

Generation of new classes of integrable quantum and statistical models

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Abstract

A scheme based on a unifying q -deformed algebra and associated with a generalized Lax operator is proposed for generating integrable quantum and statistical models. As important applications we derive known as well as novel quantum models and obtain new series of vertex models related to q -spin, q -boson and their hybrid combinations. Generic q , q roots of unity and $q \rightarrow 1$ yield different classes of integrable models. Exact solutions through algebraic Bethe ansatz is formulated for all models in a unified way.

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1 Introduction

Integrable quantum systems in $1 + 1$ -dimensions and statistical models in 2-dimensions belong to an exclusive and important class of models giving exact results. However there seems to be no established scheme for generating all such models along with their Lax operators and R -matrices in a unified way. Our aim therefore is an effort in that direction for constructing a wide range of known as well as new classes of models putting particular emphasis on statistical models.

The integrable systems may be defined by the property that they possess N -number of conserved quantities, which are independent and commute among themselves: $[c_n, c_m] = 0$, where N is to the degree of freedom of the system. Such integrability also leads to the exact solvability of the model, since taking conserved quantities as action variables one can adopt the action-angle variable description. For managing conveniently such a rich structure we introduce first a generating function

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$\tau(\lambda)$ depending on some extra parameter λ , known as the spectral parameter, such that one can recover the infinite number of conserved quantities including the Hamiltonian as the expansion coefficients of $\ln \tau(\lambda) = \sum_j c_j \lambda^j$. The crucial integrability condition may then be defined in a compact form as $[\tau(\lambda), \tau(\mu)] = 0$, from which the commutativity of c_j 's follows immediately by comparing the coefficients of different powers of λ, μ . The partition function Z of the related statistical model on the other hand can be constructed from $\tau(\lambda)$ as $Z = \text{tr}(\tau(\lambda)^M)$.

To be convinced of the amazingly wide variety and range of the integrable models, let us name a few of them [1], which allow both classical and quantum variants. They are field and lattice models of relativistic or anisotropic type such as sine-Gordon field and exact lattice [2] model and similarly Liouville model, quantum derivative nonlinear Schrödinger (DNLS) equation [3] together with their exact lattice versions, quantum spin model like XXZ chain [4], relativistic quantum Toda chain [5] etc. Models can also be of nonrelativistic or isotropic type such as NLS model [1] and its lattice variant, XXX spin chain [4], Toda chain [1] etc. Among the statistical integrable models the most simple and well known is the *6-vertex model* [6]. This model is defined on a square lattice with a random direction on each bond (left or right on horizontal, up or down on vertical) (FIG.1a)) constrained by the *ice rule*, that the number of incoming and outgoing arrows at each vertex are the same. This leaves only 6 possible configurations giving that many Boltzmann weights (BW) $\omega_{i,j;k,l}$ at each vertex point, coining the name of the model. These 6 weights may be given by the nontrivial matrix elements of the R -matrix (2.2) and the partition function is expressed as $Z = \sum_{config} \prod_{a,b,j,k} \omega_{a,j;b,k}(\lambda)$.

2 Lax operator approach

For describing the intricate integrable structures mentioned above one can not however start just from the Hamiltonian of the model as conventional in physics, since now the Hamiltonian is merely one among many commuting conserved charges. We on the other hand have to allow certain abstractions and start with an unusual type of matrix called the Lax operator $L_j(\lambda)$ defined at each site j of a 1-dimensional discretized lattice. The matrix elements of the Lax operator, unlike in an usual matrix, are themselves operators acting on some Hilbert space. For ensuring the integrability of a system the related $L_j(\lambda)$ generally should satisfy certain commutation relations given by the ultralocality condition $L_j(\lambda) \otimes L_k(\mu) = (I \otimes L_k(\mu)) \otimes (L_j(\lambda) \otimes I)$ at different lattice points and the Yang-Baxter equation

$$R(\lambda - \mu)L_j(\lambda) \otimes L_j(\mu) = (I \otimes L_j(\mu)) \otimes (L_j(\lambda) \otimes I)R(\lambda - \mu). \quad (2.1)$$

at the same points $j = 1, 2, \dots, N$. The $R(\lambda - \mu)$ -matrix has spectral parameter dependent c-number elements, which we take as a 4×4 matrix with

$$R_{11}^{11} = R_{22}^{22} = \sin \alpha(\lambda + 1), \quad R_{12}^{12} = R_{21}^{21} = \sin \alpha(\lambda), \quad R_{21}^{12} = R_{12}^{21} = \sin \alpha. \quad (2.2)$$

