Glassy Phase in the Hamiltonian Mean Field model

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We study the relaxation dynamics of a Hamiltonian system of N fully-coupled XY spins. The thermodynamics of the system predicts a ferromagnetic and a paramagnetic phase. Starting from out-of-equilibrium initial conditions, the dynamics at constant energy drives the system into quasi-stationary states (QSS) characterized by dynamical frustration. We introduce the spin polarization as a new order parameter which allows to interpret the dynamically generated QSS regime as a glassy phase of the model.

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The Hamiltonian Mean Field (HMF) model, originally introduced in Ref.[1], has been intensively studied in the last years for its extreme richness and flexibility in exploring the connections between dynamics and thermodynamics in long-range many-body systems. In fact, on one hand the model has an exact equilibrium solution, on the other hand, because of the presence of a kinetic energy term in the Hamiltonian, the dynamics can be studied by means of molecular dynamics simulations [1, 2, 3, 4]. From these investigations, many new interesting features have emerged which are common to other systems with long-range interactions [5, 6, 7]. One of the most intriguing characteristics of the dynamics is the existence of quasi-stationary states (QSS), i.e. dynamically-created states, whose lifetime diverges with the system size N [8]. In such states anomalous diffusion [3], non-Gaussian velocity distributions [8], vanishing Lyapunov exponents [8], ergodicity breaking and slow-decaying correlations [9, 10] have been observed. These features have suggested a possible application of Tsallis generalized thermodynamics [8, 11, 12, 13, 14].

In this paper we show that the HMF model in the QSS regime behaves similary to a glassy system. In fact, by means of a new order parameter, it is possible to characterize the dynamically generated QSS as a thermodynamics glassy phase of the model, despite the fact that neither disorder nor frustration are *a-priori* present in the interaction. The main idea of the paper originated from the observation of slow relaxation and aging [9, 10] in the QSS regime. Such a behavior is typical of frustrated systems, whose prototype are spin-glasses [15]: in these systems, the impossibility to minimize simultaneously the interaction energies of all the couples of spins leads the system to a very complex energetic landscape. One might imagine it as consisting of large valleys separated by high activation energies. Each valley contains many local minima, i.e. metastable states, in which the system, after quenching in his low-temperature phase, can remain trapped for a very long time, showing those strong memory effects better known as aging behavior.

The HMF model describes a system of N fully-coupled classical XY spins [1]:

$$\vec{s_i} = (cos\ \theta_i, sin\ \theta_i) \qquad i = 1, ..., N \ . \tag{1}$$

The equations of motion derive from the following Hamiltonian:

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^{N} [1 - cos(\theta_i - \theta_j)] \ , \tag{2}$$

where θ_i $(0 < \theta_i \leq 2\pi)$ is the angle and p_i the respective conjugate variable representing the rotational velocity (the mass is set equal to 1) of spin i. If we associate a particle, moving on the unit circle, to each spin, the model can be seen as a system of fully-coupled *rotators*. Though the division of the potential by a factor N (the so-called Kac's prescription) makes the Hamiltonian formally extensive [4], the latter remains nonadditive due to the long-range nature of the interaction [12].

The equilibrium solution of the model in the canonical ensemble predicts a second-order phase transition from a high temperature paramagnetic (PA) phase to a low temperature ferromagnetic (FE) one [1]. The critical temperature is $T_c = 0.5$ and corresponds to a critical energy per particle $U_c = E_c/N = 0.75$. The order parameter of this phase transition is the modulus of the *average magnetization* per spin defined as: $M = \frac{1}{N}|\sum_{i=1}^{N} \vec{s_i}|$. Above T_c, in the PA phase, the spins point in different directions and $M \sim 0$. Below T_c, in the FE phase, all the spins are aligned (the rotators are trapped in a single cluster) and $M \neq 0$.

