# Systematic Low-Energy Effective Theory for Magnons and Charge Carriers in an Antiferromagnet

F. Kämpfer, M. Moser, and U.-J. Wiese

Institute for Theoretical Physics, Bern University Sidlerstrasse 5, CH-3012 Bern, Switzerland

September 3, 2018

#### Abstract

By electron or hole doping quantum antiferromagnets may turn into high-temperature superconductors. The low-energy dynamics of antiferromagnets are governed by their Nambu-Goldstone bosons — the magnons — and are described by an effective field theory analogous to chiral perturbation theory for the pions in strong interaction physics. In analogy to baryon chiral perturbation theory — the effective theory for pions and nucleons — we construct a systematic low-energy effective theory for magnons and electrons or holes in an antiferromagnet. The effective theory is universal and makes model-independent predictions for the entire class of antiferromagnetic cuprates. We present a detailed analysis of the symmetries of the Hubbard model and discuss how these symmetries manifest themselves in the effective theory. A complete set of linearly independent leading contributions to the effective action is constructed. The coupling to external electromagnetic fields is also investigated.

## 1 Introduction

Almost 20 years after the discovery of high-temperature superconductivity in layered cuprates [1], identifying the dynamical mechanism behind it remains one of the great challenges in condensed matter physics. Ordinary low-temperature superconductors are weakly coupled electron systems in which phonon exchange mediates an attractive interaction that can overcome the Coulomb repulsion between electrons. As massless Nambu-Goldstone bosons of the spontaneously broken translation symmetry, phonons provide a natural mechanism for Cooper pair formation at low energies which is successfully quantified in BCS theory. In contrast to ordinary superconductors, layered high- $T_c$  cuprates are systems of strongly correlated electrons to which the weak coupling BCS theory is not readily applicable. Furthermore, the high transition temperatures of cuprate superconductors and the smallness of the isotope effect suggest that mechanisms other than phonon exchange may be responsible for Cooper pair formation. Since high-temperature superconductors are antiferromagnets before doping, it is natural to suspect (but not generally accepted) that magnons — the Nambu-Goldstone bosons of the spontaneously broken  $SU(2)_s$  spin symmetry — may be important for binding electrons or holes into preformed pairs.

Even if spin fluctuations were not the key to explaining high-temperature superconductivity, the dynamics of charge carriers in an antiferromagnet is an interesting topic in itself. There is a vast literature on this subject. The dynamics of holes in an antiferromagnet has been investigated, for example, in [2–36]. Understanding the dynamics of even just a single hole propagating in an antiferromagnet is a challenging problem. One can gain qualitative insight from a picture in which holes hop from site to site, leaving a string of flipped spins behind and thus locally destroying the antiferromagnetic order. Since the string costs energy proportional to its length, one might expect the holes to even be confined and thus have infinite mass. However, the locally destroyed antiferromagnetic order may be healed by appropriate hole hopping which renders the hole mass finite [7]. Angle resolved photoemission spectroscopy experiments [37–40] as well as a number of theoretical investigations [6, 7, 17, 34] indicate that the minimum of the dispersion (i.e. of the energy) of a single hole corresponds to lattice momenta  $(\pm \frac{\pi}{2}, \pm \frac{\pi}{2})$  in the Brillouin zone.

As one adds a second hole, the situation becomes more controversial. For example, there seems to be no consensus on the question if a pair of holes can form a bound state or not. If it can, the condensation of such pairs would provide a potential mechanism for high-temperature superconductivity. The effective theory to be constructed here can be used to analytically calculate the long-range magnon-mediated forces between holes using perturbation theory. It is very interesting to ask what happens when one dopes an antiferromagnet with a non-zero density of holes. At sufficient doping, experiments show that high-temperature superconductivity may arise. It has been argued on theoretical grounds that even an infinitesimal amount of doping may affect the antiferromagnetic phase and turn it into a spiral phase [6]. A systematic investigation of this question is also possible using the effective theory of this paper, but it will require the use of non-perturbative methods.

The standard models for antiferromagnets and high-temperature superconductors are the Hubbard and t-J model. Since these models are strongly coupled, they are not accessible to a systematic analytic treatment. As a consequence, analytic calculations in Hubbard-type models usually involve some uncontrolled approximations. Unfortunately, due to a severe fermion sign problem, away from half-filling these models can currently also not be simulated reliably. Hence, although they may indeed contain the relevant physics, Hubbard-type models have not yet led to a quantitative understanding of high- $T_c$  materials. An alternative to a microscopic description using Hubbard-type models is provided by phenomenological models formulated directly in terms of magnon and electron or hole fields [11, 13, 25, 32, 36]. Although they may provide qualitative insight, such models do not lead to unambiguous predictions. In this paper, for the first time we introduce a systematic low-energy effective field theory for magnons and charge carriers in an antiferromagnet. Based only on symmetries and their spontaneous breakdown, the effective theory makes universal predictions for the entire class of antiferromagnetic cuprates. Although the effective theory is not renormalizable, it yields unambiguous results in a systematic low-energy expansion. In each order of the expansion, the results depend only on a finite number of material specific low-energy parameters whose values can be determined experimentally. The effective theory is not based on a specific microscopic model Hamiltonian but is universally applicable. Furthermore, and most important, in contrast to the strongly correlated electrons of Hubbardtype models, the electrons and holes of the effective field theory are quasi-particles that are weakly coupled to the magnons. Consequently, one may expect that the effective theory is more easily solvable than the underlying microscopic models.

Possible basic applications of the effective theory to be constructed in this paper include magnon-magnon, magnon-hole, and magnon-electron scattering as well as the determination of long-range magnon-mediated forces between the charge carriers. More ambitious applications could aim at a quantitative explanation of the Mott insulator state, the reduction of the staggered magnetization upon doping, the formation of a spiral phase, or at a systematic study of potential mechanisms for the preformation of electron or hole pairs in the antiferromagnetic phase. When such pairs condense they may become the Cooper pairs of high-temperature superconductivity. Except for a derivation of the dispersion relation of charge carriers, in this paper we do not consider applications yet, but concentrate entirely on the construction of the effective theory itself.

The construction in this paper is inspired by similar developments in the theory of the strong interactions. In contrast to the high- $T_c$  problem, where the choice of a microscopic model is controversial, there is general agreement that Quantum Chromodynamics (QCD) provides the correct microscopic description of the strong interactions. Still, similar to Hubbard-type models, solving QCD is notoriously hard. At "half-filling", i.e. in the filled quark Dirac sea that represents the QCD vacuum, the  $SU(2)_L \otimes SU(2)_R$  chiral symmetry of massless up and down quarks is spontaneously broken to the isospin symmetry  $SU(2)_{L=R}$ , resulting in three massless Nambu-Goldstone pions. This is analogous to the spontaneous breaking of the  $SU(2)_s$  spin symmetry down to  $U(1)_s$  that leads to antiferromagnetism. The corresponding Nambu-Goldstone bosons — in this case two magnons — are thus analogous to the pions of the strong interactions. It is possible to study chiral symmetry breaking in the QCD vacuum in numerical simulations of lattice QCD, just as it is possible to study antiferromagnetism by simulating the Hubbard model at halffilling. However, it is very useful to also investigate these phenomena with effective field theories. The low-energy effective theory for pions was pioneered by Weinberg [41] and formulated as a systematic expansion in Gasser's and Leutwyler's chiral perturbation theory [42]. Based on symmetry considerations and the observation that chiral symmetry is spontaneously broken, chiral perturbation theory makes rigorous predictions about the pion dynamics in terms of a few low-energy parameters such as the pion decay constant, the chiral condensate, and the Gasser-Leutwyler coefficients. Once these parameters are determined, either experimentally or through lattice QCD calculations, the effective theory makes unambiguous predictions in the low-energy domain.

Chiral perturbation theory can be applied to any Nambu-Goldstone phenomenon, and has indeed been used for both ferro- [43, 44] and antiferromagnetic magnons [45–50]. To lowest order, for antiferromagnetic magnons the low-energy parameters of chiral perturbation theory are the spin stiffness  $\rho_s$  and the spin-wave velocity c. At low energies chiral perturbation theory describes all aspects of the magnon dynamics just in terms of these two parameters. For example, the low-energy physics of the Hubbard model at half-filling is completely described by the effective theory once  $\rho_s$  and c have been determined in terms of the Hubbard model parameters tand U.

A numerical challenge in high- $T_c$  physics is to simulate the Hubbard model away from half-filling. This requires a solution of the corresponding fermion sign problem. Similarly, simulating lattice QCD at non-zero baryon chemical potential, i.e. after "doping" the QCD vacuum with quarks, is prevented by a severe complex action problem. Like for high- $T_c$  materials at sufficient doping, one expects that QCD at sufficiently high baryon density becomes a superconductor, in that case for the color charge carried by quarks and gluons [51]. In contrast to high-temperature superconductivity, the mechanism responsible for color-superconductivity is well understood in terms of one-gluon exchange. Color-superconductivity requires very large baryon densities and may thus arise only in the core of compact neutron or quark stars. However, superconductivity — not of color but of ordinary electric charge — is also known to exist at more moderate baryon densities. In particular, pairing of protons or neutrons inside large nuclei or neutron stars leads to superconductivity or superfluidity. Understanding the mechanism of nucleon pairing from the microscopic QCD theory may be as hard as understanding the mechanism for high-temperature superconductivity directly from the Hubbard model. Instead it is much more useful to employ a systematic low-energy effective theory whose parameters can be determined from the underlying microscopic physics. In nuclear physics effective field theory has recently led to some progress in describing the forces between nucleons in terms of just a few low-energy parameters [52–61], while phenomenological models involve a much larger number of adjustable parameters. Also steps towards describing nuclear matter with effective field theories have already been taken [62– 66]. The goal of the present paper is to develop a similar effective theory describing the interactions between the charge carriers in an antiferromagnet through magnon exchange. Remarkably, some physical phenomena that are practically inaccessible to microscopic Hubbard-type models even by numerical simulation can be tackled analytically in the effective field theory framework.

An ambitious goal of the effective theory approach is to systematically investigate possible mechanisms for the preformation of electron or hole pairs as a potential step towards understanding high-temperature superconductivity. It is an experimental fact that antiferromagnetism is destroyed before one enters the superconducting phase. How can magnon exchange then possibly provide a mechanism relevant for Cooper pair preformation? The destruction of antiferromagnetism just means the absence of infinite-range antiferromagnetic order. Antiferromagnetic correlations, although only of finite range, exist even in the superconducting phase. The finite correlation length implies that the magnons have developed a massgap, but they may still exist as relevant low-energy degrees of freedom. In particular, in 2 + 1dimensions, as a consequence of the Hohenberg-Mermin-Wagner-Coleman theorem. magnons pick up a mass that is exponentially small in the inverse temperature [67, 68]. The generation of the massgap is a non-perturbative phenomenon that is well within the applicability range of the effective theory, although infinite-range antiferromagnetic order exists only at zero temperature. Similarly, an effective theory for magnons and electrons or holes remains valid in the superconducting phase as long as the magnons remain among the lightest degrees of freedom. Again, this is similar to QCD where pions are not exactly massless either — in that case as a result of explicit chiral symmetry breaking due to non-zero quark masses. Although pions are hence only pseudo-Nambu-Goldstone bosons, chiral perturbation theory remains perfectly well applicable.

The low-energy effective theory for magnons and charge carriers to be developed here is the condensed matter analog of baryon chiral perturbation theory in strong interaction physics [69–73]. The effective theory is based on a non-linear realization of the spontaneously broken symmetry [74, 75]. The terms in the low-energy effective Lagrangian are organized according to the number of derivatives they contain. The lowest energy physics is dominated by the terms with the smallest number of derivatives, while effects at higher energies are taken into account systematically through higher-derivative terms. A key ingredient in constructing the effective Lagrangian are symmetry considerations. At a given order of the low-energy expansion, i.e. for a given number of derivatives, all terms consistent with the symmetries must be included in the effective Lagrangian, with a low-energy parameter that determines the strength of the corresponding interaction. For cuprates the most important symmetries are the  $SU(2)_s$  spin symmetry which is spontaneously broken down to  $U(1)_s$  in the antiferromagnetic phase, as well as the  $U(1)_Q$  fermion number symmetry whose breakdown signals superconductivity. Other relevant symmetries include translation by one lattice spacing which changes the sign of the staggered magnetization, 90 degrees rotations and reflections of the square crystal lattice, as well as time-reversal. In addition to these generic symmetries of high- $T_c$  materials, the Hubbard model possesses an  $SU(2)_Q$  symmetry discussed by Yang and Zhang [76, 77] which is a non-Abelian extension of the charge symmetry  $U(1)_Q$ . This symmetry is not expected to be present in generic cuprate materials, but may still be a relevant approximate symmetry in specific samples.

In this paper we ignore phonons, assuming that they do not play an important role for high-temperature superconductivity. For example, in the Hubbard model a rigid lattice which does not have its own physical degrees of freedom is put by hand. Of course, in the actual high- $T_c$  materials a crystal lattice arises as a result of the spontaneous breakdown of translation and Galilean (or more precisely Poincaré) invariance. The corresponding Nambu-Goldstone bosons are the phonons. The role of phonons and their possible interplay with magnons can also be investigated systematically in the framework of low-energy effective field theory.