Since our intension is to establish the integrability which is a global property, we have to switch from this local picture at each site j to some global one by defining a matrix $T(\lambda) = \prod_{j=1}^N L_j(\lambda)$. Multiplying therefore the YBE (2.1) for $j = 1, 2, \dots, N$ and thanks to the ultralocality condition one arrives at the global YBE: $R_{12}(\lambda - \mu) T(\lambda) \otimes T(\mu) = (I \otimes T(\mu) T \otimes I(\lambda) R_{12}(\lambda - \mu)$ having

exactly the same structural form. Invariance of the same algebraic form also for the tensor product of the algebras as revealed here exhibits an underlying deep Hopf algebra structure. Defining further $\tau(\lambda) = \text{tr}T(\lambda)$, taking *trace* from both sides of the global YBE and canceling the R -matrices due to the cyclic rotation of matrices under the trace we arrive finally at the trace identity through $\tau(\lambda)$ defining the quantum integrability of the system.

Therefore we see that the integrable systems may be defined by their Lax operators satisfying the ultralocality condition and the YBE together with the associated R -matrix. Note that all the varied integrable models listed above have their representative L operators as 2×2 matrices, but with diverse forms expressed through different basic operators like spins $\vec{\sigma}$, bosons ψ, ψ^\dagger or real canonical operators u, p etc. However surprisingly, the R -matrix associated with all of them appears to be the same and given by the known trigonometric solution (2.2) as for the 6-vertex model (or its $q = e^{i\alpha} \rightarrow 1$, i.e rational limit). Therefore a natural question arises asking whether there can be a *unified* model associated with (2.2) such that all above models can be derived from it in a systematic way.

3 Unified model

We show that it is indeed possible to find such an integrable unified model and justify consequently the sharing of the same R -matrix by all other integrable models, which will be obtained through various reductions and realizations of this single model. Such a general model must be a quantum, lattice model, which is relativistic or anisotropic and extendible to be inhomogeneous with inbuilt quantum parameter \hbar , lattice constant Δ , deformation parameter q and the inhomogeneity parameters $\{c\}$, such that it can generate lattice/ field models, quantum/ classical models, relativistic (anisotropic)/ nonrelativistic (isotropic) variants and similarly inhomogeneous/homogeneous models at the corresponding limits of these parameters.

Since the integrable models can be represented by their Lax operators, we propose the defining form for our unified model as

$$L_t^{(anc)}(\xi) = \begin{pmatrix} \xi c_1^+ q^{S^3} + \xi^{-1} c_1^- q^{-S^3} & \epsilon S^- \\ \epsilon S^+ & \xi c_2^+ q^{-S^3} + \xi^{-1} c_2^- q^{S^3} \end{pmatrix}, \quad q = e^{i\alpha}, \quad \xi = e^{i\alpha\lambda}, \quad \epsilon = 2 \sin \alpha. \quad (3.3)$$

with the basic operators satisfying the quadratic algebra

$$[S^3, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = \left(M^+ \sin(2\alpha S^3) + M^- \cos(2\alpha S^3) \right) \frac{1}{2 \sin \alpha}, \quad [M^\pm, \cdot] = 0, \quad (3.4)$$

where M^\pm are the central elements expressed as $M^\pm = \pm \sqrt{\pm 1} (c_1^+ c_2^- \pm c_1^- c_2^+)$ through the commuting set of elements c_a^\pm . The algebra (3.4) is a novel q -deformed algebra and includes known q -spin as well as q -boson algebras as particular reductions. The quantum integrability of this model and hence all other models derived below from it is guaranteed, since (3.3) with (3.4) associated with R -matrix (2.2) satisfy YBE (2.1).

