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On Spatial Consensus Formation:
Is the Sznajd Model Different from a Voter Model?

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Abstract

In this paper, we investigate the so-called "Sznajd Model" (SM) in one dimension, which is a simple cellular automata approach to consensus formation among two opposite opinions (described by spin up or down). To elucidate the SM dynamics, we first provide results of computer simulations for the spatio-temporal evolution of the opinion distribution $L(t)$, the evolution of magnetization $m(t)$, the distribution of decision times $P(\tau)$ and relaxation times $P(\mu)$.

In the main part of the paper, it is shown that the SM can be completely reformulated in terms of a linear VM, where the transition rates towards a given opinion are directly proportional to frequency of the respective opinion of the second-nearest neighbors (no matter what the nearest neighbors are). So, the SM dynamics can be reduced to one rule, "Just follow your second-nearest neighbor". The equivalence is demonstrated by extensive computer simulations that show the same behavior between SM and VM in terms of $L(t)$, $m(t)$, $P(\tau)$, $P(\mu)$, and the final attractor statistics.

The reformulation of the SM in terms of a VM involves a new parameter σ, to bias between anti- and ferromagnetic decisions in the case of frustration. We show that σ plays a crucial role in explaining the phase transition observed in SM. We further explore the role of synchronous versus asynchronous update rules on the intermediate dynamics and the final attractors. Compared to the original SM, we find three additional attractors, two of them related to an asymmetric coexistence between the opposite opinions.

1 Introduction

The old wisdom still holds: if a single person finds a particular case important, this does not matter too much – but already if two persons are convinced of its importance, they have a good chance to convince others. This can be simulated by means of a cellular automaton (CA)

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model of consensus formation, meanwhile well known as *Sznajd model* (abbreviated as SM from now on). It is named after the two Polish authors, Katarzyna Sznajd-Weron and her father Józef Sznajd, who in 2000 published a paper on "opinion evolution in a closed community" [38]. Interestingly, the dynamics of convincing others can be applied to the adoption of the SM in the scientific community itself. The paper and the rather simple model (discussed in Sect. 2) would problably not have gained so much attention without at least one other person confident about its importance. It was Dietrich Stauffer [34, 35, 36, 37], who, after being influenced by the positive response of his collaborators, started to propagate the SM in various scientific communities: physicists, social scientists, computer scientists – and this way persuaded "neighboring" scientists [4, 5, 6, 10, 20, 25, 30] to play with it (including the authors of this paper). So the SM – which the original authors called *USDF model*: "united we stand, divided we fall" – attracted a lot of interest. In particular, the dynamics, originally given for the one-dimensional lattice, was generalized to higher dimensions [5, 37]. Bernardes *et al.* [4] used the SM to explain the distribution of political votes and in [5] applied it, together with a Barabasi network, to the Brazilian elections. In [10], the SM was also applied to small-world networks. Other applications deal with financial markets [39], with aspects of statistical physics, such as correlated percolation [20], and with different geometries [6].

While we on one hand are allured by the complex intermediate dynamics of this rather simplistic model, our interest in this subject is mainly driven by the question: Is there anything new in the SM? I.e., in what respect is the SM different from other CA models dealing with local adoption processes?

In fact, ever since CA started to invade the social sciences in the 1950's, lots of different CA-based models were proposed to describe spatial structure formation in social systems [1, 12, 13, 22, 26, 28, 29, 33]. One well established model class is known as the *voter model* (abbreviated as VM from now on). It is based on the idea that the adoption of a given "opinion" (behavior, attitude) depends on the frequency of that opinion in the neighborhood. In the *linear* VM, the transition rate of adopting an opinion is directly proportional to the given frequency, in non-linear VM also other frequency dependencies (e.g. voting against the trend) are taken into account. VM with positive frequency dependence (i.e. *majority voting rules*) can be considered as a simple prototype for modeling *herding behavior* – a phenomenon widely known in biology, economy and the social sciences. In a special class of non-linear VM, the non-linearity may result from certain economic or social considerations, for example from a payoff matrix. Hence, as long as the adoption dynamics depend on the (local) frequency, even applications in population biology or evolutionary game theory can be treated as a non-linear VM.

In addition to earlier work on mathematical analysis on VM [15], *spatial* VM have been recently investigated by means of CA concepts [7, 8, 9, 14, 19]. In our own work [3, 32], we were particularly interested in spatio-temporal pattern formation in one- and two-dimensional VM. Based

on theoretical investigations of the microscopic CA dynamics, we were able derive approximations for time-dependent macroscopic quantities, such as frequencies or spatial correlations, and critical parameters for phase transitions. A comparison of these results with the CA simulations showed a good agreement. Other investigations of the macroscopic VM dynamics by means of either pair approximation or Markov models can be found in [2, 11, 17, 18, 27].

Application of the non-linear voter dynamics to the local adoption of successful strategic behavior (such as to cooperate or to defect) revealed phase diagrams for specific spatial patterns (formation of small clusters and spatial domains, front dynamics) [31]. Similar work on the spatial adoption of game-theoretical strategies has been done in [16, 21, 23, 24].

So, given the extensive work done on VM, the question addressed in this paper is about connections between VM and SM. Evidently, both models deal with *local adoption processes*, but an in-depth analysis of SM is still lacking. If we could reveal – as we will do in this paper – that SM is just a special case of VM, then many of the techniques and results obtained earlier for linear and non-linear VM can be applied to SM, thus providing us with a wealth of analytical understanding of the SM dynamics. Hence, to appreciate the ideas behind the SM, we should also understand how the SM is related to the existing classes of CA models on consensus formation and what the difference are. To find this out, we start in Sect. 2 with a brief description of the SM and present computer simulations in one dimension. In Sect. 3, we reformulate the SM in terms of a VM, to show that the SM is in fact a *linear* VM. In Sect. 4, we further explain the nature of the phase transitions observed in SM by means of an external parameter, σ. In Sect. 5 we investigate the influence of two different update rules (asynchronous vs. synchronous) on the intermediate dynamics and the stationary states. We also show some new results on consensus formation with asymmetric coexistence. In Sect. 6, we conclude with some hints about further research.

2 Dynamics of the SM

2.1 Rules of the SM Game

In their original paper, Sznajd-Weron and Sznajd [38] proposed a one-dimensional Ising spin model with periodic boundary conditions where each spin (or lattice site) $i = 1, ..., N$ can have be found in one of two states, $\theta_i \in \{-1, +1\}$ or $\{-, +\}$ for short, which in the context of opinion formation shall refer to two opposite opinions (see Fig. 1).

For the spin interaction of neighboring spins, the following two rules are proposed:

Rule 1: If two consecutive lattice sites have the *same* opinion, (either +1 or -1), i.e. if $\theta_i\theta_{i+1} = 1$, then the two neighbouring sites $\{i-1\}$ and $\{i+2\}$ will adopt the opinion of the pair $\{i, i+1\}$,
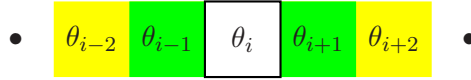
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Figure 1: One-dimensional CA, where each lattice site $i = 1, ..., N$ can have one out of two opinions, $\theta_i \in \{-1, +1\}$ or $\{-, +\}$ for short. In the SM, the opinions θ_{i-1} and θ_{i+2} depend on the opinions of the pair θ_i, θ_{i+1} as described by rules 1, 2.

i.e. $\theta_{i-1} = \theta_{i+2} = \theta_i = \theta_{i+1}$. This rule refers to *ferromagnetism*.