We also consider the coupling of antiferromagnets to external electromagnetic fields which can be used to probe the dynamics of magnons and electrons or holes. As first noted by Fröhlich and Studer, in non-relativistic condensed matter external electromagnetic fields  $\vec{E}(x)$  and  $\vec{B}(x)$  enter the dynamics in the form of non-Abelian vector potentials for the  $SU(2)_s$  spin symmetry [78]. We use this observation to couple both the microscopic Hubbard model and the effective theory to external  $\vec{E}(x)$ and  $\vec{B}(x)$  fields. As discussed in detail in [79], the electromagnetic couplings are the condensed matter analog of the weak interactions in particle physics. These are described by an  $SU(2)_L \otimes U(1)_Y$  gauge theory, which turns part of QCD's global chiral symmetry into a gauge symmetry. Remarkably, the electromagnetic couplings of non-relativistic condensed matter are described by a local  $SU(2)_s \otimes U(1)_Q$  symmetry which is the condensed matter analog of the  $SU(2)_L \otimes U(1)_Y$  symmetry in particle physics. Some correspondences between QCD and antiferromagnets are summarized in table 1. Connections between QCD and condensed matter physics have also been discussed in [80].

The rest of this paper is organized as follows. Section 2 contains a symmetry analysis of the Hubbard model as a concrete example for an underlying microscopic system. In section 3 the effective theory for magnons is reviewed and the non-linear realization of the  $SU(2)_s$  spin symmetry is constructed. In section 4 the Hubbard model is coupled to a magnon background field. In this way the fields of the effective

	QCD	Antiferromagnets
broken phase	hadronic vacuum	antiferromagnetic phase
global symmetry	chiral symmetry	spin rotations
symmetry group $G$	$SU(2)_L \otimes SU(2)_R$	$SU(2)_s$
unbroken subgroup $H$	$SU(2)_{L=R}$	$U(1)_s$
Goldstone boson	pion	magnon
Goldstone field in $G/H$	$U(x) \in SU(2)$	$\vec{e}(x) \in S^2$
order parameter	chiral condensate	staggered magnetization
coupling strength	pion decay constant $F_{\pi}$	spin stiffness $\rho_s$
propagation speed	velocity of light	spin-wave velocity
conserved charge	baryon number $U(1)_B$	electric charge $U(1)_Q$
charged particle	nucleon or antinucleon	electron or hole
long-range force	pion exchange	magnon exchange
weak probes	electroweak fields	electromagnetic fields
local symmetry of	electroweak	local
weak probes	$SU(2)_L \otimes U(1)_Y$	$SU(2)_s \otimes U(1)_Q$
dense phase	nuclear or quark matter	high- $T_c$ superconductor
microscopic description	lattice QCD	Hubbard-type models
effective description	chiral perturbation	magnon effective
of Goldstone bosons	theory	theory
effective description	baryon chiral	effective theory
of charged fields	perturbation theory	presented here

Table 1: Correspondences between QCD and antiferromagnets.

theory inherit their transformation properties under the various symmetries from the underlying microscopic degrees of freedom. In section 5 the low-energy effective theory for magnons and charge carriers is developed and the leading terms in a systematic low-energy expansion of the effective action are constructed. This section also contains an application of the effective theory to the dispersion relations of charge carriers. Section 6 treats the t-J model and its effective theory as a special case of systems with holes as the only charge carriers. In section 7 the Hubbard model as well as its effective theory are coupled to external electromagnetic fields. Finally, section 8 contains our conclusions, while some technical details are discussed in two appendices.

## 2 Symmetries of the Hubbard Model

In order to have a concrete microscopic system for which we will then construct a low-energy effective theory, we consider the Hubbard model. The Hubbard model just serves as one representative of a large class of systems, including the actual high- $T_c$  materials. Here it is essential that the Hubbard model shares important symmetries, e.g. an  $SU(2)_s$  spin symmetry and a  $U(1)_Q$  fermion number symmetry with these materials. In the Hubbard model at half-filling the  $U(1)_Q$  symmetry even extends to an  $SU(2)_Q$  symmetry. The  $SU(2)_Q$  symmetry is not exact in actual materials, but may still be approximately realized and will also be investigated in the framework of the effective theory.

#### 2.1 Hamiltonian and Generic Continuous Symmetries

The Hubbard model is defined by the Hamiltonian

$$H = -t \sum_{x,i} (c_{x\uparrow}^{\dagger} c_{x+\hat{i}\uparrow} + c_{x+\hat{i}\uparrow}^{\dagger} c_{x\uparrow} + c_{x\downarrow}^{\dagger} c_{x+\hat{i}\downarrow} + c_{x+\hat{i}\downarrow}^{\dagger} c_{x\downarrow}) + U \sum_{x} c_{x\uparrow}^{\dagger} c_{x\uparrow} c_{x\downarrow}^{\dagger} c_{x\downarrow} - \mu' \sum_{x} (c_{x\uparrow}^{\dagger} c_{x\uparrow} + c_{x\downarrow}^{\dagger} c_{x\downarrow}).$$
(2.1)

Here x denotes the sites of a 2-dimensional square lattice and  $\hat{i}$  is a vector of length a (where a is the lattice spacing) pointing in the *i*-direction. Furthermore, t is the nearest-neighbor hopping parameter, while U > 0 is the strength of the screened onsite Coulomb repulsion, and  $\mu'$  is the chemical potential for fermion number. The fermion creation and annihilation operators obey the standard anticommutation relations

$$\{c_{xs}^{\dagger}, c_{ys'}\} = \delta_{xy}\delta_{ss'}, \quad \{c_{xs}, c_{ys'}\} = \{c_{xs}^{\dagger}, c_{ys'}^{\dagger}\} = 0.$$
(2.2)

We also introduce the  $SU(2)_s$  Pauli spinor

$$c_x = \begin{pmatrix} c_{x\uparrow} \\ c_{x\downarrow} \end{pmatrix}$$
(2.3)

in terms of which (up to an irrelevant constant) the Hamiltonian takes the manifestly  $SU(2)_s$ -invariant form

$$H = -t\sum_{x,i} (c_x^{\dagger} c_{x+\hat{i}} + c_{x+\hat{i}}^{\dagger} c_x) + \frac{U}{2} \sum_x (c_x^{\dagger} c_x - 1)^2 - \mu \sum_x (c_x^{\dagger} c_x - 1).$$
(2.4)

Here  $\mu = \mu' - \frac{1}{2}U$  is the chemical potential for the fermion number relative to halffilling, i.e.  $\mu = 0$  implies an average density of one fermion per lattice site. The corresponding  $U(1)_Q$  symmetry is generated by the charge operator

$$Q = \sum_{x} Q_x = \sum_{x} (c_x^{\dagger} c_x - 1).$$
(2.5)

Again, we count fermion number relative to half-filling. The  $SU(2)_s$  symmetry is generated by the total spin

$$\vec{S} = \sum_{x} \vec{S}_x = \sum_{x} c_x^{\dagger} \frac{\vec{\sigma}}{2} c_x, \qquad (2.6)$$

where  $\vec{\sigma}$  are the Pauli matrices. It is easy to see that the above Hamiltonian conserves both fermion number and spin, i.e.  $[H,Q] = [H,\vec{S}] = 0$ , and that  $[Q,\vec{S}] = 0$ . The infinitesimal generators  $\vec{S}$  of  $SU(2)_s$  (which obey the standard commutation relations  $[S_a, S_b] = i\varepsilon_{abc}S_c$ ) can be used to construct a unitary operator

$$V = \exp(i\vec{\eta} \cdot \vec{S}),\tag{2.7}$$

which implements the corresponding symmetry transformations in the Hilbert space of the theory. In particular, the transformed annihilation operators take the form

$$c'_x = V^{\dagger} c_x V = \exp(i\vec{\eta} \cdot \frac{\vec{\sigma}}{2}) c_x = g c_x, \quad g = \exp(i\vec{\eta} \cdot \frac{\vec{\sigma}}{2}) \in SU(2)_s.$$
(2.8)

Similarly, the  $U(1)_Q$  transformations are implemented by a unitary operator

$$W = \exp(i\omega Q), \tag{2.9}$$

such that

$${}^{Q}c_{x} = W^{\dagger}c_{x}W = \exp(i\omega)c_{x}, \quad \exp(i\omega) \in U(1)_{Q}.$$
(2.10)

For large positive U, at half-filling, the repulsive Hubbard model reduces to the antiferromagnetic spin  $\frac{1}{2}$  quantum Heisenberg model with the Hamiltonian

$$H = J \sum_{x,i} \vec{S}_x \cdot \vec{S}_{x+\hat{i}}, \qquad (2.11)$$

where the exchange coupling is given by  $J = 2t^2/U$ . This follows to second order of perturbation theory in t/U. To leading order, i.e. completely ignoring the kinetic term proportional to t, there is an enormous number of degenerate ground states. Irrespective of spin, any state with exactly one fermion occupying each lattice site avoids the on-site Coulomb repulsion and thus represents a ground state for t = 0. There is no correction at order t/U. In second order of degenerate perturbation theory, a spin can virtually hop to a neighboring site occupied by a fermion with opposite spin and then hop back. On the other hand, virtual hops to sites occupied by a fermion with the same spin orientation are forbidden by the Pauli principle. This favors antiparallel spins and leads to the antiferromagnetic Heisenberg model of eq.(2.11).

#### 2.2 Discrete Symmetries

Since the Hubbard model at half-filling leads to antiferromagnetism, another important symmetry is translation by one lattice spacing (in the *i*-direction), which flips the sign of the staggered magnetization vector

$$\vec{M}_s = \sum_x (-1)^x \vec{S}_x.$$
 (2.12)

The factor  $(-1)^x = (-1)^{(x_1+x_2)/a}$  distinguishes between the sites of the even and odd sublattice. The points on the even sublattice A have  $(-1)^x = 1$  while the points on the odd sublattice B have  $(-1)^x = -1$ . The displacement symmetry is generated by a unitary operator D which acts as

$${}^{D}c_{x} = D^{\dagger}c_{x}D = c_{x+\hat{i}},$$
(2.13)

and for which [H, D] = 0. Obviously, both the  $U(1)_Q$  and the  $SU(2)_s$  symmetry commute with the displacement, i.e.  $[Q, D] = [\vec{S}, D] = 0$ . In the effective theory it will be useful to also consider a related symmetry D' which combines D with the spin rotation  $g = i\sigma_2$ . This symmetry acts as

$${}^{D'}c_x = D'^{\dagger}c_x D' = (i\sigma_2) {}^{D}c_x = (i\sigma_2)c_{x+\hat{i}}.$$
(2.14)

Also note that [H, D'] = [D, D'] = [Q, D'] = 0, but  $[\vec{S}, D'] \neq 0$ .

In non-relativistic physics orbital angular momentum and spin are separately conserved and spin plays the role of an internal quantum number. Indeed, in the Hubbard model the  $SU(2)_s$  spin symmetry is completely independent of the 90 degrees rotation invariance of the spatial lattice. The 90 degrees rotation O acts on a spatial point  $x = (x_1, x_2)$  as  $Ox = (-x_2, x_1)$ . Under the symmetry O the fermion operators transform as

$${}^{O}c_x = O^{\dagger}c_x O = c_{Ox}. \tag{2.15}$$

Parity turns x into  $(-x_1, -x_2)$  and is equivalent to a 180 degrees rotation in two dimensions. Hence, it is more useful to consider the spatial reflection R at the  $x_1$ -axis which turns x into  $Rx = (x_1, -x_2)$ . Under this transformation the fermion operators transform as

$${}^{R}c_x = R^{\dagger}c_x R = c_{Rx}. \tag{2.16}$$

The reflection at the orthogonal  $x_2$ -axis is a combination of the reflection R and the rotation O. One can also consider the reflection at an axis half between lattice points. This transformation is a combination of R with the displacement symmetry D. Similarly, a reflection at a lattice diagonal is a combination of R and O. Another important symmetry is time-reversal which is implemented by an antiunitary operator T.

It should be pointed out that, unlike the actual high- $T_c$  materials, the Hubbard model is not Galilean invariant: in the actual materials translation as well as Galilean invariance are spontaneously broken by the formation of the crystal lattice. The corresponding Nambu-Goldstone bosons are the phonons which are known to play a central role in ordinary low- $T_c$  superconductivity. In the Hubbard model, on the other hand, the lattice is imposed by hand, and thus translation and Galilean invariance are explicitly broken. In particular, phonons cannot arise because the lattice does not have its own physical degrees of freedom.

### **2.3** $SU(2)_Q$ Symmetry

As first noted by Yang and Zhang [76, 77], at half-filling (i.e. for  $\mu = 0$ ) the Hubbard model possesses a non-Abelian extension  $SU(2)_Q$  of the fermion number symmetry  $U(1)_Q$  generated by

$$Q^{+} = \sum_{x} (-1)^{x} c_{x\uparrow}^{\dagger} c_{x\downarrow}^{\dagger}, \ Q^{-} = \sum_{x} (-1)^{x} c_{x\downarrow} c_{x\uparrow}, \ Q^{3} = \sum_{x} \frac{1}{2} (c_{x\uparrow}^{\dagger} c_{x\uparrow} + c_{x\downarrow}^{\dagger} c_{x\downarrow} - 1) = \frac{1}{2} Q.$$
(2.17)

Writing  $Q^{\pm} = Q^1 \pm iQ^2$ , it is straightforward to show that, for  $\mu = 0$ , indeed  $[H, \vec{Q}] = 0$ . Also the  $SU(2)_Q$  symmetry commutes with the  $SU(2)_s$  symmetry, i.e.  $[Q^a, S^b] = 0$ , but it does not commute with the displacement symmetry because  $D^{\dagger}Q^{\pm}D = -Q^{\pm}$ . For the same reason  $[\vec{Q}, D'] \neq 0$ .