It is crucial to note that we would define the Boltzmann weights (BW) of our vertex models not by the R -matrix as conventional, but through the elements of the Lax operator (3.3) as $L_{ab}^{j,k}(\lambda) = \omega_{a,j;b,k}(\lambda)$ by using matrix representations of the unifying q -deformed algebra (3.4).

We may find an important representation of this algebra through canonical fields as

$$S^3 = u, \quad S^+ = e^{-ip}g_s(u), \quad S^- = g_s(u)e^{ip}. \quad (3.5)$$

with operator function

$$g_s(u) = (\kappa + \sin \alpha(s - u)(M^+ \sin \alpha(u + s + 1) + M^- \cos \alpha(u + s + 1)))^{\frac{1}{2}} \frac{1}{\sin \alpha} \quad (3.6)$$

containing extra free parameters κ and s . The unified model represented by (3.3) is a quantum integrable model and may be considered as a *generalized lattice SG model* through realization (3.5),(3.6). We are now in a position to generate the whole range of integrable models, known as well as new, through various choices of the central elements M^\pm as well as by mapping into different realizations. In fact (3.5) can be directly mapped through the spin- $\frac{1}{2}$ operators $\vec{\sigma}$ or realized further in bosonic operators ψ, ψ^\dagger as $\psi = e^{-ip}((s - u))^{\frac{1}{2}}, N = s - u$, etc. We may repeat similarly the whole construction at limit $\alpha \rightarrow 0$ for generating the rational cases, or consider q as roots of unity to get the restricted models. On the other hand at $\Delta \rightarrow 0$ we recover the field models, while $\hbar \rightarrow 0$ yields as usual the corresponding classical dynamical systems. As we show below, the choice for the inhomogeneity parameters c_a^\pm can be of two types: either as constant parameters reproducing generally known models or as site j dependent functions, which generates new classes of inhomogeneous or hybrid integrable models.

Motivated by the form of (3.5) we find also a matrix representation of the unifying algebra (3.4) as

$$\langle s, \bar{m} | S^3 | m, s \rangle = m \delta_{m, \bar{m}}, \quad \langle s, \bar{m} | S^\pm | m, s \rangle = f_s^\pm(m) \delta_{m \pm 1, \bar{m}}, \quad (3.7)$$

with $f_s^+(m) = f_s^-(m + 1) \equiv g_s(m)$ and using it construct the BW for our unified vertex model as

$$\omega_{\pm, k; \pm, k}(\lambda) = c_\pm^\pm e^{i\alpha(\lambda \pm m)} + c_\pm^\mp e^{-i\alpha(\lambda \pm m)}, \quad \omega_{+, k; -, k-1} = \omega_{-, k-1; +, k} = 2g_s(k - 1) \sin \alpha, \quad m = s + 1 - k, \quad (3.8)$$

where $k \in [1, D]$, depends on the dimension D of the matrix-representation of the q-algebras. Possible reductions of this unified model (3.8) would yield new series of vertex models (see FIG.1) with the familiar ice-rule is generalized here as the color conservation $a + j = b + k$ for determining the nonzero BW.

The eigenvalue solution of the transfer matrix related to the unified model can be found exactly through the algebraic Bethe ansatz, which therefore would give also exact solutions for all other quantum and statistical models we construct here in a unifying way.