Rule 2: If two consecutive lattice sites have a *different* opinion, (either +1 or -1), i.e. if $\theta_i \theta_{i+1} = -1$, then the two neighbouring sites $\{i - 1\}$ and $\{i + 2\}$ will adopt their opinion from the *second nearest neighbors* as follows: $\theta_{i-1} = \theta_{i+1}$, $\theta_{i+2} = \theta_i$. This rule refers to *anti-ferromagnetism*.

A *deterministic dynamics* is considered here, i.e. the rules apply with probability one – which is similar to an Ising system at temperature $T = 0$. But there is still randomness in the system in the sense that (i) there is an inital random distribution of the opinions with the mean values of the frequencies $f_{+1} = f_{-1} = 0.5$, and (ii) during the computer simulations, the site i for the next step is chosen randomly, i.e. the dynamics is governed by a *random sequential update*, or *asynchronous* update.

We further note that in the SM *two spins* are flipped at a time. With the two rules above, we find the following possible transitions in a neighborhood of $n = 4$:

$$
\begin{array}{cccc c cccc c}
\theta_{i-1} & \theta_i & \theta_{i+1} & \theta_{i+2} & \rightarrow & \theta_{i-1} & \theta_i & \theta_{i+1} & \theta_{i+2} & \text{rule} \\
\hline
? & + & + & ? & & + & + & + & + & (1) \\
? & - & - & ? & & - & - & - & - & (1) \\
? & + & - & ? & & - & + & - & + & (2) \\
? & - & + & ? & & + & - & + & - & (2)
\end{array}
\tag{1}
$$

It has been established through micro-simulations that the one-dimensional SM for any random initial configuration asymptotically reaches one of three possible attractors, two of which refer to ferromagnetism and one to anti-ferromagnetism. These possible attractors are reached with different probability:

- attractor *ferro$_+$*: $\{+ + + + + + + + + +\}$ with probability $p = 0.25$

- attractor *ferro$_-$*: $\{- - - - - - - - - -\}$ with probability $p = 0.25$

- attractor *anti-ferro*: $\{- + - + - + - + - +\}$ with probability $p = 0.5$

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In order to verify these probabilities, let us consider a lattice of size N with periodic boundary conditions and an initial random distribution of $+$ and $-$. Then, the number of consecutive pairs, $\{i, i+1\}$ is also N, and the initial probability of finding either a ferromagnetic or an anti-ferromagnetic pair adds up to 0.5, i.e.,

$$p_{\mathrm{f}} = p_{++} + p_{--} = 0.5 \; ; \quad p_{\mathrm{af}} = p_{-+} + p_{+-} = 0.5 \; ; \quad p_{\mathrm{f}} + p_{\mathrm{af}} = 1 \tag{2}$$

Under these conditions, what is the probability to find ferromagnetic and anti-ferromagnetic pairs in the course of time? During the first q steps, we may assume that the initial distribution is not changed much by the dynamics, i.e. eq. (2) remains valid and the probabilities are given by the binomial distribution:

$$\sum_{k=0}^{q} \binom{q}{k} p_{\mathrm{af}}^{k} \, p_{\mathrm{f}}^{q-k} = 1 \tag{3}$$

If during the first q steps more than $q/2$ antiferrormagnetic pairs are selected, then p_{af} increases since each selection will lead to two new antiferromagnetic pairs. The case of ferromagnetic pair selection can be treated similarly. For q being an even number, the probability is then given by:

$$\sum_{i=(q/2)+1}^{q} \binom{q}{k} p_{\mathrm{af}}^{k} \, p_{\mathrm{f}}^{q-k} + \frac{1}{2} \binom{q}{q/2} p_{\mathrm{af}}^{q/2} \, p_{\mathrm{f}}^{q/2} = 0.5 \tag{4}$$

where the first term denotes the probability of selecting more than $q/2$ anti-ferromagnetic pairs (favoring anti-ferromagnetism) and the second term denotes the probability of selecting exactly $q/2$ pairs (favoring both ferro- and anti-ferromagnetism with probability 0.5). Thus, we can conclude that the probability for the system to reach the *anti-ferromagnetic* attractor is given by 0.5, eq. (4).

Equation (4) is valid as long as $p_{\mathrm{f}} = p_{\mathrm{af}} = 0.5$, i.e. for $t \leq q$. After the initial time lag, the symmetry is broken and the system dynamics goes towards one of the possible anti- or ferromagnetic attractors with probability one.

2.2 Results of SM Computer Simulations

In this section, we show some results of computer simulations of the one-dimensional CA, using the asynchronous update rule. The results are basically known, but we present them here to allow for a comparison with the voter model in the following sections. Fig. 2 and Fig. 3 show the spatio-temporal evolution of the opinion distribution for the two different attractors and the respective magnetization curves over time, $m(t)$. The magnetization gives a measure for reaching the attractor and is defined as:

$$m(t) = f_{+} - f_{-} \; ; \quad f_{+} = \frac{1}{N} \sum_{i}^{N} \delta_{+1;\theta_{i}} \; ; \quad f_{-} = 1 - f_{+} \tag{5}$$

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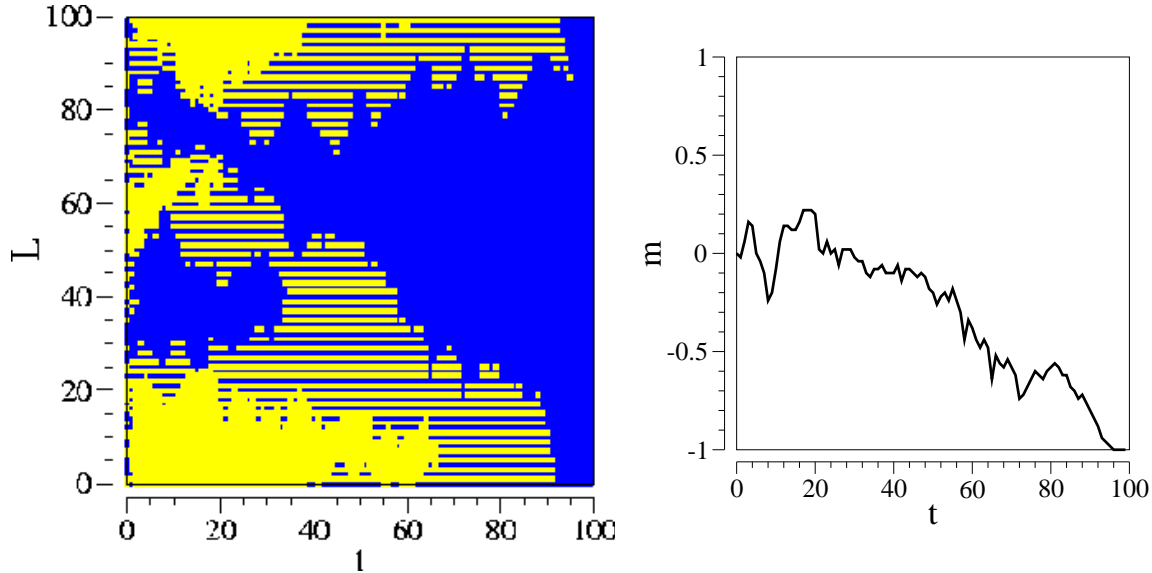and $\delta_{+;\theta_i}$ is the Kronecker delta, which is 1 only if $\theta_i = +1$ and zero otherwise.