Introducing the  $SU(2)_Q$  spinor

$$d_x = \begin{pmatrix} c_{x\uparrow} \\ (-1)^x c_{x\downarrow}^{\dagger} \end{pmatrix}, \qquad (2.18)$$

which obeys the standard anticommutation relations

$$\{d_{xa}^{\dagger}, d_{yb}\} = \delta_{xy}\delta_{ab}, \quad \{d_{xa}, d_{yb}\} = \{d_{xa}^{\dagger}, d_{yb}^{\dagger}\} = 0, \quad (2.19)$$

one writes

$$\vec{Q} = \sum_{x} \vec{Q}_x = \sum_{x} d_x^{\dagger} \frac{\vec{\sigma}}{2} d_x, \qquad (2.20)$$

where  $\vec{\sigma}$  are again the Pauli matrices now operating in  $SU(2)_Q$  space. The infinitesimal generators  $\vec{Q}$  of  $SU(2)_Q$  can be used to construct a unitary operator

$$W = \exp(i\vec{\omega} \cdot \vec{Q}), \qquad (2.21)$$

which implements the corresponding symmetry transformations in the Hilbert space of the theory. The transformed  $SU(2)_Q$  spinors are then given by

$$\vec{Q}_{d_x} = W^{\dagger} d_x W = \exp(i\vec{\omega} \cdot \frac{\vec{\sigma}}{2}) d_x = \Omega d_x, \quad \Omega = \exp(i\vec{\omega} \cdot \frac{\vec{\sigma}}{2}) \in SU(2)_Q.$$
(2.22)

In terms of the  $SU(2)_Q$  spinors the Hamiltonian takes the form

$$H = -t\sum_{x,i} (d_x^{\dagger} d_{x+\hat{i}} + d_{x+\hat{i}}^{\dagger} d_x) - \frac{U}{2} \sum_x (d_x^{\dagger} d_x - 1)^2 - \mu \sum_x d_x^{\dagger} \sigma_3 d_x.$$
(2.23)

The first two terms on the right-hand side are manifestly  $SU(2)_Q$ -invariant, while away from half-filling (i.e. for  $\mu \neq 0$ ) the chemical potential term explicitly breaks the  $SU(2)_Q$  symmetry down to  $U(1)_Q$ . Finally, we introduce a matrix-valued fermion operator

$$C_x = \begin{pmatrix} c_{x\uparrow} & (-1)^x c_{x\downarrow}^{\dagger} \\ c_{x\downarrow} & -(-1)^x c_{x\uparrow}^{\dagger} \end{pmatrix}, \qquad (2.24)$$

which displays both the  $SU(2)_s$  and the  $SU(2)_Q$  symmetries in a compact form. The first column of  $C_x$  is the  $SU(2)_s$  spinor  $c_x$ , while the second column is another  $SU(2)_s$  spinor which transforms exactly like  $c_x$ . The first row of  $C_x$  is the  $SU(2)_Q$ spinor  $d_x^T$ , while the second row is another  $SU(2)_Q$  spinor which transforms exactly like  $d_x^T$ . Under combined  $SU(2)_s$  and  $SU(2)_Q$  transformations  $C_x$  transforms as

$$\bar{Q}C'_x = gC_x\Omega^T. aga{2.25}$$

Since the  $SU(2)_s$  symmetry acts on the left while the  $SU(2)_Q$  symmetry acts on the right, it is now manifest that the two symmetry operations commute. Under the displacement symmetry one obtains

$${}^D C_x = C_{x+\hat{i}} \sigma_3. \tag{2.26}$$

The appearance of  $\sigma_3$  on the right is due to the factor  $(-1)^x$  and confirms that the displacement symmetry commutes with all  $SU(2)_s$  transformations, but only with the Abelian  $U(1)_Q$  (and not with all  $SU(2)_Q$ ) transformations. Similarly, under the symmetry D' one finds

$${}^{D'}C_x = (i\sigma_2)C_{x+\hat{i}}\sigma_3. \tag{2.27}$$

The Hamiltonian can now be expressed in a manifestly  $SU(2)_{s}$ -,  $U(1)_{Q}$ -, D-, and D'-invariant form

$$H = -\frac{t}{2} \sum_{x,i} \operatorname{Tr}[C_x^{\dagger} C_{x+\hat{i}} + C_{x+\hat{i}}^{\dagger} C_x] + \frac{U}{12} \sum_x \operatorname{Tr}[C_x^{\dagger} C_x C_x^{\dagger} C_x] - \frac{\mu}{2} \sum_x \operatorname{Tr}[C_x^{\dagger} C_x \sigma_3]. \quad (2.28)$$

The chemical potential term is only  $U(1)_Q$  invariant, while the other two terms are manifestly  $SU(2)_Q$ -invariant.

## **3** Effective Theory for Magnons

Before doping, the high- $T_c$  materials are quantum antiferromagnets in which the  $SU(2)_s$  spin symmetry is spontaneously broken down to  $U(1)_s$ . The low-energy physics of antiferromagnets is dominated by the corresponding Nambu-Goldstone bosons — the magnons. Chiral perturbation theory, which was originally developed for the Nambu-Goldstone pions of QCD, is a systematic low-energy expansion that has also been applied to magnons [43–50]. In this section we review the basic features of magnon chiral perturbation theory. As a necessary prerequisite for the coupling of magnons to charge carriers, we also construct the non-linear realization

of a spontaneously broken  $SU(2)_s$  symmetry, which then appears as a local  $U(1)_s$  symmetry in the unbroken subgroup. This is analogous to baryon chiral perturbation theory in which the spontaneously broken  $SU(2)_L \otimes SU(2)_R$  chiral symmetry of QCD is implemented on the nucleon fields as a local  $SU(2)_{L=R}$  transformation in the unbroken isospin subgroup.

#### 3.1 Continuous Symmetries of the Effective Action

The undoped precursors of high-temperature layered cuprate superconductors are quantum antiferromagnets. At half-filling, also the Hubbard model displays antiferromagnetism. In these systems, at least at zero temperature, the spin rotational symmetry  $G = SU(2)_s$  is spontaneously broken down to the subgroup  $H = U(1)_s$ by the formation of a staggered magnetization. The  $U(1)_Q$  symmetry, on the other hand, remains unbroken until one reaches the superconducting phase. In the Hubbard model even the  $SU(2)_Q$  symmetry remains unbroken at half-filling but is explicitly broken down to  $U(1)_Q$  for  $\mu \neq 0$ . As a consequence of Goldstone's theorem, there are two massless bosons — the antiferromagnetic spin-waves or magnons, which are described by a unit-vector field

$$\vec{e}(x) = (e_1(x), e_2(x), e_3(x)), \quad \vec{e}(x)^2 = 1$$
(3.1)

in the coset space  $G/H = SU(2)_s/U(1)_s = S^2$ . Here  $x = (x_1, x_2, t)$  denotes a point in Euclidean space-time. The vector  $\vec{e}(x)$  describes the direction of the local staggered magnetization. The leading order terms in the Euclidean action of the low-energy effective theory for the magnons take the form [67, 68]

$$S[\vec{e}] = \int d^2x \ dt \ \frac{\rho_s}{2} \left( \partial_i \vec{e} \cdot \partial_i \vec{e} + \frac{1}{c^2} \partial_t \vec{e} \cdot \partial_t \vec{e} \right). \tag{3.2}$$

The index  $i \in \{1,2\}$  labels the two spatial directions, while the index t refers to the Euclidean time-direction. The parameter  $\rho_s$  is the spin stiffness and c is the spin-wave velocity. For the antiferromagnetic Heisenberg model of eq.(2.11) these low-energy parameters have been determined very precisely in Monte Carlo calculations [81,82] resulting in  $\rho_s = 0.186(4)J$ , c = 1.68(1)Ja, where J is the exchange coupling of the Heisenberg model and a is the lattice spacing. The leading terms in the magnon effective action are "Poincaré"-invariant with the spin-wave velocity c playing the role of the velocity of light. Consequently, antiferromagnetic magnons have a "relativistic" spectrum. The "Poincaré" symmetry emerges only at low energies as a consequence of the discrete lattice rotation invariance. However, higher-derivative terms relevant at higher energies are in general not invariant.

In the following we prefer to work with an alternative representation of the magnon field using  $2 \times 2$  Hermitean projection matrices P(x) that obey

$$P(x)^{\dagger} = P(x), \quad \text{Tr}P(x) = 1, \quad P(x)^2 = P(x),$$
(3.3)

and are given by

$$P(x) = \frac{1}{2}(\mathbf{1} + \vec{e}(x) \cdot \vec{\sigma}) = \frac{1}{2} \begin{pmatrix} 1 + e_3(x) & e_1(x) - ie_2(x) \\ e_1(x) + ie_2(x) & 1 - e_3(x) \end{pmatrix}.$$
 (3.4)

In the above CP(1) language, the lowest-order effective action of eq.(3.2) takes the form

$$S[P] = \int d^2x \ dt \ \rho_s \text{Tr} \left[ \partial_i P \partial_i P + \frac{1}{c^2} \partial_t P \partial_t P \right]. \tag{3.5}$$

This action is invariant under the global transformations  $g \in SU(2)_s$  of eq.(2.8),

$$P(x)' = gP(x)g^{\dagger}.$$
(3.6)

Note that the magnon field P(x) is invariant under the charge symmetries  $U(1)_Q$ and  $SU(2)_Q$ , i.e.  $\vec{Q}P(x) = P(x)$ .

#### 3.2 Discrete Symmetries of Magnon Fields

Under the displacement D by one lattice spacing the staggered magnetization changes sign, i.e.

$${}^{D}\vec{e}(x) = -\vec{e}(x) \Rightarrow {}^{D}P(x) = \mathbb{1} - P(x).$$

$$(3.7)$$

Let us again combine D with the spin rotation  $g = i\sigma_2$ , which results in the transformation D' with

$${}^{D'}P(x) = (i\sigma_2) {}^{D}P(x)(i\sigma_2)^{\dagger} = (i\sigma_2)[\mathbf{1} - P(x)](i\sigma_2)^{\dagger} = P(x)^*, \qquad (3.8)$$

reminiscent of charge conjugation in particle physics.

The Hubbard model is invariant under translations by an integer multiple of the lattice spacing. As we have seen, due to the antiferromagnetic order, the displacement D by one lattice spacing (which connects the two sublattices A and B) plays a special role. In particular, in the effective theory it manifests itself as an internal symmetry that changes the sign of  $\vec{e}(x)$ . Translations by an even number of lattice spacings (which do not mix the sublattices), on the other hand, manifest themselves as ordinary translations in the effective theory. It should be noted that in the effective theory one need not distinguish between the displacement symmetries D for the two spatial directions, since they are related by an ordinary translation by two lattice spacings (one in the 1- and one in the 2-direction).

When we decompose a space-time vector  $x = (x_1, x_2, t)$  into its spatial and temporal components, the 90 degrees rotation O acts on x as  $Ox = (-x_2, x_1, t)$ . Under the symmetry O the magnon field transforms as

$$^{O}P(x) = P(Ox). ag{3.9}$$

Similarly, under the spatial reflection R at the  $x_1$ -axis, which turns x into  $Rx = (x_1, -x_2, t)$ , the magnon field transforms as

$${}^{R}P(x) = P(Rx). aga{3.10}$$

Had we not treated spin as an internal quantum number, it would also be directly affected by the spatial reflection. Since spin is a form of angular momentum, it transforms like the orbital angular momentum  $\vec{L} = \vec{r} \times \vec{p}$  of a particle, which is a pseudo-vector and thus changes into  ${}^{R}\vec{L} = (-L_1, L_2, -L_3)$  under the reflection R. This is equivalent to a 180 degrees  $SU(2)_s$  rotation around the 2-direction. Since we treat  $SU(2)_s$  as an exact internal symmetry, the pure spatial inversion R (without 180 degrees rotation of the spin) is also a symmetry.

Another important symmetry is time-reversal T which turns  $x = (x_1, x_2, t)$  into  $Tx = (x_1, x_2, -t)$ . In a Hamiltonian description time-reversal is represented by an antiunitary operator. Here we discuss time-reversal in the framework of the Euclidean path integral. Again, the spin transforms like the orbital angular momentum  $\vec{L}$  of a particle. The momentum  $\vec{p}$  changes sign under time-reversal and so does  $\vec{L}$ , i.e.  ${}^T\vec{L} = -\vec{L}.^1$  Consequently, under T the staggered magnetization vector (which is built from microscopic spins) transforms as

$${}^{T}\vec{e}(x) = -\vec{e}(Tx) \Rightarrow {}^{T}P(x) = \mathbf{1} - P(Tx) = {}^{D}P(Tx).$$
 (3.11)

Hence, time-reversal is closely related to the displacement symmetry of eq.(3.7). Just like the displacement symmetry D, time-reversal is spontaneously broken in an antiferromagnet. However, in contrast to a ferromagnet, the combination TD of time-reversal and the displacement symmetry remains unbroken. Previously we have combined the displacement symmetry D with the  $SU(2)_s$  spin rotation  $i\sigma_2$  in order to obtain the unbroken symmetry D'. In order to obtain an unbroken variant T' of time-reversal we now combine T with the spin rotation  $i\sigma_2$  which yields

$${}^{T'}P(x) = (i\sigma_2) {}^{T}P(x)(i\sigma_2)^{\dagger} = (i\sigma_2) {}^{D}P(Tx)(i\sigma_2)^{\dagger} = {}^{D'}P(Tx).$$
(3.12)

#### **3.3** Non-Linear Realization of the $SU(2)_s$ Symmetry

In order to couple electron or hole fields to the magnons one must construct a non-linear realization of the spontaneously broken  $SU(2)_s$  symmetry which then manifests itself as a local symmetry in the unbroken  $U(1)_s$  subgroup of  $SU(2)_s$ . This local transformation is constructed from the global transformation  $g \in SU(2)_s$ as well as from the local magnon field P(x) as follows: one first diagonalizes the

<sup>&</sup>lt;sup>1</sup>Note that  $-\vec{L}$  does not obey the angular momentum commutation relations. This is a consequence of the antiunitary nature of T which does not represent an ordinary symmetry (implemented by a unitary transformation) in Hilbert space.

magnon field by a unitary transformation  $u(x) \in SU(2)_s$ , i.e.