4 Construction of integrable models

4.1 Trigonometric class with generic q

1. The simplest constant choice $c_a^\pm = \mp i$ giving $M^- = 0, M^+ = 2$, reduces (3.4) to the well known $U_q(su(2))$ q-spin algebra [7] $[S^3, S^\pm] = \pm S^\pm, [S^+, S^-] = [2S^3]_q \equiv \frac{\sin(2\alpha S^3)}{\sin \alpha}$. and (3.3) to the

corresponding Lax operator

$$L_{qspin}(\lambda) = \begin{pmatrix} [\lambda + S^3]_q & S^- \\ S^+ & [\lambda - S^3]_q \end{pmatrix}, \quad (4.9)$$

expressed through the q-spin. The simplest representation $\vec{S} = \frac{1}{2}\vec{\sigma}$ reduces (4.9) further to (2.2) and recovers the well known *XXZ spin- $\frac{1}{2}$ chain* and also the 6-vertex model as the related statistical system.

On the other hand, for canonical representation (3.5) the form (3.6) reduces to $g(u) = \frac{1}{2\sin\alpha} [1 + \cos\alpha(2u+1)]^{\frac{1}{2}}$ recovering the known integrable *lattice sine-Gordon* model. However for the corresponding statistical model the BW reduced from (3.8) takes the form $\omega_{\pm,j;\pm,j}(u) = [u \pm m]_q$, $\omega_{+,j;-,j-1} = \omega_{-,j-1;+,j} = f_s^+(m)$ with $f_s^\pm(m) = ([s \mp m]_q [s \pm m + 1]_q)^{\frac{1}{2}}$, yielding a new series of integrable models, namely *q-spin (4s+2)-vertex model*, for different values of s (FIG.1b)).

2. Another constant but different set of choices: $c_1^+ = c_2^- = 1$, $c_2^+ = c_1^- = 0$ giving $M^\pm = \pm\sqrt{\pm 1}$ yield an exponentially deformed Lie algebra $[S^+, S^-] = \frac{e^{2i\alpha S^3}}{2i\sin\alpha}$. The representation (3.5) is valid now with $g(u) = \frac{(1+e^{i\alpha(2u+1)})^{\frac{1}{2}}}{\sqrt{2\sin\alpha}}$ and reproduces the *lattice version of the quantum Liouville* model. One can as well construct the related vertex model with the BW obtained as particular reductions of (3.8) similar to the above case.

3. On the other hand another constant choice $c_1^+ = c_2^+ = 1$, $c_1^- = -iq$, $c_2^- = \frac{i}{q}$ leading to $M^+ = 2\sin\alpha$, $M^- = 2\cos\alpha$ reproduces the well known *q-boson algebra* [8] $[A, N] = A$, $[A^\dagger, N] = -A^\dagger$, $[A, A^\dagger] = \frac{\cos(\alpha(2N+1))}{\cos\alpha}$ from (3.4) by denoting $S^+ = \rho A$, $S^- = \rho A^\dagger$, $S^3 = -N$, $\rho = (\cot\alpha)^{\frac{1}{2}}$ and corresponds to a new quantum integrable *q-bosonic model* with the Lax operator reduced from (3.3) as

$$L_{qboson}(\lambda) = \begin{pmatrix} e^{i\alpha\phi}[\lambda - (\hat{N} + \phi)]_q & \kappa A^\dagger \\ \kappa A & e^{-i\alpha\phi}[\lambda + (\hat{N} + \phi)]_q \end{pmatrix}, \quad (4.10)$$

Realizing the q-boson through standard boson one may construct an integrable quantum model, representing a lattice version of the quantum *derivative nonlinear Schrödinger equation* (QDNLS) and at the continuum limit the corresponding field model. Fusing two such models one can build further a quantum integrable *massive Thirring* model [3]. The QDNLS is also related to the exactly solvable interacting bose gas model with derivative δ -function potential.