Figure 2: Evolution of the one-dimensional lattice (left) and magnetization m (right) vs time in Monte-Carlo steps. At intermediate times, both anti-ferromagnetic stripes and ferromagnetic domains coexist, but asymptotically, the *consensus attractor* is reached. ($N = 100$, asynchronous update according to the Sznajd rules, eq. (1), dark gray dots indicate state -1, light gray dots state $+1$.)

In Fig. 4(left) we present the distribution $P(\tau)$ of decision times τ introduced in the original paper [38] as the time needed by an "individual" to change his/her opinion. I.e., τ is a measure of how frequently the opinion of a particular individual changes, if it is selected by the asynchronous dynamics (which is apparently not at every flip, but on average once during one Monte-Carlo step). The power-law behavior $P(\tau) \propto \tau^{-1.5}$, already found in [38] can be clearly observed in Fig. 4.

Finally, in Fig. 4(right) we also present the distribution $P(\mu)$ of relaxation times μ into one of the possible attractors. The bin size for the histogram has been chosen as $2^{n+1} - 2^n$, to allow for comparison with [37]. We find that the distribution has its maximum at about 250 MC steps, which means that an average simulation with a CA of $N = 100$ needs about this much time. But, deviations from this mean value follow approximately a *log-normal* distribution, as shown in Fig. 4(right). This agrees with the finding of Stauffer for a two-dimensional CA (for rules IIc and III) [37].
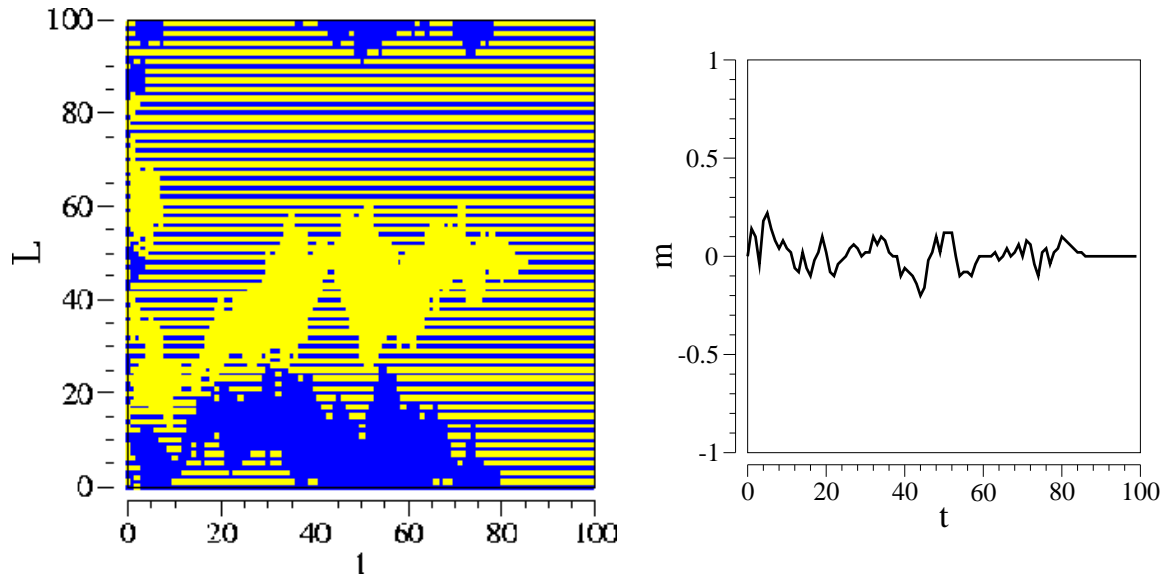
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Figure 3: Evolution of the one-dimensional lattice (left) and magnetization m (right) vs time in Monte-Carlo steps. Asymptotically, the *coexistence attractor* (or stalemate, *antagonistic* attractor) is reached. (Same parameters as in Fig. 2, but different random numbers)
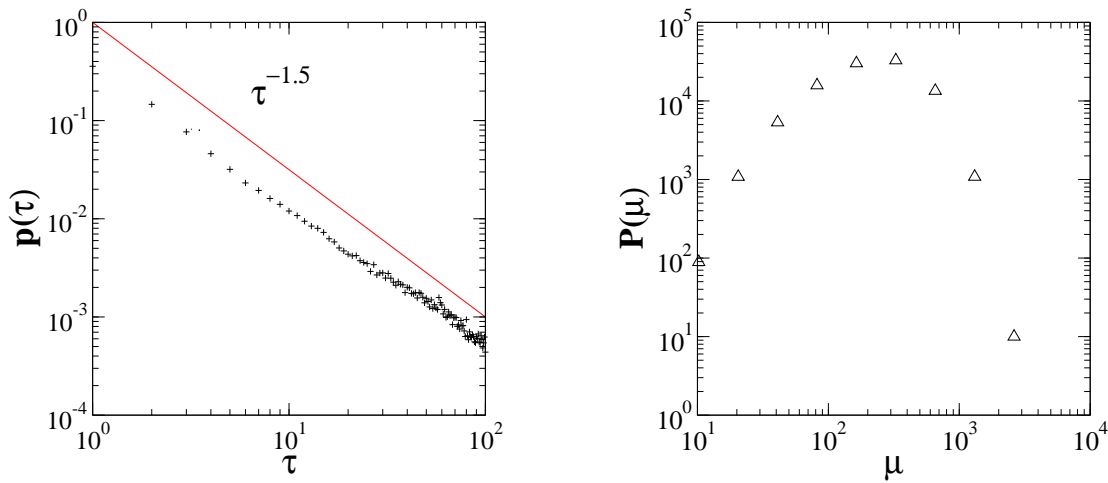

Figure 4: (Left) Distribution of decision times $P(\tau)$ averaged over 200 simulations, (right) distribution $P(\mu)$ of relaxation times μ averaged over 100.000 simulations. ($N = 100$, asynchronous update according to the Sznajd rules, eq. (1))

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3   Reformulation of the SM in terms of a VM

3.1   Rules of the VM Game

Generally, it is argued that the SM is different from the voter model (VM) in that the opinion spreads outwards instead of inwards. In this section, we reformulate the SM in terms of a VM based on the Ising spin concept and will demonstrate that the SM is in fact a *linear VM*. This derivation will occur in three steps.

First, we wish to point out the basic idea of a VM. There, the adoption of an opinion $+1$ or -1 of a given site i depends on the *local frequency* of the respective opinions in the immediate neighborhood. Usually, only nearest neighbors are taken into account. The transition rate for changing θ_i is generally given as:

$$w(\theta_i'|\theta_i) = \kappa(f)\, f_{\theta_i'} \tag{6}$$

where $f_{\theta_i'}$ is the local frequency of the *opposite* opinion in the nearest neighborhood of site i possessing opinion θ_i, and $\kappa(f)$ is a non-linear function dependent on local frequency. In the linear VM, $\kappa(f) = \text{const.}$ is chosen.

