

# Local induction approximation in the theory of superfluid turbulence.

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## Abstract

The local induction approximation (LIA) of the Biot-Savart law is often used for numerical and analytical investigations of vortex dynamics (in particular in the theory of superfluid turbulence). In this paper, using renormalization group (RG) methods, some features of the LIA is considered. The exact statistical solution of the LIA equation is presented. The problem of "marginal" terms, appearing at the Wilson's approach to the RG-procedure, is concerned.

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## 1 Introduction.

It is accepted that chaotic vortex structures, so called vortex tangles (VT), appear in volume of superfluid helium under the particular conditions. This objects are very interesting because it is well known that hydrodynamic and thermodynamic properties of HeII in many respects depend on this structures [1], [2]. Obviously, that only statistical consideration is possible for the VT evolution. The vortex tangles consist of separate vortex loops, which evolve under the vortex dynamics laws (the nonlocal Biot-Savart law plus interaction of vortices with the normal component of HeII). Besides, these loops can merge and break up during temporal evolution. These two mechanisms are not independent, there is their mutual influence. In general, any analytical investigation of the VT evolution, based on consideration of the exact vortex dynamical equations is the extremely formidable task. This connected with the absence, at this moment, of any analytical methods for solving nonlinear, nonlocal (as the Biot-Savart), stochastic equations. Practically all of the known works on this theme represent various numerical (computer) models to explore properties of VT. In turn, the numerical calculations also have large difficulties. Because the Biot-Savart interaction is the nonlocal one, to determine motion of a single vortex point, it is necessary to take into account the influence of all entire vortices to this point. Thus, it requires unreal computer resources and enormous time of the calculations.

In the initial numerical works the local induction approximation (LIA) of the Biot-Savart law was used [3], [4]. Some analytical advance also connected with this method [3]. Since then the vast list of

numerical works has been appeared. The authors obtained various results, using the Biot-Savart law as well as the local induction approximation. However, to this day the precision and the validity of the LIA is the objects of discussions.

The processes of reconnections of vortex lines even greater complicate the consideration of VT. In this paper we will not concern this problem, and only expose some properties of the LIA and, in general, RNG-based solutions of dynamical tasks.

## 2 Dynamical equations.

To describe the dynamics of a vortex filament we use the equation derived in the papers [3], [4]. To introduce stochastic evolution we use the Langevin approach, i.e. we add into the equation a random force term. That is the usual way to study turbulent-like phenomena. As the result, in the local induction approximation the equation of motion of a quantized vortex filament in HeII takes the form:

$$\frac{d\vec{s}(\xi, t)}{dt} = \beta \vec{s}' \times \vec{s}'' + \nu \vec{s}''' + \vec{f}(\xi, t) \quad (1)$$

Here  $\vec{s}(\xi, t)$  is a radius vector of a point of a line labeled by the variable  $\xi$ ,  $\vec{s}'$  is the derivative on the parameter  $\xi$ ;  $t$  is time; the quantity  $\beta$  is the coefficient of nonlinearity  $\beta = \frac{\kappa}{4\pi} \log \frac{R}{r_0}$ , with the circulation  $\kappa$  and the cutting parameters  $R$  (the external size, i.e. the averaged radius of curvature) and  $r_0$  (the vortex core size). The coefficient of dissipation  $\nu$  appears in the eq.(1) when external counterflow is absent [3], [4]. The additional term  $\vec{f}(\xi, t)$  models the external random disturbances such as the "white noise". The concrete formulation of  $\vec{f}$  will be defined later. Let's note that the eq.(1) is valid only if the normalization of the parameter  $\xi$  is the arclength. This is the additional approximation if to concern the entire consideration of vortex dynamics, since the overall length varies with time. But the aim is only to expose some characteristic features of the eq.(1).

