

New series of integrable vertex models through a unifying approach

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Abstract

Applying a unifying Lax operator approach to statistical systems a new class of integrable vertex models based on quantum algebras is proposed, which exhibits a rich variety for generic q , q roots of unity and $q \rightarrow 1$. Exact solutions are formulated through algebraic Bethe ansatz and a novel possibility of hybrid vertex models is introduced.

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Introduction:

D -dimensional quantum systems are known to be related to $(1+D)$ -dimensional classical statistical models, which is true naturally also in $D = 1$, where one finds an exclusive class of models, known as integrable systems, allowing exact solutions. Celebrated example of such relation is that between the XYZ quantum spin- $\frac{1}{2}$ chain and the 8-vertex statistical model and similarly between the XXZ chain and the 6-vertex model [1]. Hamiltonian H_s of the integrable quantum spin chain is given through its transfer matrix as $\ln \tau(u) = I + uH_s + O(u^2)$, while the partition function Z of the related vertex model is constructed from $\tau(u)$ as $Z = \text{tr}(\tau(u)^M)$. Moreover, both these models share the same quantum R -matrix and have the same representation for the transfer matrix, commutativity of which: $[\tau(u), \tau(v)] = 0$ guarantees their integrability.

It is therefore rather surprising to note that, inspite of such deep connection between these two integrable systems, their starting formulation conventionally follows two different routes. Quantum systems usually are defined by their Lax operators $L_{al}(u)$, which satisfy the quantum Yang-Baxter equation (YBE) $R_{ab}(u-v)L_{al}(u)L_{bl}(v) = L_{bl}(v)L_{al}(u)R_{ab}(u-v)$, together with its associated R -matrix. A vertex model on the other hand is described by its Boltzmann weights given generally through the elements of the R -matrix alone, which solves the YBE $R_{ab}(u-v)R_{al}(u)R_{bl}(v) = R_{bl}(v)R_{al}(u)R_{ab}(u-v)$. However such a difference in their approaches, reason of which seems to be rather historical, puts

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certain restrictions on the 2-dimensional vertex models by assuming their vertical (v) and the horizontal (h) links, which are related to the auxiliary and quantum spaces respectively, to be equivalent. As a consequence while a rich variety of integrable quantum systems with wide range of interactions involving spin, fermionic, bosonic and canonical variables does exist, the integrable vertex models are confined mostly to those quantum models that exhibit regularity property expressed through the permutation operator: $L_{al}(0) = P_{al}$ and hence correspond to local Hamiltonians with nearest neighbor (NN) interactions. Therefore the well known examples of the integrable vertex models, apart from those mentioned above, appear to be limited mainly to the models like the 5-vertex model [2], 6-vertex model in external fields [3], 19-vertex model connected with the Babujian-Takhtajan spin-1 chain [4] and the vertex models equivalent to the Hubbard model [5], supersymmetric t-J model [6], Bariev chain [7] etc. all exhibiting only NN interactions.

The basic idea of the present letter however is to exploit fully the equivalence between statistical and quantum systems and construct new class of integrable vertex models by applying a unifying scheme designed originally for quantum models [8]. In the original scheme an ancestor model was proposed for generating integrable quantum systems as its various descendant realisations. For describing our vertex models we start in analogy also with the generalised Lax operator [8]

$$L(u) = \begin{pmatrix} c_1^+ q^{S^3+u} + c_1^- q^{-(S^3+u)} & 2 \sin \alpha S^- \\ 2 \sin \alpha S^+ & c_2^+ q^{-(S^3-u)} + c_2^- q^{S^3-u} \end{pmatrix}, \quad q = e^{i\alpha}, \quad (1)$$

linked with the underlying quantum algebra

$$[S^3, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = (M^+[2S^3]_q + M^-[[2S^3]]_q), \quad [M^\pm, \cdot] = 0. \quad (2)$$