In order to derive a similar transition rate for the SM, we look at the possible local configurations in the neighborhood of site i, given in the first column of eq. (7). Here, we have to note a basic difference between SM and VM. In SM, a pair of sites $\{i-2, i-1\}$ influences its two neighbors $\{i-3, i\}$ at the same time, i.e. the dynamics of i is influenced only by *one* pair of neigbors (*either* from the left *or* from the right side). As opposed to that, in the VM, the local frequency of opinions *both* from the left *and* from the right side is taken into account. So, looking at the second nearest neighborhood on both sides, we have a total of 16 possible configurations. The respective transition rates $w(\theta_i'|\theta_i)$ for adjusting θ_i are then defined in such a way that they lead

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to the *same dynamics* as in the SM.

θ_{i-2}	θ_{i-1}	θ_i	θ_{i+1}	θ_{i+2}	$w(+|\theta_i)$	$w(-|\theta_i)$
+	+	?	+	+	1	0
−	−	?	−	−	0	1
+	+	?	−	−	0.5	0.5
−	−	?	+	+	0.5	0.5
−	+	?	+	−	0	1
+	−	?	−	+	1	0
−	+	?	−	+	0.5	0.5
+	−	?	+	−	0.5	0.5
+	+	?	−	+	1	0
−	−	?	+	−	0	1
+	−	?	+	+	1	0
−	+	?	−	−	0	1
+	+	?	+	−	σ	$1-\sigma$
−	+	?	+	+	σ	$1-\sigma$
+	−	?	−	−	$1-\sigma$	σ
−	−	?	−	+	$1-\sigma$	σ

$$(7)$$

In eq. (7), the first four transition rates are based on the ferromagnetic principle, since $\theta_{i-2}\theta_{i-1} = \theta_{i+1}\theta_{i+2} = 1$ (rule 1 of SM). The next four transition rates are based on the anti-ferromagnetic principle, since $\theta_{i-2}\theta_{i-1} = \theta_{i+1}\theta_{i+2} = -1$ (rule 2 of SM). The last eight possible configurations do not correspond to the SM, because opinion θ_i has to choose between ferromagnetism and antiferromagnetism since $\theta_{i-2}\theta_{i-1} = 1$ and $\theta_{i+1}\theta_{i+2} = -1$ and vice versa. In particular, in the last four cases frustration occurs because θ_i cannot simultaneously accomodate the opinions of both neighboring pairs. For example, if the pair $++$ appears on the left and the pair $+-$ appears on the right side, then the opinion bias from the left side would push θ_i towards $+$, while from the right side, it would push θ_i towards $-$. For those cases, we have introduced a parameter σ to bias the decision towards either the anti- or the ferromagnetic case. I.e., if $\sigma = 0$, opinion θ_i is completely biased by the anti-ferromagnetic neighbor pair, while for $\sigma = 1$ it is completely biased by the ferromagnetic neighbor pair. However, if $\sigma = 0.5$, opinion θ_i is equally balanced between ferromagnetism and antiferromagnetism spread.

Eq. (7) can be used as a lookup table for the microsimulations. But, in order to derive a generalized voter rule, we want to find a frequency dependent form for the transition rates in the form of eq. (6). This will be done in the second step, as follows. The local frequencies of the different opinions in the *nearest* neighborhood – $f_+^{(1)}$ – and in the *second nearest* neighborhood – $f_+^{(2)}$ –

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are defined as:

$$f_+^{(1)} = \frac{1}{2}(\delta_{+;\theta_{i-1}} + \delta_{+;\theta_{i+1}}) ; \quad f_+^{(2)} = \frac{1}{2}(\delta_{+;\theta_{i-2}} + \delta_{+;\theta_{i+2}}) \tag{8}$$

Using the local frequencies, we can reduce the lookup table to only nine different transition rates, as follows:

	$f_+^{(1)}$	$f_+^{(2)}$	$w(+|\theta_i)$	$w(-|\theta_i)$
1.	1	1	1	0
2.	0	0	0	1
3.	0.5	0.5	0.5	0.5
4.	1	0	0	1
5.	0	1	1	0
6.	0.5	0	0	1
7.	0.5	1	1	0
8.	1	0.5	σ	$1-\sigma$
9.	0	0.5	$1-\sigma$	σ

$$\tag{9}$$

The first three transition rates (1.-3.) apply in cases where θ_i has ferromagnetic pairs on *both* sides, while the three cases (3.-5.) apply if θ_i has anti-ferromagnetic pairs on *both* sides. The last four cases (6.-9.) apply if θ_i has one anti- and one ferromagnetic pair on each side. Again, cases 8. and 9. are special in the sense that *frustration* occurs because of the discrepancy between the opinion biases from the left and from the right side.

In the third step, we conclude the frequency dependence of the transition rates given in eq. (9) in a most concise equation:

$$w(+|\theta_i) = \kappa(f_+)\, f_+^{(2)} \quad ; \quad \kappa(f_+) = \begin{cases} 1 & \text{(no frustration)} \\ 2\sigma\, f_+^{(1)} + 2(1-\sigma)\,(1 - f_+^{(1)}) & \text{(frustration)} \end{cases}$$
$$\tag{10}$$
$$w(-|\theta_i) = \kappa(f_-)\, f_-^{(2)} \quad ; \quad \kappa(f_-) = \begin{cases} 1 & \text{(no frustration)} \\ 2\sigma\, f_-^{(1)} + 2(1-\sigma)\,(1 - f_-^{(1)}) & \text{(frustration)} \end{cases}$$

It turns out that the dynamics of the SM can be rewritten in terms of a VM where the frequency dependent transition rate basically depends on the frequencies of opinions of the *second nearest neighbors*, $f_+^{(2)}$, $f_-^{(2)}$. The opinion frequencies of the first nearest neighbors, $f_+^{(1)}$, $f_-^{(1)}$ only enter the prefactor κ, thus making the transition rate a non-linear voter rule.

We note that κ is different from 1 only in the case of *frustration*, in which the nearest neighbor frequencies $f_+^{(1)}$, $f_-^{(1)}$ can have only values of 0 or 1. However, for the special case of $\sigma = 0.5$, i.e. no bias towards either anti- or ferromagnetism, κ becomes 1 even in the case of frustration. Thus, we can reduce the dynamics of eq. (10) to a *linear* voter rule, valid for all cases:

$$w(+|\theta_i) = f_+^{(2)} ; \quad w(-|\theta_i) = f_-^{(2)} \quad \text{(for } \sigma = 0.5) \tag{11}$$

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This remarkable finding is based on theoretical investigations of the possible local configurations – but in Sect. 3.2 we will show by means of computer simulations that the linear rule of eq. (11) matches numerically with the dynamics of the SM.

To summarize, the only VM *transition rates that matter for the simulation of the SM*, are simply given by the three cases:

$$
\begin{array}{ccc}
f_+^{(2)} & w(+|\theta_i) & w(-|\theta_i) \\
\hline
1 & 1 & 0 \\
0 & 0 & 1 \\
0.5 & 0.5 & 0.5
\end{array}
\tag{12}
$$

The additional transition rates given in eq. (9) depend on σ and result from the possibility of considering frustration dynamics, which exceeds the original idea of the SM.