In the one-dimensional Fourier-representation the equation (1) takes the form:

$$i\omega s_0^\alpha = \Gamma_{120}^{\alpha\beta\gamma} s_1^\beta s_2^\gamma - \nu k^2 s_0^\alpha + f_0^\alpha, \quad (2)$$

where the operator

$$\Gamma_{120}^{\alpha\beta\gamma} \equiv i\beta \int \int \int \int dk_1 dk_2 d\omega_1 d\omega_2 k_1 k_2^2 \epsilon^{\alpha\beta\gamma} \delta(k_1 + k_2 - k_0) \delta(\omega_1 + \omega_2 - \omega_0), \quad (3)$$

and  $s_{k,\omega}$  is amplitudes (in general, complex) of the Fourier harmonics of the function  $\vec{s}(\xi, t)$ , the latin indices denote the arguments, i.e.  $s_i \equiv s(k_i, \omega_i)$ ; the Greek superscripts means the spatial components;  $\epsilon^{\alpha\beta\gamma}$  is the unit antisymmetric tensor. It is supposed, that the correlator of the random disturbances looks like:

$$\langle \vec{f}(k_1, \omega_1) \vec{f}(k_2, \omega_2) \rangle = D k^{-y} \delta(k_1 + k_2) \delta(\omega_1 + \omega_2) \quad (4)$$

### 3 Calculations with the RG method

Our application of the RG-procedure in principle will be similar to the usage of the one in the turbulence theory. The details can be found for example in [5], [6], [7].

Let's briefly describe the fundamental principles of the RG-method for VT. As usual, the RG-procedure consists of two stages. At the first stage we eliminate high (or low) harmonics from the consideration. Let us separate the modes  $s_{k,\omega}$  on low and high harmonics:  $s_{k,\omega}^<$  and  $s_{k,\omega}^>$ . Here  $s_{k,\omega}^> = s_{k,\omega}$  at  $k \in (\Lambda e^{-l}, \Lambda)$ ;  $s_{k,\omega}^< = s_{k,\omega}$  at  $k \in (K, \Lambda e^{-l})$  and  $s^>, s^< \equiv 0$  in the other cases. Thus:

$$s_{k,\omega} = s_{k,\omega}^> + s_{k,\omega}^< \quad (5)$$

The quantity  $K$  is the parameter of the infrared cutting. Since  $s_{k=0} = 0$  because of the closure of a vortex line, the value of  $K = 2\pi/L$  is defined by the first non-zero harmonic at the Fourier transforming of  $\vec{s}$ ;  $L$  denotes the overall length of the vortex filament. The parameter of the ultraviolet cutting  $\Lambda$  is defined by the vortex core size  $r_0$ , and in this case  $\Lambda \rightarrow \infty$ ; the quantity  $l$  is some positive number, determining a boundary of the modes separation.

Assume that we consider the evolution of any harmonic  $s_{k,\omega}$  accordingly to the eq.(2) and  $s_{k,\omega}$  is in the interval of low (for example) harmonics. The idea of the RG method (such as the Wilson's formulation) is not to consider the interaction of  $s_{k,\omega}$  with each of harmonics in the other interval (the interval of high harmonics in this case). Instead of this, the RG method suggests to take into account the ensemble averaged influence of the whole high region. This averaging practically is executed on realizations of  $\vec{f}$ . The interaction between harmonics in own (in this case low harmonics) interval is considered in the usual way.

Concretely for a vortex loop this first stage of the RG-procedure is realized in the following way. We substitute the rel.(5) into the eq.(2) and then averaging  $s^>$  over the ensemble of  $\vec{f}$ . As the result we obtain a new, modified equation instead of the eq.(2). Note that the value  $\Lambda$  is transformed as  $\Lambda \rightarrow \tilde{\Lambda} = \Lambda e^{-l}$ . This new equation doesn't describe behaviour of VT on small spatial scales — the region of wave vectors  $(\Lambda e^{-l} \div \Lambda)$  becomes unavailable. Such procedure is similar to the Kadanoff transformation [8]. If this new equation has the same form (perhaps approximately) as the original one and only its parameters are changed, the second stage of the RG-procedure can be done.