The molecular dynamics simulations at constant energy (microcanonical ensemble) reveals interesting properties in the energy range $U = 0.5 - 0.75$. In fact, starting from of out-of-equilibrium initial conditions [17], the system has an extremely slow relaxation to the equilibrium and show the presence of meta-equilibrium *quasi-stationary states* (QSS) with the following properties:

1) The temperature (calculated from the average kinetic energy) and the magnetization assume constant val-

ues for a time τ_{QSS}. Such values are different from the equilibrium ones and depend on the number of spins N.

2) For large N, M vanishes (as $N^{-1/6}$) and T tends to an energy-dependent value so that the QSS lie on the extension for $T < T_c$ of the high-temperature branch of the caloric curve.

3) τ_{QSS} grows linearly with the system size N [2]. For this reason the QSS regime can be interpreted as the true equilibrium if the thermodynamic limit is taken before the infinite-time limit [8].

4) The QSS are characterized by non-Gaussian velocity distributions [8], Lévy walks and anomalous diffusion [3].

5) The largest Lyapunov exponent vanishes and the system lives in a restricted part of the *a-priori* accessible phase space. Such a *weak-mixing* dynamics suggests a connection with the Tsallis generalized thermodynamics [8], but also the possibility of framing the QSS within the so-called *weak-ergodicity breaking* scenario [21], typical of glassy systems.

The last point has been recently corroborated by the discovery of aging in the QSS regime [9, 10]. In the following we show how the analogy with glassy systems and the weak ergodicity breaking scenario can be made more stringent [16] by the introduction of a new order parameter inspired by the microscopic dynamics of spin-glass models.

The materials that originally were called *spin-glasses* are alloys formed by a noble metal support (gold, silver, copper) containing randomly distributed magnetic impurities (iron or manganese). Such a configuration determines a random distribution ('quenched disorder') of the interactions: according to the distance between each pair of spins, the interaction among them may be either ferromagnetic or anti-ferromagnetic, thus generating frustration. The first theoretical spin-glass model was the short-range *Edwards-Anderson* (EA) model [18]. However, the first solvable one was the *Sherrington-Kirkpatrick* (SK) model [19], where the spins are coupled by infinite-ranged interactions independently distributed according to a Gaussian. Depending on the temperature and the parameters of the Gaussian distribution, the SK model shows three different phases, namely ferromagnetic (FE), paramagnetic (PA) and spin-glass (SG). Since the magnetization M vanishes in the SG phase as well as in the PA one, an additional order parameter q_{EA} - called *EA order parameter* - was proposed [18, 19] in order to discriminate between spin-glass disorder and paramagnetism. The physical meaning of this order parameter is that one of quantifying the degree of freezing in the SG phase. In fact the three phases are characterized by a different microscopic behavior. In order to get an intuitive picture of this behavior, let us imagine to take some snapshots of the spins configuration in each of the three phases [20]. If a snapshot is taken at one particular time, one easily would be able to recognize the FE phase, since all the spins are aligned and frozen in their equilibrium position. However it would be impossible to distinguish between the PA and the SG phase.

In fact in both of these phases the orientations of spins are random, due to the high thermal noise for the PA phase and to the quenched spatial disorder for the SG phase. The discrimination between these two phases is possible only if one takes a temporal sequence of snapshots. In fact in the PA phase the orientation of each spin at successive instants of time would be random, so the sequence of snapshots shows every time a different spatial configuration. On the other hand in the SG phase all the snapshots are identical, since each spin is frozen and retains the same orientation over very long periods of time.

As previously discussed, the HMF model at equilibrium has only two phases (PA and FE). The main goal of this paper is to show that the dynamically generated QSS can be interpreted as a glassy phase of the model. For this reason, inspired by the arguments described above, we propose to introduce a new order parameter, the *average polarization p*, in order to measure the extent of freezing of the system. The physical meaning of p is related to the elementary polarizations $\vec{p_i}$, i.e. the time averages of the successive positions of each elementary spin vector, defined as:

$$\vec{p_i} = <\vec{s_i}(t)> = \frac{1}{\tau}\int_0^\tau \vec{s_i}(t)dt \qquad i = 1, ..., N \qquad (3)$$