Finally, we emphasize that the importance of the second neighbors on the opinion dynamics is already a basic ingredient of the SM (even if not seen that way). Thus, the two rules of the SM can be simply combined into only *one* rule, namely *"just follow your second nearest neighbor"*. Specifically, using the notation of the SM, $\theta_{i-1} = \theta_{i+1}$ and $\theta_{i+2} = \theta_i$, no matter whether $\theta_i\theta_{i+1} = 1$ or $\theta_i\theta_{i+1} = -1$. To repeat this important finding, in SM the nearest neighbors of a site are just ignored in the dynamics.

3.2 Results of VM Computer Simulations

In this section, we compare the results of the VM dynamics with the known results of the SM. Therefore, we fix $\sigma = 0.5$ because only in this case the VM is equivalent to the SM, according to the previous section. The figures showing the spatio-temporal evolution of the lattice states and the respective magnetization shall be compared with the corresponding figures obtained from the SM. The basic setup chosen is the same, i.e. $N = 100$ and periodic boundary conditions for the lattice, initially uniform random distribution of the opinions, and asynchronous update rule.

As we can see in Fig. 5 and Fig. 6, the dynamics eventually reach either the coexistence or the consensus attractor, and even the intermediate coexistence of anti- and ferromagnetic domains can be observed as in the case of SM. Also, the power law of the distribution of decision times $P(\tau)$, Fig. 7(left), remains the same, as well as the distribution of relaxation times $P(\mu)$, Fig. 7(right), which follows the log-normal distribution. The only difference to be noticed is that the average relaxation time has now doubled in VM, compared to SM. This is to be expected since each update makes two flips in SM, while it makes only one flip in VM.

Thus, our microsimulations show that the proposed linear voter rule ($\sigma = 0.5$) is not different from the SM, both in terms of the dynamics and the final attractors. In the next section, we will show that this holds also for the frequency of reaching the attractors (50 percent stalemate, 25 percent up and 25 percent down).
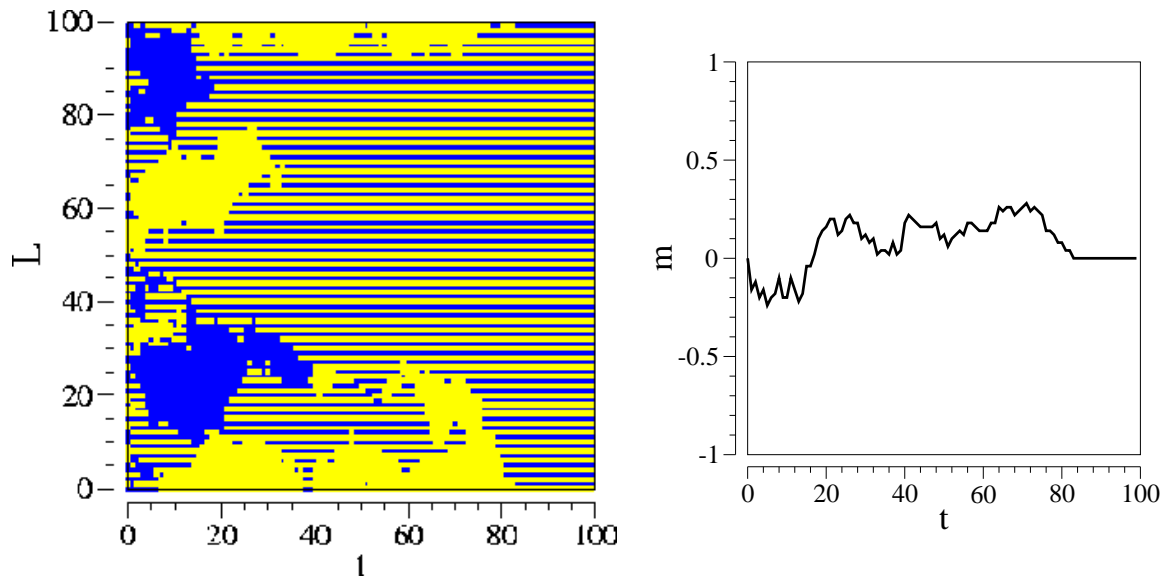
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Figure 5: Evolution of the one-dimensional lattice (left) and magnetization m (right) vs time in Monte-Carlo steps. Asymptotically, the *coexistence attractor* is reached. (Same setup as in Fig. 3, but dynamics according to the linear voter rule, eq. (11).)
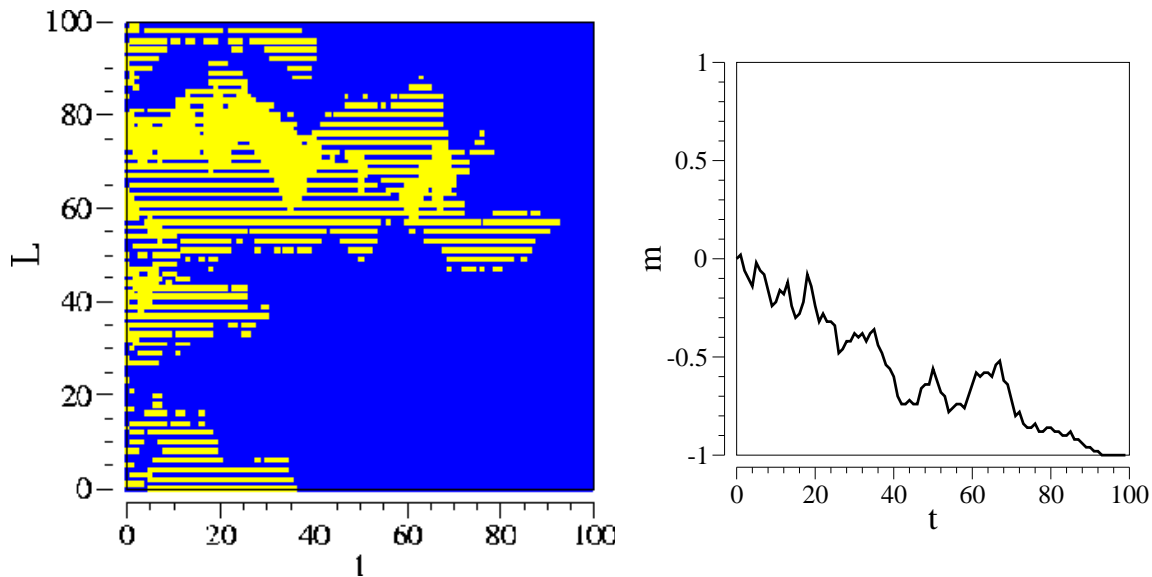

Figure 6: Evolution of the one-dimensional lattice (left) and magnetization m (right) vs time in Monte-Carlo steps. Asymptotically, the *concensus attractor* is reached. (Same setup as in Fig. 2, but dynamics according to the linear voter rule, eq. (11).)

