = \left(2 + k^2\right) u_{31} (t, \tau) + i\Omega v_{31} (t, \tau) , \tag{50}$$

$$f (t, \tau) = \left(2 + k^2\right) f_u (t, \tau) + i\Omega f_v (t, \tau) . \tag{51}$$

Eq. (49) can only be solved if the driving term f does not contain elements oscillating as $\exp(-i\Omega t)$. Upon substituting Eqs. (38) and (39) into Eq. (51), and making use of Eqs. (22), (23) and (32)–(35), the solvability condition is found to be

$$\frac{dr}{d\tau} = -\gamma_2 \left(1 + k^2\right) r + i\frac{\alpha_2 k^2}{\Omega} e^{2i\omega_2 \tau} \overline{r} - i\frac{3 + 5k^2}{\Omega} |r|^2 r. \tag{52}$$

Finally we define a new roll complex amplitude

$$R (t) = \varepsilon r (\tau) e^{-i\omega_2 \tau}, \tag{53}$$

and turn back to the original parameters by undoing the scalings (16) with the result

$$\frac{dR}{dt} = -\left[\gamma \left(1 + k^2\right) + i (\omega - \Omega)\right] R + i\frac{\alpha k^2}{\Omega} \overline{R} - i\frac{3 + 5k^2}{\Omega} |R|^2 R. \tag{54}$$

Eq. (54) is the searched roll amplitude equation. It is a Landau equation with broken phase symmetry (note the presence of the linear term proportional to $\overline{R}$). The roll solution (15) can be written in terms of R making use of Eqs. (18), (22), (23) and (53). To the leading order the roll reads

$$\psi (x, t) = e^{-i(\alpha/\omega)\sin(2\omega t)} \left[1 + w (t) \cos (kx)\right] , \tag{55}$$

$$w (t) = \left(1 - \Omega/k^2\right) R (t) e^{i\omega t} + \left(1 + \Omega/k^2\right) \overline{R} (t) e^{-i\omega t}. \tag{56}$$

Let us finally note that a straightforward linear stability analysis of the trivial solution $R = 0$ (hence $w = 0$) of Eq. (54) yields the same neutral stability curve given in (12).

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[1] K. Staliunas, S. Longhi, and G. J. de Valcárcel, *Faraday patterns in Bose–Einstein condensates* (submitted, 2002).

[2] S. Inouye, M. R. Andrews, J. Stenger, H.-J. Miesner, D.M. Stamper-Kuhn, and W. Ketterle, Nature **392**, 151 (1998).

[3] M. Marinescu and L. You, Phys. Rev. Lett. **81**, 4596 (1988).

[4] P. O. Fedichev, Yu. Kagan, G. V. Shlyapnikov, and J. T. M. Walraven, Phys. Rev. Lett. **77**, 2913 (1996).

[5] We use a very simple, yet compatible with experiments, phenomenological description of damping. See S. Choi, S. A. Morgan, and K. Burnett, Phys. Rev. A **57**, 4057 (1998).

[6] Eq. (2) holds for 1D condensates, as well as for 2D condensates whenever 1D patterns are considered, which is the concern of this work.

[7] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).

[8] In particular the Floquet exponents of the Mathieu equation (6) can be computed from the `Mathematica` built-in function `MathieuCharacteristicExponent`.

[9] A. C. Newell and J. A. Whitehead, J. Fluid. Mech. **38**, 279 (1969).