The average polarization is then obtained averaging the modulus of the elementary polarization over all the rotators:

$$p = \frac{1}{N}\sum_{i=1}^N |\vec{p_i}| \qquad . \qquad (4)$$

Such a new order parameter has to be compared to M, the modulus of the *average magnetization*, calculated as:

$$M = <M(t)> = \frac{1}{\tau}\int_0^\tau M(t)\,dt \,, \;\; M(t) = \frac{1}{N}\left|\sum_{i=1}^N \vec{s_i(t)}\right| . \tag{5}$$

In the FE phase each elementary polarization vector coincides with the correspondent spin vector, both being frozen and parallel, then the average polarization p keeps a non zero value equal to M. In the PA phase the orientation of each spin vector at every time is completely random, so this continuous motion yields a zero value both for M and p. On the other hand, if the QSS correspond to a glassy-like phase of the model, we expect to find a zero value for M, as in the PA phase, and a non zero value for p, as in the FE one. All these features are summarized in Table 1.

In fig.1 we show the modulus of the elementary polarization for each spin i. We consider a system of $N = 1000$ spins and different energy densities. The values of the

TABLE I: Values of M and p in the three phases of the HMF model

	M	p
Ferromagnetic phase (FE)	$\neq 0$	$\neq 0$
Paramagnetic phase (PA)	0	0
Glassy phase	0	$\neq 0$
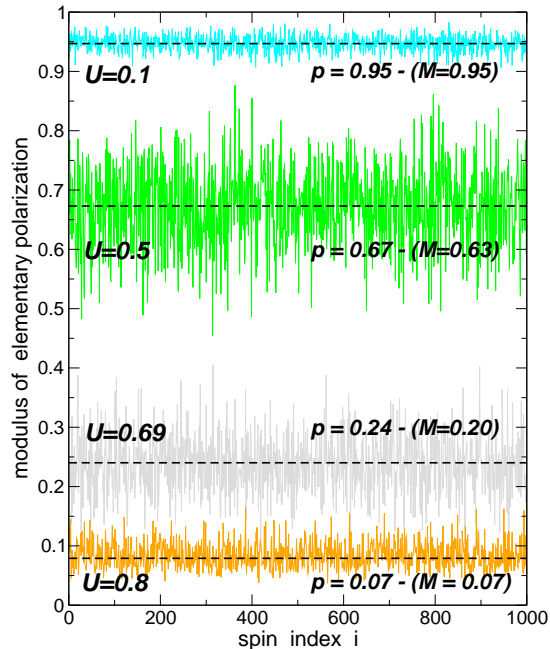
FIG. 1: The modulus of the elementary polarization $\mid \vec{p_i} \mid = \mid <\vec{s_i}(t)> \mid$ for a system with $N = 1000$ and different energies. The values of the average polarization p (dashed lines) and magnetization are also reported for comparison. Note that only for $U = 0.5$ and $U = 0.69$ we are in the QSS regime. In the other cases the system is at equilibrium

average polarization p and the average magnetization M are also reported in figure. In the simulation we have performed, the time averages of p and M are evaluated over an opportune time interval $\tau < \tau_{QSS}$, in order to stay inside the temperature plateau for those energy values where the QSS regime appears ($U = 0.5$ and $U = 0.69$). In particular we have used $\tau = 2000$ and a transient of 1000 time units. The results do not depend significatively on τ. As usual in molecular dynamics simulations, in order to make our results independent from the specific dynamical realization, we have also taken averages over a set of different realizations (events) of the same out-of-equilibrium initial conditions. As expected the two parameters p and M coincide and are close to 1 at low energy, e.g. $U = 0.1$, while both of them tend to zero for U above the critical value $U_c = 0.75$. The situation is different for $U = 0.5$ and for $U = 0.69$, two energies at which the QSS appear. In these cases the values of p and M are different: for $N = 1000$ we have respectively
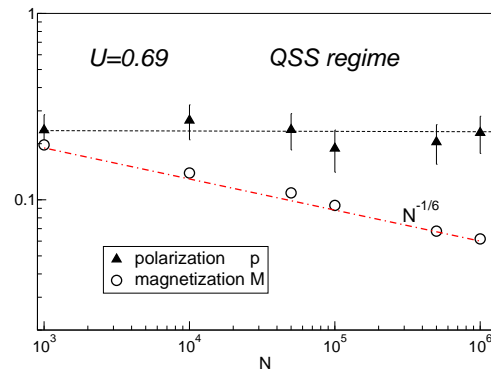