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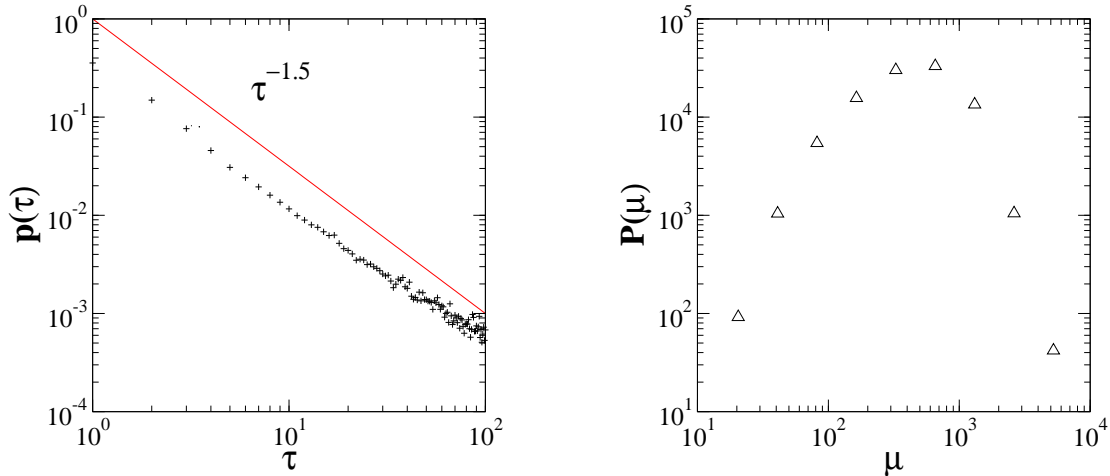accepted for publication in: *International Journal of Modern Physics C (2003)*

Figure 7: (Left) Distribution of decision times $P(\tau)$ averaged over 200 simulations, (right) distribution $P(\mu)$ of relaxation times μ averaged over 100.000 simulations (dynamics according to the linear voter rule, eq. (11).)

We conclude that it does not matter, whether "the influence flows inward from the surrounding neighbors to the center site, or spreads outwards in the opposite direction from the center to the neighbors" as argued in [10, 37], i.e. there are basically no principle differences between the SM and the VM except in the expression of the rules.

4   Influence of the Bias Parameter σ

When comparing SM and VM, we found that the decision dynamics for some local configurations is characterized by some sort of *frustration*, because of a conflict between the left and right opinion bias. In order to break the symmetry in those cases, we have introduced the bias parameter σ, which favors the anti-ferromagnetic response for $\sigma \to 0$ and the ferromagnetic response for $\sigma \to 1$. Only for the case of $\sigma = 0.5$, no bias is given – which is the case for expressing the SM in terms of a VM.

In this section, we want to pay more attention to the role of σ, which exceeds the original idea of the SM. Let us first look at the probability of reaching the different attractors. We recall from Sect. 2 that in SM three attractors exist, where the two ferromagnetic attractors are reached with probability 0.25 each, whereas the anti-ferromagnetic attractor is reached with probability 0.5. In Sect. 3.2, we have already shown that these attractors are also reached in the case of a

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linear voter model ($\sigma = 0.5$). To estimate the probability, we run 1.000 computer simulations with different values of σ (Table 1)

σ	0.00	0.25	0.45	0.50	0.55	0.75	1.00
f_{-+}	1	1	0.998	0.510	0.004	0	0
f_{++}	0	0	0.002	0.250	0.526	0.496	0.510
f_{--}	0	0	0	0.240	0.470	0.504	0.490

Table 1: Frequency of reaching the different attractors, *ferro*$_+$ (f_{++}), *ferro*$_-$ (f_{--}), and *anti-ferro* (f_{-+}) obtained from 1.000 simulations ($N = 100$, voter rules of eq. (10), asynchronous update).

For $\sigma = 0.5$, we observe that in the VM case the three attractors are reached with the same probability as in the SM case. Thus we can conclude that there are no differences between SM and VM with respect to this feature either.

Since $\sigma \to 0$ biases the dynamics towards the anti-ferromagnetic attractor while $\sigma \to 1$ biases towards the ferromagnetic one, the VM provides a simple possibility to avoid either stalemate or consensus in decision making. It is interesting to note that already *small deviations* from $\sigma = 0.5$ will lead to drastical changes in the probabilities of reaching the different attractors. I.e., already for $\sigma = 0.45$ or $\sigma = 0.55$ only one attractor is found (where the ferromagnetic one appears in two different "flavors").

The disappearance of one of the attractors basically results from a competition process between anti- and ferromagnetic domains. If a site is selected *within* a domain, nothing changes. The important events for the spatio-temporal evolution occur only at the borders *between* these domains, i.e. $\{+ + + + ? + - + -\}$ or $\{- - - - ? - + - +\}$, where frustration also occurs. Dependent on the value of σ, the following possibilities for the dynamics exist:

$$
+ + + + \; ? \; + - + - \Rightarrow
\begin{cases}
+ + + + - + - + - & \text{if } \sigma = 0.0 \\
+ + + + + + + - + - & \text{if } \sigma = 1.0 \\
+ + + + \; ? \; + - + - & \text{if } \sigma = 0.5
\end{cases}
\tag{13}
$$