Here $[x]_q \equiv \frac{\sin(\alpha x)}{\sin \alpha}$, $[[x]]_q \equiv \frac{\cos(\alpha x)}{\sin \alpha}$ and the central elements M^\pm are related to the other set of such elements appearing in the L -operator as $M^\pm = \pm \frac{1}{2} \sqrt{\pm 1} (c_\mp^+ c_\mp^- \pm c_\mp^- c_\mp^+)$. It is important to notice that (2) is a q -deformed quadratic algebra, which generalises both q -spin and q -boson algebras and in fact follows from the quantum YBE representing integrability condition. We define the Boltzmann weights (BW) of our vertex models not by the R -matrix as conventional, but through the elements of the Lax operator: $L_{ab}^{j,k}(u) = \omega_{a,j;b,k}(u)$ by using matrix representations of the general algebra (2) in (1). These generalised BW generate a unified vertex model, which through possible reductions yields new series of vertex models linked with different underlying algebras, their representations and choices of the central elements. Prominent examples of such integrable statistical systems are a rich collection of vertex models linked to q -spin and q -boson with generic q , q roots of unity and $q \rightarrow 1$. In all these models the h and v links, contrary to the usual approach, may become inequivalent and independent at every vertex point and since we consider here 2×2 Lax operators, the h links admit only 2 values: $a, b \in [+,-]$. The v links on the other hand have richer possibilities with $j, k \in [1, D]$, depending on dimension D of the matrix-representation of the q -algebras (see Fig. 1). The familiar ice-rule is generalised here as the 'colour' conservation $a + j = b + k$ for determining nonzero BW. The crucial partition function of the models however is given as usual by $Z = \sum_{config} \prod_{a,b,j,k} \omega_{a,j;b,k}(u)$.

An important point to note is that unlike traditional approach the Lax operators related to such vertex models do not coincide with their R -matrix, do not comply with the regularity condition and do

not correspond in general to quantum Hamiltonians with NN interactions. Moreover since our vertex models belonging to the same class have the same R -matrix, we can generate another rich series of integrable models, namely hybrid vertex models by combining any number of them in a row (see Fig. 1).

The eigenvalue solution of the transfer matrix needed for constructing the partition function for all these vertex models can also be found exactly through the algebraic Bethe ansatz in a unifying way.

Unified vertex model:

In accordance with our primary goal we discover an explicit matrix representation for the basic operators S^\pm, S^3 :

$$\langle s, \bar{m} | S^3 | m, s \rangle = m \delta_{m, \bar{m}}, \quad \langle s, \bar{m} | S^\pm | m, s \rangle = f_s^\pm(m) \delta_{m \pm 1, \bar{m}}, \quad (3)$$

with $f_s^+(m) = f_s^-(m+1) = (\kappa + [s-m]_q (M^+[s+m+1]_q + M^-[[s+m+1]]_q))^{\frac{1}{2}}$ having additional parameters κ, s . It may be checked that (3) indeed gives an exact representation of the general q -deformed algebra (2) for arbitrary values of the central elements M^\pm . Therefore the BW may be constructed from the matrix representation of the generalised Lax operator (1) by using (3) in the form

$$\omega_{\pm, j; \pm, j}(u) = c_\pm^+ e^{i\alpha(u \pm m)} + c_\pm^- e^{-i\alpha(u \pm m)}, \quad \omega_{+, j; -, j-1} = \omega_{-, j-1; +, j} = 2f_s^+(m) \sin \alpha, \quad (4)$$

with $m = s + 1 - j$, $j = 1, 2, \dots, D$. The BW parameterised as (4) would generate now a unified $(4D-2)$ -vertex model, representing a new series with arbitrary parameters c_\pm^\pm , s and κ . These models and naturally all others obtained below through various reductions are integrable statistical models and share the same R -matrix, which is given by that of the well known 6-vertex model [1].

Note that though in general the dimension D of the matrices (3) is infinite, it may get truncated through possible appearance of zero-normed states. To analyse this important effect we observe that since $[0]_q = 0$, one gets $f_s^+(m=s) = 0$ for $\kappa = 0$, recovering the familiar 'vacuum' state: $S^+ | s, s \rangle = 0$. However due to the presence of the second term one gets here $f_s^-(m=-s) = ([2s+1]_q (M^+[0]_q + M^-[[0]]_q))^{\frac{1}{2}} \neq 0$ and unlike the spin representation we have $S^- | m, s \rangle \neq 0$ for all m . This creates therefore an infinite tower of states by the action of the lowering operator S^- , as typical for the bosonic representation. This also signals the fact that algebra (2) includes both q -spin and q -boson and therefore their representations must be derivable from (3) as particular cases.

q -spin vertex model:

It is straightforward to check that for $M^+ = 1, M^- = 0$, our unifying algebra (2) reduces to the well known $U_q(su(2))$ quantum spin algebra [9] and at the same time (3) reproduces the known q -spin representation. Therefore the corresponding BW may be obtained from (4) for a consistent choice $c_\mp^\pm = c_\pm^\pm = \mp i$, as

$$\omega_{\pm, j; \pm, j}(u) = [u \pm m]_q, \quad \omega_{+, j; -, j-1} = \omega_{-, j-1; +, j} = f_s^{+(qspin)}(m), \quad m = s + 1 - j$$

with $f_s^{\pm(qspin)}(m) = ([s \mp m]_q [s \pm m + 1]_q)^{\frac{1}{2}}$. In this case the truncation $S^\pm | m = \pm s, s \rangle = 0$ typical for spin models and hence the familiar $D = 2s + 1$ dimensional representation naturally arise, which

produces therefore a series of q-spin $(8s + 2)$ -vertex models. The 6-vertex model is clearly recovered at $s = \frac{1}{2}$, while $s = 1, \frac{3}{2}, 2, \frac{5}{2}, \dots$ yield new 10, 14, 18, 22, \dots -vertex models (Fig. 1 a,b)).

The quantum systems related to such statistical models may be represented in general by interacting q-spins with nonlocal interactions. In particular, since the well known sine-Gordon model is a realisation of the q-spin [10], the vertex models constructed with nonzero κ and having infinite D will be related to the quantum integrable lattice sine-Gordon model [11].

q-boson vertex model:

We find that q-bosonic algebra [12] can also be derived as a subalgebra of (2) for the complementary choice $M^+ = \sin \alpha, M^- = \cos \alpha$ by denoting $S^+ = \rho A, S^- = \rho A^\dagger, S^3 = -N, \rho = (\cot \alpha)^{\frac{1}{2}}$. The same choice derives therefore the matrix representation for the q-boson directly from (3) with the assumption that $\kappa = s = 0$ and $n = -m$, yielding

$$f_0^{-(qbos)}(n) = ([1+n]_q[[-n-1]_q])^{\frac{1}{2}} = \frac{1}{\sqrt{2}}[1+n]_{q^2}^{\frac{1}{2}}, \quad f_0^{+(qbos)}(n) = f_0^{-(qbos)}(n-1) = \frac{1}{\sqrt{2}}[n]_{q^2}^{\frac{1}{2}}.$$

Consequently for a consistent solution $c_{\pm}^{\dagger} = 1, c_{\pm}^{-} = \mp i e^{\pm i\alpha}$ we can derive from (4) the BW as

$$\begin{aligned} \omega_{\pm, j; \pm, j}(u) &= i e^{\pm i\alpha\phi} [u \mp (j + \phi - 1)]_q, \quad \phi = \frac{1}{2} \left(1 + \frac{\pi}{2\alpha}\right), \\ \omega_{+, j; -, j-1} &= \omega_{-, j-1; +, j} = f_0^{+(qbos)}(j-1) = \frac{1}{\sqrt{2}} [j-1]_{q^2}^{\frac{1}{2}}. \end{aligned} \quad (5)$$

It is obvious that apart from the vacuum state $|0\rangle$ with $f_0^{+(qbos)}(0) = \frac{1}{\sqrt{2}} [0]_{q^2}^{\frac{1}{2}} = 0$ we can have no other zero-normed states and the q-bosonic representation like the standard boson is semi-infinite with $D = n + 1$. The integrable $(4n + 2)$ -vertex model linked to the q-boson (Fig. 1c)) that we construct using (5) would therefore be related to the lattice version of the quantum derivative nonlinear Schrödinger model (DNLS), which exhibits a q-bosonic realisation [13].

Vertex models with q roots of unity:

An excellent possibility for regulating the dimension of the matrix representation opens up when $q = e^{i\alpha}$ is chosen as solutions of $q^p = \pm 1$ with parameter α taking discrete values $\alpha_a = \frac{\pi a}{p}, a = 1, 2, \dots, p-1$ [14]. Note however that when some values of a becomes a factor of p one faces a situation with $q^{\frac{p}{a}} = \pm 1$. Therefore to avoid such complications we suppose p to be prime in our present discussion. For further analysis we focus on the action of S^- assuming $\kappa = 0$ in (3): $S^- |m = -\bar{s}, s\rangle = ([s + \bar{s} + 1]_q (M^+[s - \bar{s}]_q + M^-[[s - \bar{s}]_q]))^{\frac{1}{2}}$. and observe that due to $[p]_q = \sin \alpha_a p = 0$, unlike generic q we can get now $S^- |-\bar{s}, s\rangle = 0$ at $\bar{s} = p - (s + 1)$, which reduces matrix (3) to a finite dimensional representation. Therefore the BW obtained from (4) for this case would generate another series of unified K-vertex model having finite $K = 4p - 2$ configurations at every vertex point. Moreover, since for a fixed p there can be $p - 1$ different α_a , each of these discrete values describes a different set of BW and hence a novel model.