On the second stage we reconvert the quantity  $\tilde{\Lambda}$  to the original value, i.e. we are returning to the description on the initial scale. In the other words, we are performing the transformation  $\tilde{\Lambda} \rightarrow \Lambda = \tilde{\Lambda} e^l$ .

Let's analyse some possible transformations of the parameters at the first stage. For example, in the turbulence theory the RG-procedure gives the renormalization of viscosity. This value has a positive addition after averaging high modes. In the other words, the influence of small spatial scales (or high harmonics) leads to increasing the viscosity. That, in turn, means that the dissipation of energy is increased. The mechanism of that phenomena is obvious. There are two processes for the evolution of any harmonic. The first is the thermal dissipation, the second is the nonlinear interaction with other harmonics, or, in the other words, the transfer of the energy of turbulent pulses across the spectrum in the wave-number space. (We don't take into account the external random disturbance  $\vec{f}$ ). Averaging high

harmonics lead to increasing the dissipation of low harmonics. This mean the existence of the transfer of the harmonic amplitudes from the low-frequency part of the spectrum to the high-frequency one. And just the transfer is the mechanism of the additional dissipation. Ones again emphasize: the additional dissipation (increasing viscosity) connected with the influence of the opposite spatial region, in the other words, the flux across the spectrum is responsible for increasing dissipation. And vice versa, a change of the viscosity under the averaging, means, in turn, the presence of the flux across the spectrum in a considering dynamical system. For turbulence theory this result is well known and trivial. Let's assume that we have solved some dynamical task and in an analogous case, we have obtained a *negative* addition to a dissipative coefficient. It must mean that the transfer across the wave-number spectrum has the opposite direction (i.e. from high modes to low ones). The zero addition must mean the absence of the transfer.

Now let's study the eq.(1)-eq.(2) in this context. The substitution of the rel.(5) into the eq.(2), gives the result:

$$i\omega s_0^{\alpha<} = \Gamma_{120}^{\alpha\beta\gamma} \{ s_1^{\beta<} s_2^{\gamma<} + \langle s_1^{\beta<} s_2^{\gamma>} \rangle + \langle s_1^{\beta>} s_2^{\gamma<} \rangle + \langle s_1^{\beta>} s_2^{\gamma>} \rangle \} - \nu k^2 s_0^{\alpha<} + f_0^{\alpha<} \quad (6)$$

The brackets  $\langle \dots \rangle$  mean the averaging of only high harmonics. To do this averaging, turn to the functional integral formalism [9]. In this approach, any correlator (for example the fourth term in the right hand side of the eq.(6)) is represented as a product of the functional derivatives (below denoted as  $\frac{\delta}{i\delta\eta}$  or  $\frac{\delta}{i\delta\hat{\eta}}$ ) on auxiliary fields  $(\eta, \hat{\eta})$  from the characteristic (generating) functional of a dynamical system. Not going into details [9], we write the expression to average the fourth term:

$$\begin{aligned} \langle s_1^{\beta>} s_2^{\gamma>} \rangle = & \frac{\delta}{i\delta\eta^\beta(1)} \frac{\delta}{i\delta\eta^\gamma(2)} \left\{ e^{\int d0 \Gamma_{120}^{\alpha\beta\gamma} \left\{ \frac{\delta}{i\delta\hat{\eta}^\alpha(0)} \frac{\delta}{i\delta\eta^\beta(1)} \frac{\delta}{i\delta\eta^\gamma(2)} \right\}} \right\} \\ & e^{\int d1 i\eta^\alpha(1) G^{(0)\alpha\beta}(1) \hat{\eta}^\beta(-1) - 1/2\eta^\alpha(1) C^{(0)\alpha\beta}(1) \eta^\beta(-1)} \Big|_{\eta=\hat{\eta}=0} \end{aligned} \quad (7)$$