That means that for $\sigma \to 0$ the anti-ferromagnetic domain will always increase at the cost of the ferromagnetic one, while for $\sigma \to 1$ the opposite will occur. Only for $\sigma = 0.5$, both cases occur with the same probability, i.e. in half of the cases the system may eventually reach (one of) the ferromagnetic attractors and in half of the cases the anti-ferromagnetic one. We note that this insight, how one domain may invade the other one, became clear only in the VM picture through investigation of the frustration dynamics (while it was not apparent in the SM view). Thus, we conclude that σ plays a crucial role in explaining the *phase transition* known in SM, from the initial random distribution to either antagonistic or consensus attractor.
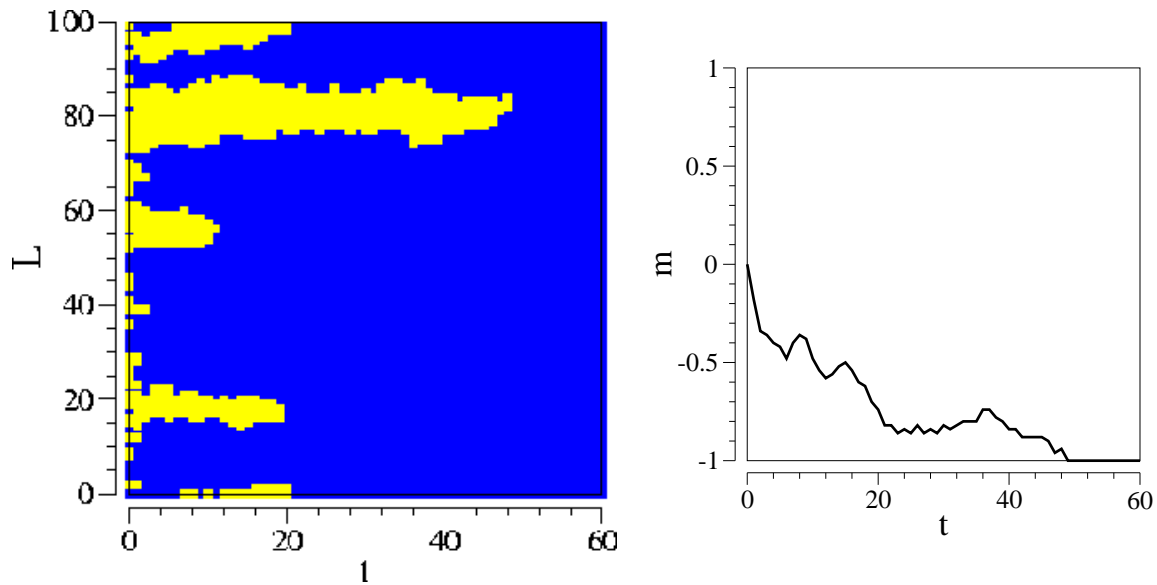
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Figure 8: Evolution of the one-dimensional lattice (left) and magnetization m (right) vs time in Monte-Carlo steps. Asymptotically, the *coexistence attractor* is reached. (Same setup as in Fig. 2, but dynamics according to the voter rules, eq. (9) with $\sigma = 1.0$)
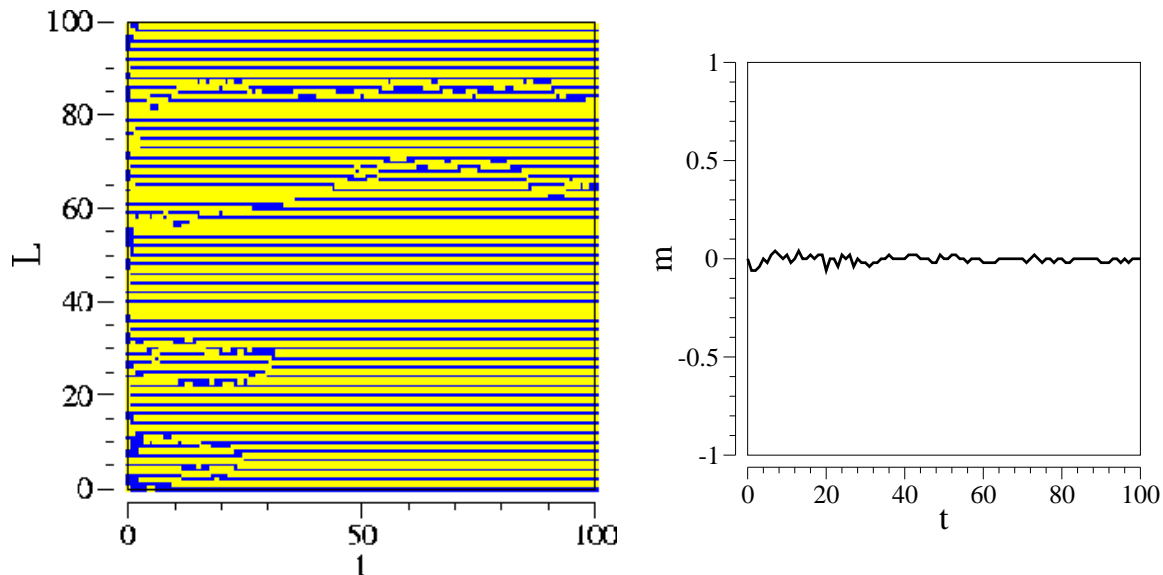

Figure 9: Evolution of the one-dimensional lattice (left) and magnetization m (right) vs time in Monte-Carlo steps. Asymptotically, the *coexistence attractor* is reached. (Same setup as in Fig. 3, but dynamics according to the voter rules, eq. (9) with $\sigma = 0.0$)

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Finally, we will have a look at the evolution of the spatio-temporal patterns for the two extreme cases, $\sigma \to 0$, Fig. 8, and $\sigma \to 1$, Fig. 9, and compare them with the case of $\sigma = 0.5$, Figs. 5, 6 where the VM dynamics are equivalent to the SM dynamics. As already noticed above, the asymptotic distributions are "preselected" by the choice of σ, so it is not surprising that either the coexistence or the consensus attractors are reached. However, looking at the intermediate dynamics, we realize that there are no anti-ferromagnetic domains ("striped patterns") for $\sigma = 1.0$, while there are no ferromagnetic domains ("filled patterns") for $\sigma = 0.0$. I.e., there is no coexistence between ferromagnetic and anti-ferromagnetic domains in the intermediate dynamics in the biased case, while it can be observed in the non-biased case.

5 Synchronous vs. Asynchronous Update

So far, we have used the so-called asynchronous (i.e. random sequential) update dynamics both for the SM and the VM, which means that at each time step one lattice site is randomly updated and changes are immediately processed to the neighborhood. VM, however, were first considered in a biological context, where time is measured in *generations* and changes of the lattice states become effective only *after* a generation is completed (i.e., usually after all sites are selected). The information generated will be thus processed in *parallel*, which is known as synchronous update.

In this section, we want to investigate whether the different update rules, i.e. the different ways of information processing, may affect the outcome of the SM/VM dynamics. Therefore, we have fixed $\sigma = 0.5$. First, we have a look again at the possible attractors of the dynamics, as shown in Table 2.

attractor	local configuration	f_+/f_-	frequency
1	$+ + + + + + + + + + ++$	$1/0$	0.056
2	$- - - - - - - - - - --$	$0/1$	0.060
3	$- + - + - + - + - + -+$	$0.5/0.5$	0.125
4	$+ + + - + + + - + + +-$	$0.75/0.25$	0.250
5	$- - - + - - - + - - -+$	$0.25/0.75$	0.252
6	$- - + + - - + + - - ++$	$0.5/0.5$	0.257

Table 2: Attractors of the VM dynamics and frequencies of reaching them (obtained from 10.000 simulations). f_+/f_- gives the frequencies of each opinion in the asymptotic configuration. ($N = 100$, voter rules of eq. (10), $\sigma = 0.5$, **synchronous** update).

Compared to the asynchronous VM/SM dynamics, we notice the appearance of three more

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attractors in the synchronous case. Two of these are different from the known ones in that they are characterized by a *asymmetric* coexistence of the two different opinions. I.e., we find stable configurations (in the absense of noise) with 75 percent of one of the opinions. The third new attractor is again a stalemate, or antagonistic attractor, but the local configuration is different from the known anti-ferromagnetic one, (*anti-ferro*), attractor 3 in Table 2.