Consequently at particular reductions as analysed above, we obtain the corresponding series of new vertex models linked with q-spin or q-boson, but now having finite configuration space determined by p . Thus for the q-spin with fixed $p, 0 < p < 2s + 1$, in place of a $(8s + 2)$ -vertex model for generic q ,

one obtains $p - 1$ number of different $(4p - 2)$ -vertex models and the related representation including the case $p > 2s + 1$ become more involved [14]. The corresponding BW defining these models should however be given by their same generic form, though using discrete α_a values. As for example in case of $s = \frac{5}{2}$ with $q^5 = -1$, instead of a 22-vertex model one obtains 4 different 18-vertex models for distinct values of $\alpha_a = \pi \frac{a}{5}, a = 1, 2, 3, 4$. Noticeably, as a quantum model the q-spin with q roots of unity are realised as the restricted sine-Gordon model [15].

The situation becomes more interesting when applied to the q-boson with finite p , since now together with the standard vacuum we get also $A^\dagger |n = \frac{p}{2} - 1 \rangle = \frac{1}{\sqrt{2}} [\frac{p}{2}]_{q^2}^{\frac{1}{2}} | \frac{p}{2} \rangle = 0$, yielding finite $(\frac{p}{2} \times \frac{p}{2})$ matrix representations for the q-bosonic operators A, A^\dagger . As a result we obtain an intriguing series of $(2p - 2)$ -vertex models with BW described by the same form (5) as for the generic q-bosonic case, but with different possible parameter values $q = e^{i\alpha_a}, a = 1, 2, \dots, p - 1$. The quantum model corresponding to such q-boson vertex models can be realised as the restricted DNLS model, which supports finite quasi-particle bound states [16].

Rational class of vertex models :

At $q \rightarrow 1 (\alpha \rightarrow 0)$ on the other hand, the associated R -matrix goes to its known rational limit and the underlying algebra becomes undeformed one with $M^\pm \rightarrow m^\pm$, reducing at the same time the unified model to its rational form. Consequently, taking carefully the limits we may construct in a similar way the corresponding set of vertex models belonging to the rational class. Not going into details we mention only that the BW of these vertex models can be obtained from the limiting values of (4) yielding $f_s^+(m) \rightarrow ((s - m)(m^+(m + s + 1) + m^-))^{\frac{1}{2}}$. It is easy to check that the BW for the vertex models related to the undeformed spin as well as the standard boson correspond to the particular values of the central elements: $m^+ = 1, m^- = 0$ and $m^+ = 0, m^- = 1$, respectively. Remarkably, the spin vertex model constructed in this way coincides with the similar higher s model obtained earlier through fusion method [4], whereas the bosonic-vertex model apparently is a new model, linked to a quantum integrable lattice NLS model [17].

Hybrid vertex models:

In constructing our vertex models we have flatly assumed that in any model the same BW must be defined at every vertex point. An immediate generalisation is therefore possible by relaxing this condition and considering the central elements c_\pm^\pm as well as the spin parameters s appearing in (4) to be different at different sites. As we have already stressed, vertex models obtained as various reductions of the same integrable unified model belong to the same class sharing the same R -matrix. Thus the q-spin and q-boson vertex models are members of the trigonometric class, while the normal spin and boson models belong to the rational class. Based on this fact therefore we can construct a rich collection of hybrid models by combining different vertex models of the same class and inserting their defining BW along the vertex points $l = 1, 2, \dots, N$ in a row, in any but fixed manner. Due to the association with the same R -matrix the integrability of such statistical models would be naturally preserved.

Thus for example an alternate insertion of 10 and 6 vertex models results to a hybrid model, which is related to the known quantum model [18] involving spin-1 and spin- $\frac{1}{2}$ operators with next-NN

interactions. More exotic hybrid models can be formed by arranging the BW for the q-spin and q-boson vertex models, alternatively or in any other way at different vertex points (Fig. 1). Similarly one can construct a spin-boson hybrid vertex model by combining their individual vertex models, which would correspond to a quantum chain of interacting spins and bosons involving next-NN couplings.