$$u(x)P(x)u(x)^{\dagger} = \frac{1}{2}(\mathbb{1} + \sigma_3) = \begin{pmatrix} 1 & 0\\ 0 & 0 \end{pmatrix}, \quad u_{11}(x) \ge 0.$$
(3.13)

Note that, due to its projector properties, P(x) has eigenvalues 0 and 1. In order to make u(x) uniquely defined, we demand that the element  $u_{11}(x)$  is real and nonnegative. Otherwise the diagonalizing matrix u(x) would be defined only up to a  $U(1)_s$  phase. Using eq.(3.4) and spherical coordinates for  $\vec{e}(x)$ , i.e.

$$\vec{e}(x) = (\sin\theta(x)\cos\varphi(x), \sin\theta(x)\sin\varphi(x), \cos\theta(x)), \qquad (3.14)$$

one obtains

$$u(x) = \frac{1}{\sqrt{2(1+e_3(x))}} \begin{pmatrix} 1+e_3(x) & e_1(x) - ie_2(x) \\ -e_1(x) - ie_2(x) & 1+e_3(x) \end{pmatrix}$$
  
=  $\begin{pmatrix} \cos(\frac{1}{2}\theta(x)) & \sin(\frac{1}{2}\theta(x))\exp(-i\varphi(x)) \\ -\sin(\frac{1}{2}\theta(x))\exp(i\varphi(x)) & \cos(\frac{1}{2}\theta(x)) \end{pmatrix}$ . (3.15)

Under a global  $SU(2)_s$  transformation g the diagonalizing field u(x) transforms as

$$u(x)' = h(x)u(x)g^{\dagger}, \quad u_{11}(x)' \ge 0,$$
(3.16)

which implicitly defines the non-linear symmetry transformation

$$h(x) = \exp(i\alpha(x)\sigma_3) = \begin{pmatrix} \exp(i\alpha(x)) & 0\\ 0 & \exp(-i\alpha(x)) \end{pmatrix} \in U(1)_s.$$
(3.17)

The transformation h(x) is uniquely defined since we demand that  $u_{11}(x)'$  is again real and non-negative. Note that with this definition of h(x) indeed

$$u(x)'P(x)'u(x)'^{\dagger} = \frac{1}{2}(\mathbf{1} + \sigma_3).$$
(3.18)

Interestingly, the global  $SU(2)_s$  transformation g manifests itself in the form of a local transformation  $h(x) \in U(1)_s$  which inherits its x-dependence from the magnon field P(x).

We still need to show that the  $SU(2)_s$  group structure  $g = g_2g_1$  is inherited by the non-linear  $U(1)_s$  realization, i.e.  $h(x) = h_2(x)h_1(x)$ . First, we perform the global  $SU(2)_s$  transformation  $g_1$ , i.e.

$$P(x)' = g_1 P(x) g_1^{\dagger}, \quad u(x)' = h_1(x) u(x) g_1^{\dagger}, \tag{3.19}$$

which defines the non-linear realization  $h_1(x)$ . Then we perform the subsequent global transformation  $g_2$  which defines the non-linear realization  $h_2(x)$ , i.e.

$$P(x)'' = g_2 P(x)' g_2^{\dagger} = g_2 g_1 P(x) (g_2 g_1)^{\dagger} = g P(x) g^{\dagger},$$
  

$$u(x)'' = h_2(x) u(x)' g_2^{\dagger} = h_2(x) h_1(x) u(x) (g_2 g_1)^{\dagger} = h(x) u(x) g^{\dagger}.$$
 (3.20)

This indeed implies the correct group structure  $h(x) = h_2(x)h_1(x)$ .

Under the displacement symmetry D the sign-change of the staggered magnetization  $\vec{e}(x)$  implies

$${}^{D}u(x) = \frac{1}{\sqrt{2(1-e_{3}(x))}} \begin{pmatrix} 1-e_{3}(x) & -e_{1}(x)+ie_{2}(x) \\ e_{1}(x)+ie_{2}(x) & 1-e_{3}(x) \end{pmatrix}$$
  
$$= \begin{pmatrix} \sin(\frac{1}{2}\theta(x)) & -\cos(\frac{1}{2}\theta(x))\exp(-i\varphi(x)) \\ \cos(\frac{1}{2}\theta(x))\exp(i\varphi(x)) & \sin(\frac{1}{2}\theta(x)) \end{pmatrix}$$
  
$$= \tau(x)u(x), \qquad (3.21)$$

where

$$\tau(x) = \frac{1}{\sqrt{e_1(x)^2 + e_2(x)^2}} \begin{pmatrix} 0 & -e_1(x) + ie_2(x) \\ e_1(x) + ie_2(x) & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -\exp(-i\varphi(x)) \\ \exp(i\varphi(x)) & 0 \end{pmatrix}.$$
(3.22)

Note that  ${}^{D}\tau(x) = -\tau(x) = \tau(x)^{\dagger}$ , such that

$$^{DD}u(x) = {}^{D}\tau(x) {}^{D}u(x) = \tau(x)^{\dagger}\tau(x)u(x) = u(x),$$
 (3.23)

as one would expect for the displacement symmetry. It should also be noted that — like the  $SU(2)_s$  symmetry — the displacement symmetry is also spontaneously broken and hence realized in a non-linear (i.e. magnon-field-dependent) manner. Similarly, under the displacement symmetry D' one finds  ${}^{D'}u(x) = h(x) {}^{D}u(x)g^{\dagger}$ with  $g = i\sigma_2$ . For this particular g the local transformation takes the form  $h(x) = (i\sigma_2)\tau(x)^{\dagger}$ , such that  ${}^{D'}u(x) = u(x)^*$ (2.24)

$$D'u(x) = u(x)^*.$$
 (3.24)

In contrast to the displacement symmetry D, the symmetry D' is not spontaneously broken and is thus realized in a linear (i.e. magnon-field-independent) manner.

In the next step we consider the anti-Hermitean field

$$v_{\mu}(x) = u(x)\partial_{\mu}u(x)^{\dagger}, \qquad (3.25)$$

which transforms under  $SU(2)_s$  as

$$v_{\mu}(x)' = h(x)u(x)g^{\dagger}\partial_{\mu}[gu(x)^{\dagger}h(x)^{\dagger}] = h(x)[v_{\mu}(x) + \partial_{\mu}]h(x)^{\dagger}.$$
 (3.26)

Writing

$$v_{\mu}(x) = iv_{\mu}^{a}(x)\sigma_{a} = i \begin{pmatrix} v_{\mu}^{3}(x) & v_{\mu}^{+}(x) \\ v_{\mu}^{-}(x) & -v_{\mu}^{3}(x) \end{pmatrix}, \quad v_{\mu}^{\pm}(x) = v_{\mu}^{1}(x) \mp iv_{\mu}^{2}(x)$$
(3.27)

and using eq.(3.17) this implies

$$v^{3}_{\mu}(x)' = v^{3}_{\mu}(x) - \partial_{\mu}\alpha(x), \quad v^{\pm}_{\mu}(x)' = \exp(\pm 2i\alpha(x))v^{\pm}_{\mu}(x).$$
 (3.28)

Hence,  $v^3_{\mu}(x)$  transforms like an Abelian gauge field for  $U(1)_s$ , while  $v^{\pm}_{\mu}(x)$  represent vector fields "charged" under  $U(1)_s$ . For later convenience we also introduce the Hermitean charged vector field

$$V_{\mu}(x) = v_{\mu}^{1}(x)\sigma_{1} + v_{\mu}^{2}(x)\sigma_{2} = v_{\mu}^{+}(x)\sigma_{+} + v_{\mu}^{-}(x)\sigma_{-} = \begin{pmatrix} 0 & v_{\mu}^{+}(x) \\ v_{\mu}^{-}(x) & 0 \end{pmatrix}, \quad (3.29)$$

where  $\sigma_{\pm} = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$  are raising and lowering operators of spin. Under the  $SU(2)_s$  symmetry the charged vector field transforms as

$$V_{\mu}(x)' = h(x)V_{\mu}(x)h(x)^{\dagger}.$$
(3.30)

The magnon action can also be written as

$$S[v_{\mu}] = \int d^2x \ dt \ 2\rho_s \left( v_i^+ v_i^- + \frac{1}{c^2} v_t^+ v_t^- \right) = \int d^2x \ dt \ \rho_s \operatorname{Tr} \left[ V_i^{\dagger} V_i + \frac{1}{c^2} V_t^{\dagger} V_t \right].$$
(3.31)

It should be pointed out that the fields  $v^a_{\mu}(x)$  do not represent independent degrees of freedom, but are composed of magnon fields. In particular, what looks like a mass term for a charged vector field is indeed just the kinetic term of a massless Nambu-Goldstone boson.

#### 3.4 Discrete Symmetries of Composite Fields

Under the displacement symmetry D the composite vector field transforms as

$${}^{D}v_{\mu}(x) = \tau(x)[v_{\mu}(x) + \partial_{\mu}]\tau(x)^{\dagger} \Rightarrow {}^{D}v_{\mu}^{3}(x) = -v_{\mu}^{3}(x) + \partial_{\mu}\varphi(x),$$
  
$${}^{D}v_{\mu}^{\pm}(x) = -\exp(\mp 2i\varphi(x))v_{\mu}^{\mp}(x), \quad {}^{D}V_{\mu}(x) = \tau(x)V_{\mu}(x)\tau(x)^{\dagger}.$$
 (3.32)

Similarly, under the symmetry D' one finds

$${}^{D'}v_{\mu}(x) = v_{\mu}(x)^{*} \Rightarrow {}^{D'}v_{\mu}^{3}(x) = -v_{\mu}^{3}(x),$$
  
$${}^{D'}v_{\mu}^{\pm}(x) = -v_{\mu}^{\mp}(x), \quad {}^{D'}V_{\mu}(x) = -V_{\mu}(x)^{*}.$$
 (3.33)

This is exactly how an ordinary non-Abelian gauge field behaves under charge conjugation in particle physics.

Under the 90 degrees spatial rotation O the composite field  $v_{\mu}(x)$  transforms as

$${}^{O}v_i(x) = \varepsilon_{ij}v_j(Ox), \quad {}^{O}v_t(x) = v_t(Ox), \tag{3.34}$$

while under the reflection R one obtains

$${}^{R}v_{1}(x) = v_{1}(Rx), \quad {}^{R}v_{2}(x) = -v_{2}(Rx), \quad {}^{R}v_{t}(x) = v_{t}(Rx).$$
 (3.35)

Finally, under the time-reversal symmetry T the field  $v_{\mu}$  transforms as

$${}^{T}v_{i}(x) = {}^{D}v_{i}(Tx), {}^{T}v_{t}(x) = - {}^{D}v_{t}(Tx) \Rightarrow$$
  
$${}^{T}v_{i}^{3}(x) = -v_{i}^{3}(Tx) + \partial_{i}\varphi(Tx), {}^{T}v_{t}^{3}(x) = v_{t}^{3}(Tx) - \partial_{t}\varphi(Tx),$$
  
$${}^{T}v_{i}^{\pm}(x) = -\exp(\mp 2i\varphi(Tx))v_{i}^{\mp}(Tx), {}^{T}v_{t}^{\pm}(x) = \exp(\mp 2i\varphi(Tx))v_{t}^{\mp}(Tx),$$
  
$${}^{T}V_{i}(x) = \tau(Tx)V_{i}(Tx)\tau(Tx)^{\dagger}, {}^{T}V_{t}(x) = -\tau(Tx)V_{t}(Tx)\tau(Tx)^{\dagger},$$
(3.36)

and under its unbroken variant T' one finds

Note that the upper index T on the right denotes transpose, while on the left it denotes time-reversal. The above relations are equivalent to time-reversal of an ordinary non-Abelian gauge field.

#### 3.5 Alternative Representation of Magnon Fields

We have used two equivalent representations of the magnon field in terms of the unitvector  $\vec{e}(x)$  and in terms of the projection matrix P(x). There is a third equivalent representation in terms of a complex doublet

$$z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}, \quad z(x)^{\dagger} = (z_1(x)^*, z_2(x)^*),$$
  
$$z(x)^{\dagger} z(x) = |z_1(x)|^2 + |z_2(x)|^2 = 1,$$
 (3.38)

which is related to the other two representations by

$$\vec{e}(x) = z(x)^{\dagger} \vec{\sigma} z(x) \Rightarrow$$

$$e_{1}(x) = z_{1}(x)^{*} z_{2}(x) + z_{2}(x)^{*} z_{1}(x),$$

$$e_{2}(x) = i[z_{2}(x)^{*} z_{1}(x) - z_{1}(x)^{*} z_{2}(x)],$$

$$e_{3}(x) = |z_{1}(x)|^{2} - |z_{2}(x)|^{2},$$

$$P(x) = z(x)z(x)^{\dagger} = \begin{pmatrix} |z_{1}(x)|^{2} & z_{1}(x)z_{2}(x)^{*} \\ z_{2}(x)z_{1}(x)^{*} & |z_{2}(x)|^{2} \end{pmatrix}.$$
(3.39)

The field z(x) is defined in terms of  $\vec{e}(x)$  or P(x) only up to a  $U(1)_s$  gauge transformation

$$z(x)' = \exp(i\beta(x))z(x). \tag{3.40}$$

It is therefore necessary to also introduce the auxiliary real-valued  $U(1)_s$  gauge field

$$a_{\mu}(x) = \frac{1}{2i} [z(x)^{\dagger} \partial_{\mu} z(x) - \partial_{\mu} z(x)^{\dagger} z(x)], \qquad (3.41)$$

which under the symmetry of eq.(3.40) transforms as

$$a_{\mu}(x)' = a_{\mu}(x) + \partial_{\mu}\beta(x). \tag{3.42}$$

The complex doublet z(x) is closely related to the field u(x). Fixing the gauge freedom of eq.(3.40) such that  $z_1(x)$  is real and non-negative, it is easy to show that

$$u(x) = \begin{pmatrix} z_1(x) & z_2(x)^* \\ -z_2(x) & z_1(x) \end{pmatrix}, \quad v^3_\mu(x) = a_\mu(x).$$
(3.43)

Hence, the description in terms of complex doublets z(x) and an additional auxiliary gauge field  $a_{\mu}(x)$  is physically equivalent to what we described before. It should again be pointed out that  $a_{\mu}(x)$  (or equivalently  $v_{\mu}^{3}(x)$ ) does not represent a dynamical Abelian gauge field, but is simply a composite field constructed from the underlying magnon field P(x).