For constructing the related vertex model we require matrix representation of the q-bosonic operators, which with the present choice of the central elements and assuming $\kappa = s = 0$, $n = -m$, may be derived from (3.7) as $\langle \bar{n}|N|n \rangle = n\delta_{n,\bar{n}}$, $\langle \bar{n}|A^\dagger|n \rangle = f_0^-(n)\delta_{n+1,\bar{n}}$, $\langle \bar{n}|A|n \rangle = f_0^+(n)\delta_{n-1,\bar{n}}$ with $f_0^-(n) = ([1+n]_q[[-n-1]_q])^{\frac{1}{2}} = \frac{1}{\sqrt{2}}[1+n]_{q^2}^{\frac{1}{2}}$, $f_0^+(n) = f^-(n-1) = \frac{1}{\sqrt{2}}[n]_{q^2}^{\frac{1}{2}}$. Consequently the BW of the related statistical model reduced from (3.8) take the form as

$$\omega_{\pm,j;\pm,j}(u) = ie^{\pm i\alpha\phi}[u \mp (j + \phi - 1)]_q, \quad \omega_{+,j;-,j-1} = \omega_{-,j-1;+,j} = f^+(j-1) = \frac{1}{\sqrt{2}}[j-1]_{q^2}^{\frac{1}{2}}, \quad (4.11)$$

with $\phi = \frac{1}{2}(1 + \frac{\pi}{2\alpha})$ which generates new *q-boson (4n+2)-vertex model* (FIG.1c)).

4. It is easy to see that for all the following parameter choices: i) $c_a^+ = 1$, $a = 1, 2$, or ii) $c_1^\mp = \pm 1$, or iii) $c_1^+ = 1$, with rest of the c 's being zero, we get $M^\pm = 0$ with the underlying algebra

reducing to $[S^+, S^-] = 0$, $[S^3, S^\pm] = \pm S^\pm$ and (3.5) simply to $S^3 = -ip$, $S^\pm = \alpha e^{\mp u}$. This yields the *relativistic quantum Toda chain* recovering also its different Lax operator constructions.

The statistical systems related to this case however seem to be give uninteresting vertex models with infinite matrix traces.

4.2 Models with q roots of unity

It is obvious that the integrability of a system remains intact even if q is chosen as solutions of $q^p = 1$. with parameter α taking discrete values $\alpha_a = 2\pi\frac{a}{p}$, $a = 1, 2, \dots, p-1$ [12]. However this opens up an excellent possibility for regulating the dimension of the representation of the underlying algebra. All quantum models constructed above will yield their restricted versions when q is taken as the roots of unity. Among these models only a few appear to have been studied earlier, which include *restricted sine-Gordon model* connected with the q -spin and *restricted DNLS* associated with the q -boson.

The related vertex models generate now completely new series, since the number of the possible configurations can be regulated through integer p . To analyze this fact we focus on the action of S^- in (3.7) assuming $\kappa = 0$: and observe that due to $[p]_q = \sin \alpha_a p = 0$, unlike generic q we can get now $S^- |-\bar{s}, s\rangle = 0$ at $\bar{s} = p - (s + 1)$, which reduces matrix (3.7) to a finite dimensional representation. As a consequence, for q -spin with fixed p , $0 < p < 2s + 1$, we get now $p - 1$ number of different $(4p - 2)$ -vertex models for different discrete values of α_a .

The situation becomes more interesting when applied to the q -boson with finite p , since now in place of its unbounded representation we obtain a finite dimensional matrix representation leading to an intriguing series of $(2p - 2)$ -vertex models with BW described by the same form (4.11) as for the generic q -boson case, but now with different possible discrete parameter values $q = e^{i\alpha_a}$, $a = 1, 2, \dots, p - 1$.

4.3 Rational class

At the limit $\alpha \rightarrow 0$ or $q \rightarrow 1$ the Lax operator (3.3), the BM (3.8) as well as the R -matrix reduce to their corresponding rational limits with the underlying algebra becoming

$$[s^+, s^-] = 2m^+s^3 + m^-, \quad [s^3, s^\pm] = \pm s^\pm, \quad [m^\pm, \cdot] = 0, \quad (4.12)$$

with the central elements $M^\pm \rightarrow m^\pm$. We can find as before various realization of this algebra directly from (3.5) using however a limiting form of (3.6) as $g_0(u) = i((s - u)(m^+(u + s + 1) + m^-))^{1/2}$ and construct in the similar way as above different integrable quantum and statistical models belonging to this rational class.