Unified solution:

The construction of the unified vertex model through the generalised Lax operator suggests also a scheme for exactly solving the eigenvalue problem for the transfer matrix. Since the partition functions in turn can be determined from the knowledge of these eigenvalues, all vertex models obtained as particular cases and linked to (un-)deformed spin or (un-)deformed boson can also be solved in a unified way. There is a well formulated algebraic Bethe ansatz method for exactly solving the eigenvalue problem of the transfer matrix: $\tau(u) = \text{tr}_h(\prod_l^N L_l(u))$, when the associated Lax operator as well as the R -matrix are given [19]. Therefore, since we have defined the BW through matrix representations of the Lax operator and the R -matrix in our case is given by that of the well known 6-vertex model, we can derive the exact eigenvalues for the transfer matrix of our models as

$$\Lambda(u) = \omega_{+,1;+,1}^N(u) \prod_k^n g(u_k - u) + \omega_{-,-,1;-,-,1}^N(u) \prod_k^n g(u - u_k), \quad g(u) = \frac{[u+1]_q}{[u]_q}. \quad (6)$$

with all possible solutions of $\{u_k\}$ to be determined from the Bethe equations

$$\left(\frac{\omega_{+,1;+,1}(u_l)}{\omega_{-,-,1;-,-,1}(u_l)} \right)^N = \prod_{k \neq l}^n \frac{[u_l - u_k + 1]_q}{[u_l - u_k - 1]_q}, \quad l = 1, 2, \dots, n. \quad (7)$$

By analysing the structure of these equations we conclude that, the factors involving BW in both of them come from the action of the Lax operator on the pseudovacuum, which is chosen as the direct product of the highest weight states with $j = 1$ i.e. $|m = s \rangle$. The rest of the factors on the other hand are originated from the R -matrix elements, which arise during diagonalisation of the transfer matrix due to the use of the quantum YBA. Therefore it is crucial to note that, the only part given by BW is model-dependent and defined for the vertex models by the diagonal entries in (4) with $j = 1$, while the remaining parts contributed by the R -matrix are the same for all our models from the same class. Consequently the exact solutions for all models constructed here can be found in a systematic way from (6) and (7) by using corresponding reductions of the unified model (4).

The total number of solutions $\{\Lambda_\gamma(u)\}$ for the eigenvalues (6) should be D^N , which coincides with the number of possible eigenstates and gives the dimension of the vector space on which the transfer matrix acts. The partition function of the vertex models may therefore be given at the thermodynamic limit by $Z = \lim_{M,N \rightarrow \infty} \text{tr}_v(\tau^M(u)) = \lim_{M,N \rightarrow \infty} \sum_{\gamma=1}^{D^N} \Lambda_\gamma^M(u)$. At this important limit, the Bethe equations (7) turn into an integral equation $V(u) = 2\pi\rho(u) + \int \rho(v)K(v,u)dv$, with known kernel of the 6-vertex model [20]. Interestingly, all information about a particular model is encoded in the driving term only, which is expressed through $\frac{\omega_{+,1;+,1}(u)}{\omega_{-,-,1;-,-,1}(u)} = r e^{iP(u)}$ as $V(u) = P'(u)$ and therefore knowing the explicit form of BM one can derive easily the equations for individual models.

For extracting the solutions of the hybrid vertex models however the BW dependent parts in the above equations should be slightly modified by generalising the factors inhomogeneously as $\prod_{\beta}((\omega_{\pm,1;\pm,1}^{(\beta)}(u))^{N_{\beta}}$, where N_{β} is the number of vertices of type β appearing in a row with the constraint $N = \sum_{\beta} N_{\beta}$.

Detail investigation of individual models and identification of their most probable states are important problems to be pursued.

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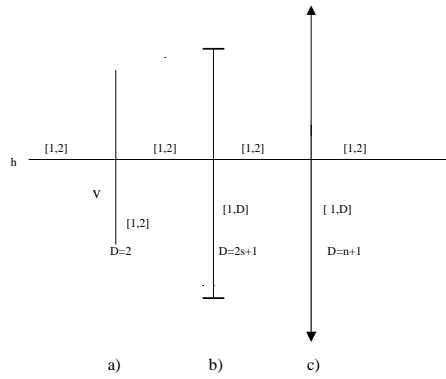


Figure 1: Integrable vertex models with horizontal (h) links taking 2 values, while the vertical (v) ones may have D possible values. a) 6-vertex b) q-spin vertex and c) q-boson vertex models. Combining a,b,c) an integrable hybrid model may be formed. $q^p = 1$ gives $D = p$ in b) and c)