FIG. 2: We plot the values of the polarization p and the magnetization M calculated in the QSS regime for $U = 0.69$ as a function of the size N of the system. While p assumes a constant value $\sim 0.24 \pm 0.02$, M decreases as $N^{-1/6}$.
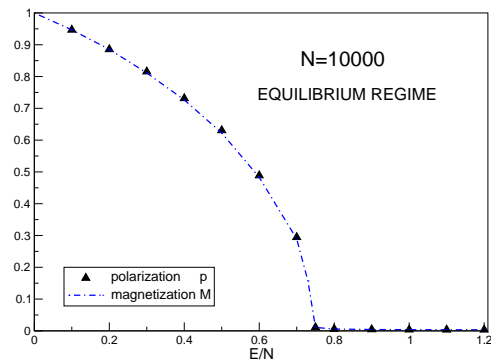

FIG. 3: For $N = 10000$, we show the polarization p and magnetization M vs energy per particle U once the equilibrium regime has been reached.

$p = 0.67$, $M = 0.63$ and $p = 0.24$, $M = 0.20$. We have checked that the difference between p and M increases with the system size N. In particular for large N, in the QSS regime, we expect a vanishing average magnetization M and an average polarization p different from zero.

In fig.2 we study the behavior of p and M with the size of the system. We report only the case $U = 0.69$ where the anomalous effects of QSS are more evident. As expected, while M vanishes as $N^{-1/6}$, p is independent of N (within the error) and equal to 0.24 ± 0.02.

Finally in fig.3 we consider a system with $N = 10000$ and we compare magnetization M and polarization p at equilibrium for different energies. In order to let the system reach equilibrium for the energy range $0.5 \leq U \leq U_c$ we ran the simulations for a time much larger than τ_{QSS}. In this way every trace of metastability, and consequently also of the glassy phase behavior, disappears. The numerical values of M and p reported in figure coincide,

in agreement with the previous statement about equivalence between M and p in the pure FE and PA phase.

Our numerical results support the interpretation of the QSS regime as a dynamically-created glassy phase of the HMF model. In the QSS regime the simulations show the formation of a dynamical clustering [10]. The rotators feel the attraction of the dynamically-generated clusters in competition within each other. Each rotator remains trapped in a cluster for a while and then eventually succeed in escaping from it [22]. This is also the cause of the anomalous diffusion and Lévy walks observed in Ref.[3]. Such a competition between the different clusters in the QSS regime therefore realizes a *dynamical frustration* that slows down the dynamics and prevents the system from exploring all the potentially available phase space. Such a behavior is also related to the aging phenomenon observed in refs. [9, 10] and can be interpreted in the framework of the weak-ergodicity breaking scenario [21]. When, at the end of the QSS regime, the system relaxes to the equilibrium of the pure FE phase, all the rotators concentrate in a single cluster which rotates with the same phase of the average magnetization vector, i.e. $\phi = tan^{-1}(M_y/M_x)$ [23], and all the anomalies disappear.

In conclusion the results of this paper show that the most remarkable features of the long-range HMF model, namely the dynamically-generated metastable states, can be interpreted as a thermodynamical glassy phase of the model. If the system is started sufficiently far from equilibrium, the long-range character of the interaction produces dynamically a very complex configurational landscape typical of glassy systems. We have introduced the polarization p as a new order parameter to characterize the degree of freezing of the spins due to the presence of the dynamical competition among clusters in the metastable state. Considering that the HMF model is paradigmatic of a large class of long-range Hamiltonian systems, it seems very interesting to search for further connections with glassy dynamics, which likely could help understanding some of the open problems in this field.

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[22] Although the system has a long-range interaction, one can define clusters by considering rotators close in angle which move with almost the same velocity[3, 10].

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