#### 3.6 Baby-Skyrmions

It is interesting to note that magnon fields support topological solitons known as baby-Skyrmions — a lower-dimensional variant of the Skyrme soliton which represents a baryon in the low-energy pion effective theory for QCD [83]. Baby-Skyrmions are solitons whose topological charge

$$B = \frac{1}{8\pi} \int d^2x \ \varepsilon_{ij} \vec{e} \cdot (\partial_i \vec{e} \times \partial_j \vec{e}), \qquad (3.44)$$

defined at every instant in time, is an element of the homotopy group  $\Pi_2[S^2] = \mathbb{Z}$ . The corresponding topological current

$$j_{\mu}(x) = \frac{1}{8\pi} \varepsilon_{\mu\nu\rho} \vec{e}(x) \cdot [\partial_{\nu} \vec{e}(x) \times \partial_{\rho} \vec{e}(x)]$$
(3.45)

is conserved, i.e.  $\partial_{\mu} j_{\mu} = 0$ , independent of the equations of motion. Baby-Skyrmions are massive excitations inaccessible to the systematic low-energy expansion of chiral perturbation theory. Still, the existence of the conserved current  $j_{\mu}(x)$  may have physical consequences even for the pure magnon dynamics.

Under the various symmetries the topological current transforms as

$$SU(2)_{s}: \quad j_{\mu}(x)' = j_{\mu}(x),$$
  

$$SU(2)_{Q}: \quad {}^{\vec{Q}}j_{\mu}(x) = j_{\mu}(x),$$
  

$$D: \quad {}^{D}j_{\mu}(x) = -j_{\mu}(x),$$
  

$$D': \quad {}^{D'}j_{\mu}(x) = -j_{\mu}(x),$$
  

$$O: \quad {}^{O}j_{t}(x) = j_{t}(Ox), \quad {}^{O}j_{i}(x) = \varepsilon_{ij}j_{j}(Ox),$$
  

$$R: \quad {}^{R}j_{t}(x) = -j_{t}(Rx), \quad {}^{R}j_{1}(x) = -j_{1}(Rx), \quad {}^{R}j_{2}(x) = j_{2}(Rx),$$
  

$$T: \quad {}^{T}j_{t}(x) = -j_{t}(Tx), \quad {}^{T}j_{1}(x) = j_{1}(Tx), \quad {}^{T}j_{2}(x) = j_{2}(Rx),$$
  

$$T': \quad {}^{T'}j_{t}(x) = -j_{t}(Tx), \quad {}^{T'}j_{1}(x) = j_{1}(Tx), \quad {}^{T'}j_{2}(x) = j_{2}(Tx). \quad (3.46)$$

One might be tempted to add a term  $j_{\mu}(x)v_{\mu}^{3}(x)$  to the magnon Lagrangian because this is how an Abelian gauge field couples to a conserved current. Indeed, this term is invariant under  $SU(2)_{s}$ ,  $SU(2)_{Q}$ , D, D', and O. However, it violates the reflection and time-reversal symmetries R, T, and T' and is hence forbidden.

There is another non-trivial homotopy group,  $\Pi_3[S^2] = \mathbb{Z}$ , which is relevant for baby-Skyrmions. It implies that space-time-dependent magnon fields fall into distinct topological classes characterized by the Hopf number  $H[\vec{e}] \in \Pi_3[S^2] = \mathbb{Z}$ . In 2 + 1 dimensions baby-Skyrmions can be quantized as anyons characterized by a statistics angle  $\theta$  [84]. The cases  $\theta = 0$  and  $\theta = \pi$  correspond to bosons and fermions, respectively. Including the Hopf term, the magnon path integral takes the form

$$Z = \int \mathcal{D}\vec{e} \, \exp(-S[\vec{e}]) \exp(i\theta H[\vec{e}]).$$
(3.47)

The Hopf term also changes sign under R, T, and T'. Hence,  $\exp(i\theta H[\vec{e}])$  is invariant only if  $\theta$  is 0 or  $\pi$ . Consequently, in an antiferromagnet with exact R, T, or T'symmetries baby-Skyrmions can only be quantized as bosons or fermions. For the antiferromagnetic quantum Heisenberg model it has been argued that no Hopf term is generated [85–89]. Hence, in that case the baby-Skyrmions should be bosons.

## 4 The Hubbard Model in a Magnon Background Field

The half-filled ground state of the Hubbard model plays a similar role as the Dirac sea in a relativistic quantum field theory. In particular, any fermion added to a half-filled state will be denoted as an electron, while any fermion removed from such a state represents a hole. In this section we couple a background magnon field to the microscopic degrees of freedom of the Hubbard model. In this way composite operators are constructed which transform exactly like the fields of the effective theory. Hence, the effective fields inherit their transformation properties under symmetry operations from the Hubbard model degrees of freedom.

#### 4.1 Fermion Operators in a Magnon Background Field

In order to analyze the transformation properties of the electron and hole fields, as an intermediate step between the microscopic and effective descriptions, we first add a continuum magnon background field P(x) to the Hubbard model by hand. The corresponding diagonalizing unitary matrix field u(x) is used to turn the matrixvalued Hubbard model operator  $C_x$  of eq.(2.24) into new operators  $\Psi_x^A$  and  $\Psi_x^B$  defined on the even and odd sublattices, respectively

$$\Psi_x^A = u(x)C_x = u(x) \begin{pmatrix} c_{x\uparrow} & c_{x\downarrow}^{\dagger} \\ c_{x\downarrow} & -c_{x\uparrow}^{\dagger} \end{pmatrix} = \begin{pmatrix} \psi_{x+}^A & \psi_{x-}^{A\dagger} \\ \psi_{x-}^A & -\psi_{x+}^{A\dagger} \end{pmatrix}, \quad x \in A,$$
$$\Psi_x^B = u(x)C_x = u(x) \begin{pmatrix} c_{x\uparrow} & -c_{x\downarrow}^{\dagger} \\ c_{x\downarrow} & c_{x\uparrow}^{\dagger} \end{pmatrix} = \begin{pmatrix} \psi_{x+}^B & -\psi_{x-}^{B\dagger} \\ \psi_{x-}^B & \psi_{x+}^{B\dagger} \end{pmatrix}, \quad x \in B. \quad (4.1)$$

In order to achieve a consistent representation of the underlying antiferromagnetic structure, it is unavoidable to explicitly split the degrees of freedom according to their location on sublattice A or B. In this context it may be interesting to consider the electron-hole representation of the Hubbard model operators discussed in appendix A. The operators  $\psi_{x\pm}^{A,B}$  obey standard anticommutation relations. It should be noted that here the continuum field u(x) is evaluated only at discrete lattice points x.

The new lattice operators inherit their transformation properties from the operators of the Hubbard model. According to eqs.(3.16) and (2.8), under the  $SU(2)_s$ symmetry one obtains

$$\Psi_x^{A,B'} = u(x)'C'_x = h(x)u(x)g^{\dagger}gC_x = h(x)\Psi_x^{A,B}.$$
(4.2)

In components this relation takes the form

$$\psi_{x\pm}^{A,B'} = \exp(\pm i\alpha(x))\psi_{x\pm}^{A,B}.$$
 (4.3)

The components  $\psi_{x\pm}^{A,B}$  do not simply correspond to spin up and spin down with respect to an arbitrarily chosen global quantization axis. Instead they correspond to spin parallel (+) or antiparallel (-) to the local staggered magnetization. This follows from considering global symmetry transformations  $g \in U(1)_s$  in the unbroken subgroup of  $SU(2)_s$  which describe rotations around the spontaneously selected direction of the staggered magnetization vector. In that case, according to eq.(3.16), h(x) = g becomes a global transformation as well and eq.(4.3) shows that  $\psi_{x\pm}^{A,B}$  indeed has spin parallel or antiparallel to the direction of the staggered magnetization.

Similarly, under the  $SU(2)_Q$  symmetry one obtains

$${}^{\vec{Q}}\Psi_x^{A,B} = {}^{\vec{Q}}u(x){}^{\vec{Q}}C_x = u(x)C_x\Omega^T = \Psi_x^{A,B}\Omega^T.$$
 (4.4)

In particular, under the  $U(1)_Q$  subgroup of  $SU(2)_Q$  the components transform as

$${}^{Q}\psi^{A,B}_{x\pm} = \exp(i\omega)\psi^{A,B}_{x\pm}.$$
(4.5)

Under the displacement symmetry the new operators transform as

$${}^{D}\Psi_{x}^{A,B} = {}^{D}u(x+\hat{i})C_{x+\hat{i}}\sigma_{3} = \tau(x+\hat{i})u(x+\hat{i})C_{x+\hat{i}}\sigma_{3} = \tau(x+\hat{i})\Psi_{x+\hat{i}}^{B,A}\sigma_{3}, \quad (4.6)$$

where  $\tau(x)$  is the field introduced in eq.(3.21). Expressed in components this implies

$${}^{D}\psi^{A,B}_{x\pm} = \mp \exp(\mp i\varphi(x+\hat{i}))\psi^{B,A}_{x+\hat{i},\mp}.$$
(4.7)

Similarly, under the symmetry D' one finds

$${}^{D'}\Psi_x^{A,B} = {}^{D'}u(x+\hat{i})(i\sigma_2)C_{x+\hat{i}}\sigma_3 = u(x+\hat{i})^*(i\sigma_2)C_{x+\hat{i}}\sigma_3 = (i\sigma_2)\Psi_{x+\hat{i}}^{B,A}\sigma_3.$$
(4.8)

Here we have used  $u(x + \hat{i})^*(i\sigma_2) = (i\sigma_2)u(x + \hat{i})$ . Again, expressed in components this relation takes the form

$${}^{D'}\psi^{A,B}_{x\pm} = \pm \psi^{B,A}_{x+\hat{i},\mp}.$$
(4.9)

We have seen before that the symmetry D' acts on the composite field  $v_{\mu}(x)$  exactly like charge conjugation in particle physics. However, it should be noted that D' acts on the electron and hole fields in a different way than the usual charge conjugation of a relativistic Dirac fermion which interchanges electrons and positrons. In particular, D' does not interchange electrons and holes. Instead, it flips the spin of both electrons and holes from + to - and vice versa. Indeed, the spin is the "charge" that couples to the composite gauge field of eq.(3.25) constructed from the magnon field.

In the condensed matter literature on high-temperature superconductivity the concept of spin-charge separation (whose existence is established for some systems in one spatial dimension) has often been invoked. The idea is that there may be quasi-particles — so-called holons — which carry charge but no spin, as well as so-called spinons which are neutral and carry spin  $\frac{1}{2}$ . In order to avoid confusion between holons and the holes of our effective theory, we like to make a few comments: one might think that the fermion operator  $\Psi_x^{A,B}$  does not carry spin since it does not transform with the global spin transformation  $g \in SU(2)_s$ . However, the spin symmetry is non-linearly realized and hence the fermion operator transforms with the local  $h(x) \in U(1)_s$ . Consequently,  $\Psi_x^{A,B}$  still carries spin and hence does not represent a holon. It should also be pointed out that in the weakly coupled effective theory of magnons and holes there are no linearly confining forces that could form a spinless holon out of  $\Psi_x^{A,B}$  and the magnon field z(x) of eq.(3.38).

### 4.2 Formal Continuum Limit of the Hubbard Model in a Magnon Background Field

In terms of the new operators the Hubbard model Hamiltonian takes the form

$$H = -\frac{t}{2} \sum_{x \in A, i} \operatorname{Tr}[\Psi_x^{A\dagger} \mathcal{V}_{x,i} \Psi_{x+\hat{i}}^B + \Psi_{x+\hat{i}}^{B\dagger} \mathcal{V}_{x,i}^{\dagger} \Psi_x^A] - \frac{t}{2} \sum_{x \in B, i} \operatorname{Tr}[\Psi_x^{B\dagger} \mathcal{V}_{x,i} \Psi_{x+\hat{i}}^A + \Psi_{x+\hat{i}}^{A\dagger} \mathcal{V}_{x,i}^{\dagger} \Psi_x^B]$$

$$+\frac{U}{12}\sum_{x\in A}\operatorname{Tr}[\Psi_{x}^{A\dagger}\Psi_{x}^{A}\Psi_{x}^{A\dagger}\Psi_{x}^{A}] + \frac{U}{12}\sum_{x\in B}\operatorname{Tr}[\Psi_{x}^{B\dagger}\Psi_{x}^{B}\Psi_{x}^{B\dagger}\Psi_{x}^{B}] \\ -\frac{\mu}{2}\sum_{x\in A}\operatorname{Tr}[\Psi_{x}^{A\dagger}\Psi_{x}^{A}\sigma_{3}] - \frac{\mu}{2}\sum_{x\in B}\operatorname{Tr}[\Psi_{x}^{B\dagger}\Psi_{x}^{B}\sigma_{3}], \qquad (4.10)$$

where we have introduced the parallel transporter

$$\mathcal{V}_{x,i} = u(x)u(x+\hat{i})^{\dagger} \in SU(2)_s, \tag{4.11}$$

which transforms under  $SU(2)_s$  as

$$\mathcal{V}'_{x,i} = h(x)\mathcal{V}_{x,i}h(x+\hat{i})^{\dagger}.$$
(4.12)

For smooth magnon fields we can put

$$u(x) = u(x + \frac{\hat{i}}{2}) - \frac{a}{2}\partial_i u(x + \frac{\hat{i}}{2}) + \frac{a^2}{8}\partial_i^2 u(x + \frac{\hat{i}}{2}) + \mathcal{O}(a^3),$$
  
$$u(x + \hat{i}) = u(x + \frac{\hat{i}}{2}) + \frac{a}{2}\partial_i u(x + \frac{\hat{i}}{2}) + \frac{a^2}{8}\partial_i^2 u(x + \frac{\hat{i}}{2}) + \mathcal{O}(a^3), \quad (4.13)$$

where a is the lattice spacing. Similar expressions hold for the other fields. Using the unitarity of  $u(x + \frac{i}{2})$  one can show that the lattice parallel transporter reduces to

$$\mathcal{V}_{x,i} = \mathbf{1} + av_i(x + \frac{\hat{i}}{2}) + \frac{a^2}{2}v_i(x + \frac{\hat{i}}{2})^2 + \mathcal{O}(a^3), \qquad (4.14)$$

with  $v_i(x)$  given by eq.(3.25). Note that both the continuum field  $v_i(x)$  and the lattice parallel transporter field  $\mathcal{V}_{x,i}$  transform locally only with the unbroken  $U(1)_s$  subgroup and not with the full  $SU(2)_s$  symmetry.