1. Thus at $m^+ = 1, m^- = 0$, when one gets the standard $su(2)$ algebra, the quantum models produced are the *XXX spin chain* and the *lattice NLS* model, while the corresponding integrable vertex models related to the spin- s operators recover those obtained earlier through fusion technique [9].

2. A complementary choice $m^+ = 0, m^- = 1$, on the other hand yields the bosonic algebra generates another quantum *simple lattice NLS* model [10]. The corresponding bosonic vertex model represents a nontrivial integrable statistical model, apparently never studied before.

3.A trivial choice $m^\pm = 0$ reproduces the algebra related to the relativistic Toda chain considered above, which however in the rational limit gives its well known *nonrelativistic* variant. The corresponding vertex model as before seem to be not physically interesting.

5 Inhomogeneous and hybrid models

An immediate generalization of all the above models is possible by considering the central elements c 's appearing in the Lax operator (3.3) to be site dependent functions. This would lead to a new class of integrable inhomogeneous extensions of the above models and may be interpreted as models with impurities, varying external fields, incommensuration etc.

Thus we can construct for example variable mass sine-Gordon, Liouville models, variable coefficient NLS, Toda chain models etc similarly the corresponding integrable q-spin and q-boson vertex models in varying external field.

Another intriguing class of models can be formed by regulating the inhomogeneity such that the Lax operators of different models are arranged to sit at different lattice sites along the chain defining the transfer matrix of the model as $\tau(u) = \text{tr} \left(\prod_{\beta} \prod_{j=1}^{N^{(\beta)}} L_j^{(\beta)}(u) \right)$, where $L_j^{(\beta)}(u)$ indicates Lax operators of different models belonging to the same class with the same R -matrix and occurring $N^{(\beta)}$ times in the total number of sites $N = \sum_{\beta} N^{(\beta)}$.

Thus one can construct new series of exotic integrable models like hybrid sine-Gordon-Liouville model, hybrid NLS-Toda chain or spin-boson model, describing different types of nonlinear interactions at different domains of the coordinate space. Similarly one obtains hybrid integrable statistical models like (un-)deformed spin-boson models etc. combining different types of vertex models belonging to the same class (FIG.1).

Considering the continuum limit $\Delta \rightarrow 0$ we recover the quantum field models from their respective lattice versions constructed above, while their field Lax operator $\mathcal{L}(x, \lambda)$ takes the form $L_j(\lambda) \rightarrow I + i\Delta\mathcal{L}(x, \lambda) + O(\Delta^2)$. The associated R -matrix however remains the same as its discrete counterpart. At the limit $\hbar \rightarrow 0$ one obtains the corresponding classical models with the R -matrix reducing also to its classical form $R(\lambda) = I + \hbar r(\lambda) + O(\hbar^2)$. For statistical models on the other hand the most relevant is the thermodynamic limit: $N \rightarrow \infty$, with Δ, \hbar fixed.

6 Unified solutions

: There is a well formulated algebraic Bethe ansatz method for exactly solving the eigenvalue problem of the transfer matrix $\tau(u) = \text{tr}(\prod_l^N L_l(u))$, when the L, R matrices are given [11]. Therefore for our unified model represented by the generalized Lax operator (3.3) and the R -matrix (2.2) we get

$$\Lambda(\lambda) = (\langle 0 | \hat{L}^{11}(u) | 0 \rangle)^N (u) \prod_k^n f(u_k - u) + (\langle 0 | \hat{L}^{22}(u) | 0 \rangle)^N (u) \prod_k^n f(u - u_k), \quad f(u) = \frac{[u+1]_q}{[u]_q}. \quad (6.13)$$

with all possible solutions of $\{u_k\}$ to be determined from the Bethe equations

$$\left(\frac{\langle \langle 0 | \hat{L}^{11}(u_j) | 0 \rangle \rangle}{\langle \langle 0 | \hat{L}^{22}(u_j) | 0 \rangle \rangle} \right)^N = \prod_{k \neq j}^n \frac{[u_j - u_k + 1]_q}{[u_j - u_k - 1]_q}, \quad j = 1, 2, \dots, n. \quad (6.14)$$