Here  $G^{(0)}(1)$  is the zero-order response function  $G^{(0)\alpha\beta}(1) = \frac{\delta^{\alpha\beta}}{i\omega_1 + \nu k_1^2}$ ;  $C^{(0)}(1)$  is the zero-order correlation function  $C^{(0)\alpha\beta}(1) = \frac{D\delta^{\alpha\beta}}{(\omega_1^2 + \nu^2 k_1^4)k\nu}$  (to clarify  $\frac{D}{k\nu}$  see the rel.(4)).

It is well known that the Taylor-series of the first exponent gives usual Feynman's diagram series. In our case this takes the form:

$$\begin{aligned} \langle s_1^{\beta>} s_2^{\gamma>} \rangle = & \frac{\delta}{i\delta\eta^\beta(1)} \frac{\delta}{i\delta\eta^\gamma(2)} \left\{ 1 + \Gamma_{120}^{\alpha\beta\gamma} \left\{ \frac{\delta}{i\delta\hat{\eta}^\alpha(0)} \frac{\delta}{i\delta\eta^\beta(1)} \frac{\delta}{i\delta\eta^\gamma(2)} \right\} + \Gamma_{120}^{\alpha\beta\gamma} \{ \dots \} \Gamma_{120}^{\delta\theta\zeta} \{ \dots \} + \dots \right\} \\ & e^{i\eta G^{(0)\hat{\eta}-1/2\eta C^{(0)}\eta} \Big|_{\eta=\hat{\eta}=0}} \end{aligned} \quad (8)$$

The essential integrations in the last formulae are assumed. Thus, the result for  $\langle \vec{s} \vec{s} \rangle$  represents the infinite series of terms, each of that is a product of  $C^{(0)}$ ,  $G^{(0)}$  functions (corresponding to the lines, if diagrams to consider) and essential integrations. Every line  $C^{(0)}$  or  $G^{(0)}$  is the result of coupling the derivatives:  $\left\{ \frac{\delta}{i\delta\eta} \frac{\delta}{i\delta\eta} \right\}$  or  $\left\{ \frac{\delta}{i\delta\eta} \frac{\delta}{i\delta\hat{\eta}} \right\}$  accordingly. Obviously only terms, containing an even number of the derivatives is not equal to zero. Otherwise,  $\eta$  or  $\hat{\eta}$  remains as the efficient in the product and the term equal to zero because  $\eta = \hat{\eta} = 0$ . Besides if  $C^{(0)\alpha\beta}$  or  $G^{(0)\alpha\beta}$  ( $\alpha \neq \beta$ ) is presented in a product, this term is equal to zero, because  $\delta^{\alpha\beta}$  is presented in the definition of  $G^{(0)\alpha\beta}$  and  $C^{(0)\alpha\beta}$ . It is easy to

observe that *all* terms of the series for the exact correlator  $\langle s^\alpha s^\beta \rangle$  ( $\alpha \neq \beta$ ) are identically equal to zero. Since  $\alpha \neq \beta$  in the external differentiation, in the internal differentiation an *odd* number of the derivatives having  $\alpha$  or  $\beta$ -components are remained and thus the line  $G^{(0)\alpha\beta}$  or  $C^{(0)\alpha\beta}$  is *necessarily* appeared. *Thus,  $\langle s^\alpha s^\beta \rangle$  ( $\alpha \neq \beta$ ) equal to zero and this is the exact result.*

Similar argumentation leads to  $\langle s^\alpha \rangle$  is equal to zero. Hence all additional terms in the eq.(6) exactly equal to zero. The latter means the eq.(6) doesn't change under the averaging. Thus, the transfer within the spectrum in the LIA is absent. And this result is exact.