In the continuum limit we make the replacements

$$\sum_{x \in A}, \sum_{x \in B} \rightarrow \frac{1}{2a^2} \int d^2 x, \quad \Psi_x^{A,B} \rightarrow \sqrt{2}a \Psi^{A,B}(x).$$

$$(4.15)$$

The factor  $\frac{1}{2}$  in front of the integral accounts for the fact that each sublattice covers only half of the space. Similarly the factor  $\sqrt{2}a$  in the definition of the continuum field  $\Psi^{A,B}(x)$  arises because there is only one degree of freedom of a given type A or B per area  $2a^2$ . The components  $\psi^{A,B}_{\pm}(x)$  of  $\Psi^{A,B}(x)$  again obey standard anticommutation relations, however, with the Dirac  $\delta$ -function of the continuum theory instead of the Kronecker  $\delta$ -function of the lattice. It should be noted that, due to the antiferromagnetic order, the number of degrees of freedom per continuum point is twice as large as the number per lattice point. Taking the formal continuum limit  $a \to 0$  (and ignoring an irrelevant constant) the Hamiltonian of eq.(4.10) takes the form

$$H = \int d^{2}x \left\{ M \operatorname{Tr}[\Psi^{A\dagger}\Psi^{B}] + \frac{1}{2M'} \operatorname{Tr}[D_{i}\Psi^{A\dagger}D_{i}\Psi^{B}] + iK \operatorname{Tr}[D_{i}\Psi^{A\dagger}V_{i}\Psi^{B} + D_{i}\Psi^{B\dagger}V_{i}\Psi^{A}] + N \operatorname{Tr}[\Psi^{A\dagger}V_{i}V_{i}\Psi^{B}] + \frac{G}{12} \operatorname{Tr}[\Psi^{A\dagger}\Psi^{A}\Psi^{A\dagger}\Psi^{A} + \Psi^{B\dagger}\Psi^{B}\Psi^{B\dagger}\Psi^{B}] - \frac{\mu}{2} \operatorname{Tr}[\Psi^{A\dagger}\Psi^{A}\sigma_{3} + \Psi^{B\dagger}\Psi^{B}\sigma_{3}] \right\}.$$

$$(4.16)$$

It should be noted that, due to the structure of  $\Psi^{A,B}(x)$ , the individual terms are Hermitean. In the above expression  $V_{\mu}(x)$  is the field defined in eq.(3.29) and the covariant derivatives are given by

$$D_{\mu}\Psi^{A,B}(x) = (\partial_{\mu} + iv_{\mu}^{3}(x)\sigma_{3})\Psi^{A,B}(x),$$
  
$$D_{\mu}\Psi^{A,B\dagger}(x) = [D_{\mu}\Psi^{A,B}(x)]^{\dagger} = \partial_{\mu}\Psi^{A,B\dagger}(x) - \Psi^{A,B\dagger}(x)iv_{\mu}^{3}(x)\sigma_{3}.$$
 (4.17)

In terms of the fundamental parameters t and U and the lattice spacing a of the Hubbard model one obtains

$$M = -4t, \quad M' = \frac{1}{2ta^2}, \quad K = ta^2, \quad N = ta^2, \quad G = 2Ua^2.$$
 (4.18)

It should be noted that (in contrast to a relativistic theory) the kinetic mass M' is in general different from the rest mass M. The Hamiltonian from above resembles some (but not all) terms in the action of the effective theory to be constructed below. However, the coupling constants resulting from the formal continuum limit get renormalized and will hence be replaced by a priori unknown low-energy parameters in the effective action. The values of the low-energy parameters can be determined in experiments with cuprate materials or through numerical simulations of a microscopic Hubbard-type model.

## 5 Effective Theory for Magnons and Charge Carriers

The low-energy effective theory for magnons is analogous to chiral perturbation theory for pions in QCD. In QCD the baryon number B is a conserved quantity. Thus one can investigate the low-energy QCD dynamics separately in each baryon number sector. Ordinary chiral perturbation theory operates in the B = 0 sector. The low-energy physics in the B = 1 sector involves a single nucleon interacting with soft pions. The low-energy effective theory describing these dynamics is known as baryon chiral perturbation theory [69–73]. Similar effective theories have been constructed for the B = 2 [52, 53] and B = 3 sectors [55, 60] in the context of nuclear physics. Even nuclear matter (i.e. a system with non-zero baryon density) has been studied with effective theories [62–66]. The condensed matter analog of baryon number is electron (or hole) number (or equivalently electric charge) which is obviously also conserved. In analogy to QCD it is hence possible to construct a low-energy effective theory describing the interactions of soft magnons with charge carriers doped into an antiferromagnet. Most high- $T_c$  materials result by hole-doping of quantum antiferromagnets, but the effective theory also applies to electron-doping. The key observation is that the spontaneously broken  $SU(2)_s$  spin symmetry is non-linearly realized on the electron or hole fields and appears as a local  $U(1)_s$  symmetry in the unbroken subgroup. This is analogous to baryon chiral perturbation theory in which

the spontaneously broken  $SU(2)_L \otimes SU(2)_R$  chiral symmetry of QCD is implemented on the nucleon fields as a local  $SU(2)_{L=R}$  transformation in the unbroken isospin subgroup.

#### 5.1 Effective Fields for Charge Carriers

In the low-energy effective theory we will use a Euclidean path integral description instead of the Hamiltonian description used in the Hubbard model. Consequently, the Hermitean conjugate lattice operators  $\psi_{x\pm}^{A,B\dagger}$  are then replaced by Grassmann numbers  $\psi_{\pm}^{A,B\dagger}(x)$  which are completely independent of  $\psi_{\pm}^{A,B}(x)$ . Therefore, in the effective theory the electron and hole fields are represented by eight independent Grassmann numbers  $\psi_{\pm}^{A,B}(x)$  and  $\psi_{\pm}^{A,B\dagger}(x)$  which can be combined to

$$\Psi^{A}(x) = \begin{pmatrix} \psi^{A}_{+}(x) & \psi^{A\dagger}_{-}(x) \\ \psi^{A}_{-}(x) & -\psi^{A\dagger}_{+}(x) \end{pmatrix}, \quad \Psi^{B}(x) = \begin{pmatrix} \psi^{B}_{+}(x) & -\psi^{B\dagger}_{-}(x) \\ \psi^{B}_{-}(x) & \psi^{B\dagger}_{+}(x) \end{pmatrix}.$$
(5.1)

In order to avoid confusion with relativistic theories, we do not denote the conjugate fields by  $\overline{\psi}_{\pm}^{A,B}(x)$ . For notational convenience we also introduce the fields

$$\Psi^{A\dagger}(x) = \begin{pmatrix} \psi^{A\dagger}_{+}(x) & \psi^{A\dagger}_{-}(x) \\ \psi^{A}_{-}(x) & -\psi^{A}_{+}(x) \end{pmatrix}, \quad \Psi^{B\dagger}(x) = \begin{pmatrix} \psi^{B\dagger}_{+}(x) & \psi^{B\dagger}_{-}(x) \\ -\psi^{B}_{-}(x) & \psi^{B}_{+}(x) \end{pmatrix}.$$
(5.2)

It should be noted that  $\Psi^{A,B\dagger}(x)$  is not an independent field, but consists of the same Grassmann fields  $\psi^{A,B}_{\pm}(x)$  and  $\psi^{A,B\dagger}_{\pm}(x)$  as  $\Psi^{A,B}(x)$ .

It should be pointed out that, since they emerge dynamically, the continuum fields of the low-energy effective theory can not be derived explicitly from the lattice operators of the microscopic Hubbard model. Still, the Grassmann fields  $\Psi^{A,B}(x)$  describing electrons and holes in the low-energy effective theory transform just like the lattice operators  $\Psi^{A,B}_x$  discussed before. In contrast to the lattice operators, the fields  $\Psi^{A,B}(x)$  are defined in the continuum. Hence, under the displacement symmetries D and D' one no longer distinguishes between the points x and  $x + \hat{i}$ . As a result, the transformation rules of the various symmetries take the form

$$SU(2)_{s}: \quad \Psi^{A,B}(x)' = h(x)\Psi^{A,B}(x),$$
  

$$SU(2)_{Q}: \quad {}^{\vec{Q}}\Psi^{A,B}(x) = \Psi^{A,B}(x)\Omega^{T},$$
  

$$D: \quad {}^{D}\Psi^{A,B}(x) = \tau(x)\Psi^{B,A}(x)\sigma_{3},$$
  

$$D': \quad {}^{D'}\Psi^{A,B}(x) = (i\sigma_{2})\Psi^{B,A}(x)\sigma_{3}.$$
(5.3)

In components the symmetry transformations read

$$SU(2)_{s}: \qquad \psi_{\pm}^{A,B}(x)' = \exp(\pm i\alpha(x))\psi_{\pm}^{A,B}(x), U(1)_{Q}: \qquad {}^{Q}\psi_{\pm}^{A,B}(x) = \exp(i\omega)\psi_{\pm}^{A,B}(x), D: \qquad {}^{D}\psi_{\pm}^{A,B}(x) = \mp \exp(\mp i\varphi(x))\psi_{\mp}^{B,A}(x), D': \qquad {}^{D'}\psi_{\pm}^{A,B}(x) = \pm \psi_{\mp}^{B,A}(x).$$
(5.4)

Under the space-time symmetries, i.e. under the 90 degrees rotation O, the reflection R, time-reversal T, and its unbroken variant T' the fermion fields transform as

$$O: \quad {}^{O}\Psi^{A,B}(x) = \Psi^{A,B}(Ox), R: \quad {}^{R}\Psi^{A,B}(x) = \Psi^{A,B}(Rx), T: \quad {}^{T}\Psi^{A,B}(x) = \tau(Tx)(i\sigma_{2})\Psi^{A,B\dagger}(Tx)^{T}\sigma_{3}, \quad {}^{T}\Psi^{A,B\dagger}(x) = -\sigma_{3}\Psi^{A,B}(Tx)^{T}(i\sigma_{2})^{\dagger}\tau(Tx)^{\dagger}, T': \quad {}^{T'}\Psi^{A,B\dagger}(x) = -\Psi^{A,B\dagger}(Tx)^{T}\sigma_{3}, \quad {}^{T'}\Psi^{A,B\dagger}(x) = \sigma_{3}\Psi^{A,B}(Tx)^{T}.$$
(5.5)

Again an upper index T on the right denotes transpose, while on the left it denotes time-reversal. The form of the time-reversal symmetry T in the effective theory with non-linearly realized  $SU(2)_s$  symmetry follows from the usual form of timereversal in the Euclidean path integral of a non-relativistic theory in which the spin symmetry is linearly realized. The fermion fields in the two formulations just differ by a factor u(x). In components the previous relations take the form

$$O: \quad {}^{O}\psi_{\pm}^{A,B}(x) = \psi_{\pm}^{A,B}(Ox), R: \quad {}^{R}\psi_{\pm}^{A,B}(x) = \psi_{\pm}^{A,B}(Rx), T: \quad {}^{T}\psi_{\pm}^{A,B}(x) = \exp(\mp i\varphi(Tx))\psi_{\pm}^{A,B\dagger}(Tx), \quad {}^{T}\psi_{\pm}^{A,B\dagger}(x) = -\exp(\pm i\varphi(Tx))\psi_{\pm}^{A,B}(Tx), T': \quad {}^{T'}\psi_{\pm}^{A,B}(x) = -\psi_{\pm}^{A,B\dagger}(Tx), \quad {}^{T'}\psi_{\pm}^{A,B\dagger}(x) = \psi_{\pm}^{A,B}(Tx).$$
(5.6)

It should be noted that the components + and - (denoting spin parallel and antiparallel to the direction of the staggered magnetization) are not interchanged under time-reversal. While both the spin of the fermion and the staggered magnetization change sign under time-reversal, the projection of one onto the other does not.

The action to be constructed in the next section must be invariant under the internal symmetries  $SU(2)_s$ ,  $U(1)_Q$  (or even  $SU(2)_Q$ ), D and D', as well as under space-time translations and the other space-time symmetries O, R, and T (or equivalently T').