Here $|0\rangle$ is the pseudo-vacuum and the only model dependent parts in both the above equations are given by the actions of the upper and lower diagonal operator elements L^{11}, L^{22} of the Lax operator. Note that since in our scheme the R -matrix is the same for all models and their L -operators are given through various reductions of (3.3), the eigenvalue form (6.13) together with the Bethe equation (6.14) give a unifying scheme for exactly solving all the integrable quantum as well as statistical models constructed here.

For our vertex models the Lax operator elements in the above equations should be replaced by their matrix representations expressed through the BW (3.8) as $\langle \langle 0 | \hat{L}^{11}(u) | 0 \rangle \rangle = \omega_{+,1;+,1} \langle \langle 0 | \hat{L}^{22}(u) | 0 \rangle \rangle = \omega_{-,1;-,-,1}$. The total number of independent solutions for the eigenstates should be equal to the dimension of the vector space on which the transfer matrix acts. For the vertex models in our construction this is $K = D^N$ and the partition function should be given through exact eigenvalue solutions as $Z = \lim_{M,N \rightarrow \infty} \text{tr}_v(\tau^M) = \lim_{M,N \rightarrow \infty} \sum_k^K \Lambda_k^M$, in the thermodynamic limit. Note that though in general the dimension of the transfer matrix may even be infinite (including degeneracies), the corresponding partition function must be well defined. The Hamiltonian of the quantum models related to such statistical systems would generally involve nonlocal interactions which however are not relevant for the associated vertex models we are concerned with. Though the Bethe equation gives the form for deriving exact solutions, only in the thermodynamic limit one usually expects to find such solutions by converting this algebraic equation into an integral equation [4]. For our unified model assuming $c_2^\pm = \frac{r_2}{r_1}(c_1^\pm)^*$, for physical reasons we can similarly derive from (6.14)

$$V(c_1^\pm, u) = 2\pi\rho(u) - \int dv \rho(v) \frac{\sin 2\alpha s}{\cos 2\alpha s - \cosh(u-v)}. \quad (6.15)$$

It is important to note that the rhs of the above equation totally coincides with that for the 6-vertex model [4], while the lhs represents the model-dependent part and is expressed through BM $\omega_{+,1;+,1} = c_1^+ e^{u+is\alpha} + c_1^- e^{-(u+is\alpha)} \equiv r_1 \omega(u)$ given by (3.8) as $V(c_1^\pm, u) = \frac{\omega' \omega^* - \omega \omega'^*}{|\omega|^2}$. Therefore the integral equations related to all vertex models including the q-spin and q-boson vertex models can be obtained from the single equation (6.15) at proper choices of c_1^\pm , which should also naturally include the well known case of 6-vertex model [4]. Hybrid models, models with q roots of unity or at $q \rightarrow 1$ can also be covered by the above unifying scheme or its extensions.

7 Concluding remarks

Thus we have prescribed an unifying scheme for constructing as well as solving integrable quantum and vertex models of certain classes, which covers lattice and field models of (non-)relativistic and (isotropic)anisotropic types as well as the corresponding integrable statistical models. Along with the known models one can construct new models, including a novel series of q-spin and q-boson vertex models related to q at roots of unity. Inhomogeneous and hybrid models constitute new exotic classes

of integrable models. Using algebraic Bethe ansatz we can also find exact solutions for all these models in a unifying way. For some other details concerning our unified scheme the readers may consult the related works [13, 14, 15, 16].

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FIG. 1. Integrable vertex models with horizontal (h) links taking 2 values, while the vertical (v) ones may have D possible values. a) 6-vertex b) q -spin vertex and c) q -boson vertex models. Combining a,b,c) an integrable hybrid model may be formed. $q^p = 1$ gives $D = p$ in b) and c)

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