## 4 Discussion and conclusions.

Thus we obtained that the interaction of the form:  $\vec{s}' \times \vec{s}''$  doesn't change the spectral structure of the value  $\vec{s}$ . In the other words, though there is the nonlinear interaction in the system, harmonics doesn't affect to each other. This is the extremely unexpected result. The usual scenario looks quite otherwise. Usually, because nonlinearity is present, interaction of only two harmonics leads to the appearance of additional harmonics. They, in turn, interact between each other and with original ones. As the result, after some time, the full spectrum is presented in the system. In our case in the system remain only originally excited modes and their amplitudes are not changed. At the same time, their phases ( $s_{k\omega}$  is a complex number) must be changed. Otherwise it should be full correspondense to a linear equation. However, it is well known the solution (that used the LIA) of the task about the decay of kink [10], that is not possible in a linear case.

Obviously, it is possible to invent an infinite number of similar equations, not having a flow in the wave-number space. Perhaps, all of them present pure abstract interest. Nevertheless, the single practical note can be done. As mentioned above, the LIA is often used to approximate the full Biot-Savart law in numerical models of superfluid turbulence. The question about the validity of this approach was dicussed repeatedly and the exact opinion is absent. Let's consider this problem in the context of this work. Transfer within the spectrum is certainly presented in the entire Biot-Savart law, at that, probably, the nonlocal transfer. (Real vortex tangle is described by the Biot-Savart law). Besides, the transfer within the spectrum is the very important feature of nonlinear dynamical systems. For example, whether is it possible to imagine any approach to a turbulence theory, where the transfer of turbulent pulses from large scales to small ones is absent... But just the same picture takes place, when the LIA is used for numerical or analytical considerations of real VT.

The last remark concerns to the usage of RG-procedures for dynamical problems. At this moment this question is not fully clear. Usual way emploing this method based on a perturbation theory. As a result, at the Wilson's formulation of RG, an infinite series of additional terms is appeared in the renormalized equations. So-called "marginal" terms represent, at this case, the main problem. The total value of this additions is uncontrolled when the Wilson's RG-procedure is carried out. The detal consideration of this question can be found in the paper [11].

In principle, we can expand  $s^>$ ,  $s^<$  to a perturbation series. In the other words, to employ the usual

approach to the RG-procedure. If we have done this, the following result should be obtained. There is transferring amplitudes of harmonics across the spectrum is existed. And it is interesting to note, that the flux of harmonics is directed from small spatial scales to large ones, i.e. to the opposite side, in comparison with the turbulence theory. It is very unexpected result for superfluid turbulence, if it were right. In real, the LIA does not describe the flux of amplitudes of harmonics within the spectrum.

Returning to the RG-method and comparing perturbation theory with the presented above exact solution, one can conclude that the marginal additions are really uncontrolled, even if  $\epsilon$ -expansion to use.

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## References

- [1] S. K. Nemirovskii and W. Fiszdon, *Rev. Mod. Phys.* **67**, 37 (1995).
- [2] Donnelly R. J., *Quantized Vortices in Helium II*, Cambridge University, Cambridge, England (1991).
- [3] K. W. Schwarz, *Phys. Rev. B* **18**, 245 (1978).
- [4] K. W. Schwarz, *Phys. Rev. B* **38**, 2398 (1988).
- [5] W. D. McComb, *The Physics of Fluid Turbulence*, Clarendon Press, Oxford (1990).
- [6] D. Foster, D. Nelson, M. Stephen, *Phys. Rev. A* **16**, 732 (1977).
- [7] V. Yakhot, S. A. Orszag *J. Sci. Comput.* **1**, 1 (1986).
- [8] S. K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, Reading, MA 1976).
- [9] R. V. Jensen *Journal of Statistical Physics* **25**, 2 (1981).
- [10] T. F. Buttke *Journal of Computational Physics* **76**, pp. 301-326, (1988).
- [11] G. L. Eyink *Phys. of Fluids* **6**, 9, (1994).