The fundamental forces underlying condensed matter physics are Poincaré-invariant. However, some of the space-time symmetries may be spontaneously broken by the formation of a crystal lattice. The resulting Nambu-Goldstone bosons are the phonons, which play a central role in ordinary low-temperature superconductors by providing the force that binds Cooper pairs. In high- $T_c$  superconductors, on the other hand, it is expected that phonons alone cannot provide the mechanism for Cooper pair formation. In the Hubbard model (and also in our effective theory) phonons are explicitly excluded because one imposes a rigid lattice by hand. This does not only break continuous translations and rotations down to their discrete counterparts; it also breaks space-time rotations. In a relativistic context these would be the boosts of the Poincaré group. In a non-relativistic theory the lattice explicitly breaks Galilean boost invariance, thus providing a preferred rest frame (a condensed matter "ether"). As a consequence, the magnon-mediated forces between a pair of electrons or holes may depend on the center of mass momentum of the pair. In the actual high- $T_c$  materials Galilean (or more precisely Poincaré symmetry) is spontaneously (and not explicitly) broken. If phonons play an important role in the understanding of high-temperature superconductivity, one should construct an effective theory of spontaneously broken (and thus non-linearly realized)  $SU(2)_s$  and Galilean symmetry which would automatically include both magnons and phonons. This is indeed possible and presently under investigation using the techniques of lowenergy effective field theory. In the present paper we assume that phonons play no major role in the cuprates. In that case, it is legitimate to break Galilean invariance explicitly instead of spontaneously.

#### 5.2 Effective Action for Magnons and Charge Carriers

We now construct the leading terms in the effective action of magnons and electrons or holes. The effective theory provides a systematic low-energy expansion organized according to the number of derivatives in the terms of the effective action. We decompose the effective Lagrangian into an  $SU(2)_Q$ -invariant part  $\mathcal{L}$  and an  $SU(2)_Q$ breaking (but still  $U(1)_Q$ -invariant) part  $\tilde{\mathcal{L}}$ . The contributions  $\mathcal{L}_{n_t,n_i,n_{\psi}}$  and  $\tilde{\mathcal{L}}_{n_t,n_i,n_{\psi}}$ to the effective Lagrangian are classified according to the number of time-derivatives  $n_t$ , the number of spatial derivatives  $n_i$ , and the number of fermion fields  $n_{\psi}$  they contain. The total action is then given by

$$S[\psi_{\pm}^{A,B\dagger},\psi_{\pm}^{A,B},P] = \int d^2x \ dt \ \sum_{n_t,n_i,n_{\psi}} (\mathcal{L}_{n_t,n_i,n_{\psi}} + \widetilde{\mathcal{L}}_{n_t,n_i,n_{\psi}})$$
(5.7)

and the partition function takes the form

$$Z = \int \mathcal{D}\psi_{\pm}^{A,B\dagger} \mathcal{D}\psi_{\pm}^{A,B} \mathcal{D}P \exp(-S[\psi_{\pm}^{A,B\dagger},\psi_{\pm}^{A,B},P]).$$
(5.8)

Until now we have constructed the effective action in the Q = 0 sector, i.e. for a half-filled system which is described entirely in terms of magnons. Since antiferromagnetic magnons have a "relativistic" dispersion relation (with the spin-wave velocity c playing the role of the velocity of light), in pure magnon chiral perturbation theory one counts temporal and spatial derivatives as being of the same order. The leading contributions of eq.(3.5) take the form

$$\mathcal{L}_{2,0,0} = \frac{\rho_s}{c^2} \text{Tr}[\partial_t P \partial_t P], \quad \mathcal{L}_{0,2,0} = \rho_s \text{Tr}[\partial_i P \partial_i P].$$
(5.9)

Next we consider terms quadratic in the fermion fields. These contribute to the scattering of magnons off electrons or holes in the  $Q = \pm 1$  sectors and they generally

describe the propagation of charge carriers in an antiferromagnet with  $|Q| \geq 1$ . In contrast to magnons, electrons or holes are massive and have a non-relativistic dispersion relation. Hence, it is natural to count one temporal and two spatial derivatives as being of the same order. In order to count derivatives consistently, in the  $Q \neq 0$  sectors it may thus be necessary to also consider the pure magnon term  $\mathcal{L}_{2,0,0}$  with two temporal derivatives as being of higher order. The leading order terms without any derivatives which are Hermitean and invariant under  $SU(2)_s$ ,  $SU(2)_Q$ , D, and D' as well as under the space-time symmetries O, R, T, and T'take the form

$$\mathcal{L}_{0,0,2} = M_1 \text{Tr}[\Psi^{A\dagger} \Psi^B] + \frac{M_2}{2} \text{Tr}[\Psi^{A\dagger} \sigma_3 \Psi^A - \Psi^{B\dagger} \sigma_3 \Psi^B]$$
  
=  $M_1(\psi^{A\dagger}_+ \psi^B_+ + \psi^{A\dagger}_- \psi^B_- + \psi^{B\dagger}_+ \psi^A_+ + \psi^{B\dagger}_- \psi^A_-)$   
+ $M_2(\psi^{A\dagger}_+ \psi^A_+ - \psi^{A\dagger}_- \psi^A_- - \psi^{B\dagger}_+ \psi^B_+ + \psi^{B\dagger}_- \psi^B_-).$  (5.10)

The mass parameters  $M_1$  and  $M_2$  (as well as all other low-energy parameters to be introduced below) take real values in order to ensure Hermiticity of the corresponding Hamiltonian. It should be noted that

$$\operatorname{Tr}[\Psi^{A\dagger}\Psi^{A}] = \operatorname{Tr}[\Psi^{B\dagger}\Psi^{B}] = 0,$$
  
$$\operatorname{Tr}[\Psi^{A\dagger}\Psi^{B}] = \operatorname{Tr}[\Psi^{B\dagger}\Psi^{A}],$$
(5.11)

due to the anticommutativity of the Grassmann fields. When we impose only the generic  $U(1)_Q$  but not the full  $SU(2)_Q$  symmetry, one more fermion mass term can be added

$$\widetilde{\mathcal{L}}_{0,0,2} = \frac{m}{2} \operatorname{Tr}[\Psi^{A\dagger} \Psi^{A} \sigma_{3} + \Psi^{B\dagger} \Psi^{B} \sigma_{3}] 
= m(\psi^{A\dagger}_{+} \psi^{A}_{+} + \psi^{A\dagger}_{-} \psi^{A}_{-} + \psi^{B\dagger}_{+} \psi^{B}_{+} + \psi^{B\dagger}_{-} \psi^{B}_{-}).$$
(5.12)

This term can be absorbed into a redefinition of the chemical potential. Remarkably, no other fermion mass terms (consistent with the  $SU(2)_s$ ,  $U(1)_Q$ , D, D', T, and T'symmetries) exist. In particular, it is useful to note that

$$\operatorname{Tr}[\Psi^{A\dagger}\sigma_{3}\Psi^{A}\sigma_{3}] = \operatorname{Tr}[\Psi^{B\dagger}\sigma_{3}\Psi^{B}\sigma_{3}] = 0.$$
(5.13)

The terms with one temporal derivative are given by

$$\mathcal{L}_{1,0,2} = \frac{1}{2} \text{Tr}[\Psi^{A\dagger} D_t \Psi^A + \Psi^{B\dagger} D_t \Psi^B] + \frac{\Lambda_1}{2} \text{Tr}[\Psi^{A\dagger} V_t \Psi^A + \Psi^{B\dagger} V_t \Psi^B] + \Lambda_2 \text{Tr}[\Psi^{A\dagger} \sigma_3 V_t \Psi^B] = \psi^{A\dagger}_+ D_t \psi^A_+ + \psi^{A\dagger}_- D_t \psi^A_- + \psi^{B\dagger}_+ D_t \psi^B_+ + \psi^{B\dagger}_- D_t \psi^B_- + \Lambda_1 (\psi^{A\dagger}_+ v^+_t \psi^A_- + \psi^{A\dagger}_- v^-_t \psi^A_+ + \psi^{B\dagger}_+ v^+_t \psi^B_- + \psi^{B\dagger}_- v^-_t \psi^B_+) + \Lambda_2 (\psi^{A\dagger}_+ v^+_t \psi^B_- + \psi^{B\dagger}_- v^-_t \psi^A_+ - \psi^{B\dagger}_+ v^+_t \psi^A_- - \psi^{A\dagger}_- v^-_t v^B_+).$$
(5.14)

Here  $V_t$  is the field defined in eq.(3.29) and the covariant derivatives are those of eq.(4.17). In components they take the form

$$D_{\mu}\psi_{\pm}^{A,B}(x) = (\partial_{\mu} \pm iv_{\mu}^{3}(x))\psi_{\pm}^{A,B}(x),$$
  
$$D_{\mu}\psi_{\pm}^{A,B\dagger}(x) = (\partial_{\mu} \mp iv_{\mu}^{3}(x))\psi_{\pm}^{A,B\dagger}(x).$$
 (5.15)

Note that  $v_t^3$  as well as  $v_t^{\pm}$  (and hence  $V_t$ ) count like one temporal derivative because these composite fields indeed contain one time-derivative of the magnon field.

When one derives the Euclidean path integral from the Hamiltonian formulation of the effective theory, the term  $\psi_{+}^{A\dagger}\partial_{t}\psi_{+}^{A}+\psi_{-}^{A\dagger}\partial_{t}\psi_{-}^{A}+\psi_{+}^{B\dagger}\partial_{t}\psi_{+}^{B}+\psi_{-}^{B\dagger}\partial_{t}\psi_{-}^{B}$  arises from the pairs of anticommuting fermion operators. It should be noted that there are two more  $SU(2)_{Q}$ -breaking but  $U(1)_{Q}$ -invariant terms with a single time-derivative

$$\frac{1}{2} \operatorname{Tr} [\Psi^{A\dagger} \sigma_{3} D_{t} \Psi^{A} \sigma_{3} - \Psi^{B\dagger} \sigma_{3} D_{t} \Psi^{B} \sigma_{3}] 
= \psi^{A\dagger}_{+} D_{t} \psi^{A}_{+} - \psi^{A\dagger}_{-} D_{t} \psi^{A}_{-} - \psi^{B\dagger}_{+} D_{t} \psi^{B}_{+} + \psi^{B\dagger}_{-} D_{t} \psi^{B}_{-}, 
\frac{1}{2} \operatorname{Tr} [\Psi^{A\dagger} D_{t} \Psi^{B} \sigma_{3} + \Psi^{B\dagger} D_{t} \Psi^{A} \sigma_{3}] 
= \psi^{A\dagger}_{+} D_{t} \psi^{B}_{+} + \psi^{A\dagger}_{-} D_{t} \psi^{B}_{-} + \psi^{B\dagger}_{+} D_{t} \psi^{A}_{+} + \psi^{B\dagger}_{-} D_{t} \psi^{A}_{-}. \quad (5.16)$$

These terms need not be included in the effective Lagrangian, since they would not imply canonical anticommutation relations in the Hamiltonian formulation. In any case, as discussed in appendix B, if one does include these terms they can again be removed by an appropriate field redefinition.

Interestingly, there is only one more term that violates the  $SU(2)_Q$  symmetry but still respects the  $U(1)_Q$  symmetry

$$\widetilde{\mathcal{L}}_{1,0,2} = \lambda \operatorname{Tr}[\Psi^{A\dagger} V_t \Psi^B \sigma_3] 
= \lambda (\psi^{A\dagger}_+ v^+_t \psi^B_- + \psi^{B\dagger}_- v^A_t + \psi^{B\dagger}_+ v^+_t \psi^A_- + \psi^{A\dagger}_- v^-_t \psi^B_+).$$
(5.17)

Further potential contributions are absent because, for example,

$$Tr[\Psi^{A\dagger}D_t\Psi^A\sigma_3] = Tr[\Psi^{B\dagger}D_t\Psi^B\sigma_3] = 0,$$
  

$$Tr[\Psi^{A\dagger}V_t\Psi^A\sigma_3] = Tr[\Psi^{B\dagger}V_t\Psi^B\sigma_3] = 0,$$
  

$$Tr[\Psi^{A\dagger}\sigma_3V_t\Psi^A\sigma_3] = Tr[\Psi^{B\dagger}\sigma_3V_t\Psi^B\sigma_3] = 0.$$
(5.18)