Furthermore, from the frequencies with which the attractors are reached, Table 2, we note that the three new attractors are reached with the same probability of about 0.25, while the "old" attractors (1-3) alltogether only have a probability of 0.25. Again, within that share, the probability of the anti-ferromagentic attractor is equal to the probability of both the ferromagnetic ones, 0.125. But, given the absolute probability, pure anti- (3) and ferromagnetic configurations (1,2) become rare in synchronous update.
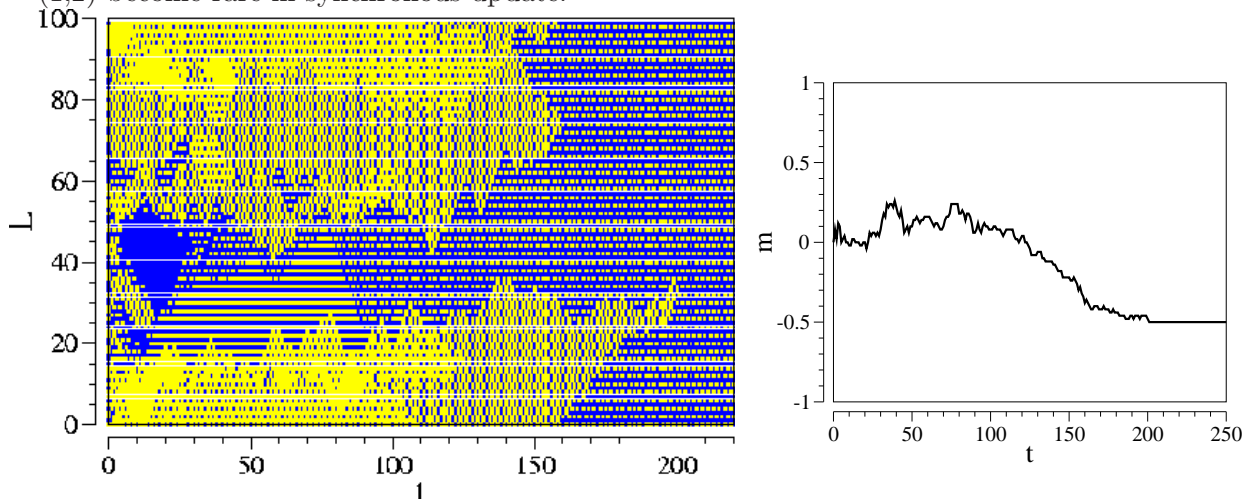

Figure 10: Evolution of the one-dimensional lattice (left) and magnetization m (right) vs time in generations. Asymptotically, an *asymmetric* coexistence attractor is reached. (Same setup as in Fig. 2, but *synchronous* update according to the voter rules, eq. (10), with $\sigma = 0.5$)

The influence of the synchronous update rule on the intermediate dynamics and the stationary distributions is shown in Fig. 10. We see that the spatio-temporal distribution now shows the intermediate coexistence of six different domains, characterized by the local configurations given in Table 2. Eventually, attractor 5, displaying the *asymmetric coexistence* of the opposite opinions, is reached, which can also be confirmed by looking at the magnetization $m(t)$.

Finally, we investigate how the synchronous update rule affects the distribution of decision times, $P(\tau)$. As shown in Fig. 11(left), for $\sigma = 0.5$ we do *not* find a power law for the synchronous case. This is due to the fact that during synchronous update, mostly (i.e. with probability 0.75) domains from the three new attractors (4-6) appear, even during the intermediate dynamics (cf. Fig. 10). Their local configurations, however, force the "individuals" to change their opinion during every time step. We can observe that the mean value of τ is about 0.93. However, if

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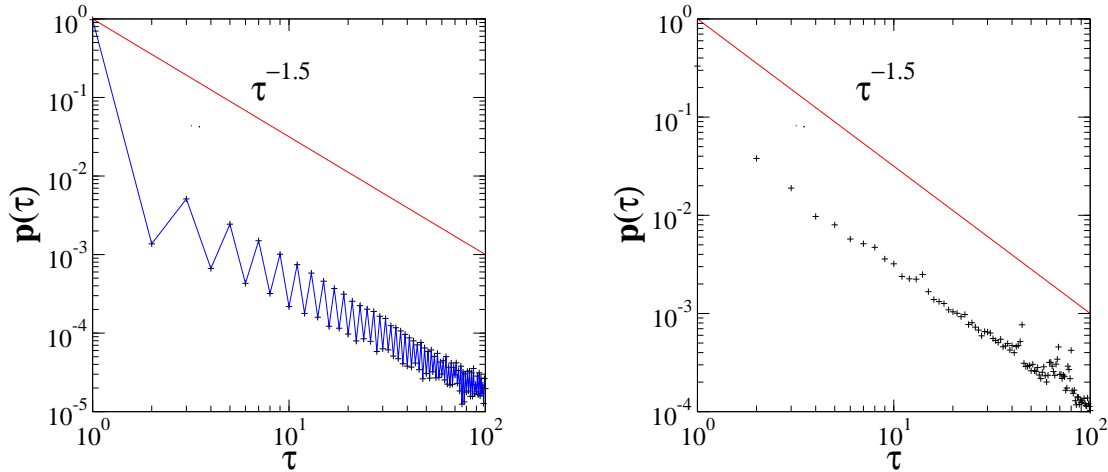accepted for publication in: *International Journal of Modern Physics C (2003)*

Figure 11: Distribution of decision times $P(\tau)$ averaged over 200 simulations: (left) $\sigma = 0.5$, (right) $\sigma in \{0; 1\}$. ($N = 100$, synchronous update according to the voter rules, eq. (10))

$\sigma \in \{0, 1\}$ then the three new attractors do not appear and the known power law $P(\tau) \propto \tau^{-1.5}$ can be recovered as in the asynchronous case, see Fig. 11(right).

6   Conclusions

In this paper we investigate similarities and differences between the previously established Sznajd model (SM) and the well known voter model (VM) in one dimension. It is shown that the SM can be completely reformulated in terms of a *linear* VM, where the transition rates towards a given opinion are directly proportional to the second-nearest neighborhood frequency of the respective opinion, eq. (11). The equivalence of the dynamics is demonstrated by extensive computer simulations that show the same behavior (i) for the spatial-temporal evolution of the lattice, $L(t)$, $m(t)$, (ii) for the power law distribution of decision times $P(\tau)$, (iii) for the log-normal distribution of relaxation times $P(\mu)$, and (iv) for the final attractor statistics.

We basically conclude that there are no differences between SM and VM with respect to these indicators. In particular, it does not matter whether the information flows from inward out (as in SM) or from outward in (as in VM). Also the fact that in SM dynamics *two* opinions are changed at the same time, while in VM only *one* opinion is changed, does not change the dynamic behavior, except that the average time scale of relaxation is doubled.

So, given that we can reduce the SM dynamics to linear VM dynamics, what are the advantages of such a reduction? *First*, we could reveal that in SM only the *second nearest neighbors* matter for the opinion dynamics, no matter what the nearest neighbors are. *Second*, in VM we could find a parameter σ that expands the original SM dynamics by considering the case of *frustration*. We further show that σ plays a crucial role in explaining the *phase transition* known in SM, from the initial random distribution to either antagonistic or consensus attractors. *Third*, since the SM is basically a linear VM, all the techniques developed for VM to describe the spatial structure formation, e.g. *pair approximations* of the spatial correlations or Markov chain analysis, can be adapted also for the analysis of the SM. This will be done in a forthcoming paper.

In this paper, we have also expanded the original SM dynamics by considering *synchronous* update rules. We show that this will lead to three additional attractors, which are reached with probability 0.75, while the original three attractors are reached only with probability 0.25. In the synchronous case, we find a *asymmetric coexistence* of the different opinions, i.e. the existence of a majority/minority different from 1/0, which is not found in the original SM.

Finally we address the issue of extending the proposed VM dynamics to a two-(and higher) dimensional CA. This extension has been done for the SM already in [37]. Also the proposed VM can be easily extended to two-dimensional problems, based on considerations e.g. in [19, 32]. We just have to adjust the second-nearest neighbor frequencies to the different neighborhood definitions (such as Von-Neumann neighborhood with eight, or Moore neighborhood with sixteen second-nearest neighbors). This shall be also done in a forthcoming paper.

Acknowledgement

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