Terms with a single spatial derivative are forbidden due to the reflection symmetry R and the 90 degrees rotation symmetry O of the quadratic spatial lattice of the underlying microscopic system. The terms with two spatial derivatives are given by

$$\mathcal{L}_{0,2,2} = \frac{1}{2M_1'} \operatorname{Tr}[D_i \Psi^{A\dagger} D_i \Psi^B] + \frac{1}{4M_2'} \operatorname{Tr}[D_i \Psi^{A\dagger} \sigma_3 D_i \Psi^A - D_i \Psi^{B\dagger} \sigma_3 D_i \Psi^B] + iK_1 \operatorname{Tr}[D_i \Psi^{A\dagger} V_i \Psi^B + D_i \Psi^{B\dagger} V_i \Psi^A]$$

$$\begin{aligned} &+iK_{2}\mathrm{Tr}[D_{i}\Psi^{A\dagger}\sigma_{3}V_{i}\Psi^{A}-D_{i}\Psi^{B\dagger}\sigma_{3}V_{i}\Psi^{B}] \\ &+N_{1}\mathrm{Tr}[\Psi^{A\dagger}V_{i}V_{i}\Psi^{B}] + \frac{N_{2}}{2}\mathrm{Tr}[\Psi^{A\dagger}V_{i}\sigma_{3}V_{i}\Psi^{A}-\Psi^{B\dagger}V_{i}\sigma_{3}V_{i}\Psi^{B}] \\ = & \frac{1}{2M_{1}'}(D_{i}\psi^{A\dagger}_{+}D_{i}\psi^{B}_{+}+D_{i}\psi^{B\dagger}_{+}D_{i}\psi^{A}_{+}+D_{i}\psi^{A\dagger}_{-}D_{i}\psi^{B}_{-}+D_{i}\psi^{B\dagger}_{-}D_{i}\psi^{A}_{-}) \\ &+\frac{1}{2M_{2}'}(D_{i}\psi^{A\dagger}_{+}D_{i}\psi^{A}_{+}-D_{i}\psi^{A\dagger}_{-}D_{i}\psi^{A}_{+}+D_{i}\psi^{B\dagger}_{-}D_{i}\psi^{B}_{+}+D_{i}\psi^{B\dagger}_{-}D_{i}\psi^{B}_{-}) \\ &+iK_{1}(D_{i}\psi^{A\dagger}_{+}v^{i}_{i}\psi^{B}_{-}-\psi^{B\dagger}_{-}v^{-}_{i}D_{i}\psi^{A}_{+}+D_{i}\psi^{A\dagger}_{-}v^{-}_{i}\psi^{A}_{+}-\psi^{A\dagger}_{+}v^{i}_{i}D_{i}\psi^{B}_{-}) \\ &+iK_{2}(D_{i}\psi^{A\dagger}_{+}v^{i}_{i}\psi^{A}_{-}-\psi^{A\dagger}_{-}v^{-}_{i}D_{i}\psi^{A}_{+}+D_{i}\psi^{B\dagger}_{-}v^{-}_{i}\psi^{A}_{+}+\psi^{A\dagger}_{+}v^{i}_{i}D_{i}\psi^{B}_{-}) \\ &+iK_{2}(D_{i}\psi^{A\dagger}_{+}v^{i}_{i}\psi^{A}_{-}+\psi^{B\dagger}_{-}v^{-}_{i}D_{i}\psi^{A}_{+}+D_{i}\psi^{B\dagger}_{-}v^{-}_{i}\psi^{A}_{+}+\psi^{A\dagger}_{+}v^{i}_{i}D_{i}\psi^{B}_{-}) \\ &+N_{1}(\psi^{A\dagger}_{+}v^{i}_{i}\psi^{A}_{-}+\psi^{B\dagger}_{-}v^{i}_{i}v^{i}_{i}\psi^{A}_{-}+\psi^{B\dagger}_{+}v^{i}_{i}v^{-}_{i}\psi^{A}_{+}+\psi^{B\dagger}_{-}v^{i}_{i}\psi^{A}_{-}) \\ &-N_{2}(\psi^{A\dagger}_{+}v^{i}_{i}v^{-}_{i}\psi^{A}_{+}-\psi^{A\dagger}_{-}v^{i}_{i}v^{i}_{+}\psi^{A}_{-}-\psi^{B\dagger}_{+}v^{i}_{i}v^{i}_{i}\psi^{B}_{+}+\psi^{B\dagger}_{-}v^{i}_{i}\psi^{B}_{+}+\psi^{B\dagger}_{-}v^{i}_{i}\psi^{B}_{+}).(5.19) \end{aligned}$$

Note that the imaginary unit *i* in front of the terms proportional to  $K_1$  and  $K_2$  is necessary to ensure that the corresponding Hamiltonian is Hermitean. In principle, terms containing  $D_i D_i$  and  $D_i V_i$  could also be written down. However, upon partial integration, up to irrelevant surface terms they lead to the same Euclidean action as the terms constructed here. Since the doped electrons or holes are non-relativistic, there is no reason why the kinetic mass parameters  $M'_1$  and  $M'_2$  should agree with the rest mass parameters  $M_1$  and  $M_2$ . In addition, there are again terms that break the  $SU(2)_Q$  symmetry but leave the  $U(1)_Q$  symmetry intact

$$\begin{split} \widetilde{\mathcal{L}}_{0,2,2} &= \frac{1}{4m'} \mathrm{Tr}[D_i \Psi^{A\dagger} D_i \Psi^A \sigma_3 + D_i \Psi^{B\dagger} D_i \Psi^B \sigma_3] \\ &+ i\kappa_1 \mathrm{Tr}[D_i \Psi^{A\dagger} \sigma_3 V_i \Psi^B \sigma_3 + D_i \Psi^{B\dagger} V_i \sigma_3 \Psi^A \sigma_3] \\ &+ i\kappa_2 \mathrm{Tr}[D_i \Psi^{A\dagger} V_i \Psi^A \sigma_3 + D_i \Psi^{B\dagger} V_i \Psi^B \sigma_3] \\ &+ \frac{\nu}{2} \mathrm{Tr}[\Psi^{A\dagger} V_i V_i \Psi^A \sigma_3 + \Psi^{B\dagger} V_i V_i \Psi^B \sigma_3] \\ &= \frac{1}{2m'} (D_i \psi^{A\dagger}_+ D_i \psi^A_+ + D_i \psi^{A\dagger}_- D_i \psi^{A\dagger}_+ D_i \psi^{B\dagger}_+ D_i \psi^{B\dagger}_- D_i \psi^{B}_-) \\ &+ i\kappa_1 (D_i \psi^{A\dagger}_+ v_i^+ \psi^B_- - \psi^{B\dagger}_- v_i^- D_i \psi^A_+ - D_i \psi^{A\dagger}_- v_i^- \psi^A_+ + \psi^{A\dagger}_+ v_i^+ D_i \psi^B_-) \\ &+ i\kappa_2 (D_i \psi^{A\dagger}_+ v_i^+ \psi^A_- - \psi^{A\dagger}_- v_i^- D_i \psi^A_+ + D_i \psi^{B\dagger}_- v_i^- \psi^A_+ - \psi^{A\dagger}_+ v_i^+ D_i \psi^A_- \\ &+ D_i \psi^{B\dagger}_+ v_i^+ \psi^B_- - \psi^{B\dagger}_- v_i^- D_i \psi^A_+ + D_i \psi^{B\dagger}_- v_i^- \psi^A_+ - \psi^{A\dagger}_+ v_i^+ D_i \psi^A_- \\ &+ D_i \psi^{B\dagger}_+ v_i^+ \psi^B_- - \psi^{B\dagger}_- v_i^- D_i \psi^B_+ + D_i \psi^{B\dagger}_- v_i^- \psi^B_+ + \psi^{B\dagger}_- v_i^- \psi^B_-) \\ &+ \nu (\psi^{A\dagger}_+ v_i^+ v_i^- \psi^A_+ + \psi^{A\dagger}_- v_i^- v_i^+ \psi^A_- + \psi^{B\dagger}_+ v_i^- \psi^B_+ + \psi^{B\dagger}_- v_i^- v_i^+ \psi^B_-). \ (5.20) \end{split}$$

Next we consider terms quartic in the fermion fields which describe short-range interactions between the charge carriers. To lowest order there are five linearly independent 4-fermion contact interaction terms

$$\mathcal{L}_{0,0,4} = \frac{G_1}{12} \operatorname{Tr}[\Psi^{A\dagger} \Psi^A \Psi^{A\dagger} \Psi^A + \Psi^{B\dagger} \Psi^B \Psi^{B\dagger} \Psi^B] + \frac{G_2}{2} \operatorname{Tr}[\Psi^{A\dagger} \Psi^A \Psi^{B\dagger} \Psi^B]$$

$$\begin{aligned} &+ \frac{G_3}{2} \mathrm{Tr} [\Psi^{A\dagger} \Psi^B \Psi^{B\dagger} \Psi^A] + \frac{G_4}{2} \mathrm{Tr} [\Psi^{A\dagger} \sigma_3 \Psi^A \Psi^{B\dagger} \sigma_3 \Psi^B] \\ &+ G_5 \mathrm{Tr} [\Psi^{A\dagger} \sigma_3 \Psi^A \Psi^{A\dagger} \Psi^B - \Psi^{B\dagger} \sigma_3 \Psi^B \Psi^{B\dagger} \Psi^A] \\ = & G_1 (\psi^{A\dagger}_+ \psi^A_+ \psi^{A\dagger}_- \psi^A_- + \psi^{B\dagger}_+ \psi^{B\dagger}_+ \psi^{B\dagger}_- \psi^B_- + \psi^{A\dagger}_- \psi^A_- \psi^{B\dagger}_+ \psi^A_+ \psi^A_- \psi^{B\dagger}_- \psi^B_- + \psi^{A\dagger}_- \psi^A_- \psi^{B\dagger}_+ \psi^A_- \psi^{B\dagger}_- \psi^A_- + \psi^{A\dagger}_- + \psi^{A\dagger}_- \psi^A_- + \psi^{A\dagger}_- \psi^A_- + \psi^{A\dagger}_- \psi^A_- + \psi^{A\dagger}_- \psi^A_- + \psi^{A\dagger}_- + \psi^{A}_- + \psi^{A\dagger}_- + \psi^{A}_- + \psi^{A$$

It is interesting to note that

$$Tr[\Psi^{A\dagger}\Psi^{B}\Psi^{B\dagger}\Psi^{A} + \Psi^{A\dagger}\Psi^{A}\Psi^{B\dagger}\Psi^{B} + 2\Psi^{A\dagger}\Psi^{B}\Psi^{A\dagger}\Psi^{B}] = 0,$$
  

$$Tr[\Psi^{A\dagger}\Psi^{B}\Psi^{B\dagger}\Psi^{A} + \Psi^{A\dagger}\sigma_{3}\Psi^{B}\Psi^{B\dagger}\sigma_{3}\Psi^{A} + 2\Psi^{A\dagger}\sigma_{3}\Psi^{A}\Psi^{B\dagger}\sigma_{3}\Psi^{B}] = 0,$$
  

$$Tr[\Psi^{A\dagger}\Psi^{B}\Psi^{B\dagger}\Psi^{A} - \Psi^{A\dagger}\Psi^{A}\Psi^{B\dagger}\Psi^{B}] = 2(Tr[\Psi^{A\dagger}\Psi^{B}])^{2}.$$
(5.22)

Together with other relations similar to these ones, this implies that the terms listed above form a maximal linearly independent set.

Again, there are additional terms that are invariant under  $U(1)_Q$  but not under  $SU(2)_Q$ 

$$\begin{aligned} \widetilde{\mathcal{L}}_{0,0,4} &= \frac{g_1}{4} \mathrm{Tr} [\Psi^{A\dagger} \sigma_3 \Psi^A \Psi^{B\dagger} \Psi^B \sigma_3 - \Psi^{B\dagger} \sigma_3 \Psi^B \Psi^{A\dagger} \Psi^A \sigma_3] + \frac{g_2}{2} \mathrm{Tr} [\Psi^{A\dagger} \Psi^A \sigma_3 \Psi^{B\dagger} \Psi^B \sigma_3] \\ &+ g_3 \mathrm{Tr} [\Psi^{A\dagger} \Psi^A \sigma_3 \Psi^{A\dagger} \Psi^B + \Psi^{B\dagger} \Psi^B \sigma_3 \Psi^{B\dagger} \Psi^A] \\ &= g_1 (\psi^{A\dagger}_+ \psi^A_+ \psi^{B\dagger}_- \psi^A_- \psi^{A\dagger}_- \psi^A_- \psi^{B\dagger}_+ \psi^B_+) \\ &+ g_2 (\psi^{A\dagger}_+ \psi^A_+ \psi^{B\dagger}_+ \psi^B_+ + \psi^{A\dagger}_+ \psi^A_- \psi^{B\dagger}_- \psi^B_- + \psi^{A\dagger}_- \psi^A_- \psi^{B\dagger}_+ \psi^B_+ \psi^A_- \psi^{B\dagger}_- \psi^A_- \psi^{B\dagger}_- \psi^A_- \\ &+ 2\psi^{A\dagger}_+ \psi^B_+ \psi^{A\dagger}_- \psi^B_- + 2\psi^{B\dagger}_+ \psi^A_- \psi^B_- \psi^A_- \psi^A_- \psi^A_+ \psi^B_- \psi^A_- \\ &- g_3 (\psi^{A\dagger}_+ \psi^A_- \psi^A_- + \psi^{A\dagger}_+ \psi^B_- \psi^B_- + \psi^{A\dagger}_- \psi^A_- \psi^A_+ \psi^A_+ \psi^A_- \psi^A_- + \psi^{A\dagger}_- \psi^B_- \psi^A_- + \psi^{A\dagger}_- \psi^B_- \psi^A_- \psi^A_+ \psi^A_- \psi^A_- \psi^A_+ \psi^B_+ \psi^A_- \psi^A_- \psi^A_- \psi^A_+ \psi^B_- \psi^A_- \psi^A_- \psi^A_- \psi^A_- \psi^A_- \psi^B_- \psi^A_- \psi^A_-$$

One may note that, for example,

$$\operatorname{Tr}[\Psi^{A\dagger}\Psi^{A}\Psi^{B\dagger}\Psi^{B} + \Psi^{A\dagger}\Psi^{A}\sigma_{3}\Psi^{B\dagger}\Psi^{B}\sigma_{3} + 2\Psi^{A\dagger}\Psi^{B}\sigma_{3}\Psi^{B\dagger}\Psi^{A}\sigma_{3}] = 0.$$
(5.24)

Together with further relations of a similar kind, this implies that there are no other linearly independent 4-fermion terms that obey the relevant symmetries. For completeness, let us also construct the terms containing six fermion fields and no derivatives. The  $SU(2)_Q$ -invariant 6-fermion terms can be written as

$$\mathcal{L}_{0,0,6} = \frac{H_{1}}{4} \operatorname{Tr}[\Psi^{A\dagger}\sigma_{3}\Psi^{A}\Psi^{A\dagger}\sigma_{3}\Psi^{A}\Psi^{B\dagger}\sigma_{3}\Psi^{B} - \Psi^{B\dagger}\sigma_{3}\Psi^{B}\Psi^{B\dagger}\sigma_{3}\Psi^{B}\Psi^{A\dagger}\sigma_{3}\Psi^{A}] \\ + \frac{H_{2}}{3} \operatorname{Tr}[\Psi^{A\dagger}\Psi^{B}\Psi^{A\dagger}\Psi^{B}\Psi^{A\dagger}\Psi^{B}\Psi^{A}\Psi^{B}] \\ = H_{1}(\psi^{A\dagger}_{+}\psi^{A}_{+}\psi^{B\dagger}_{+}\psi^{B}_{+}\psi^{B}_{-}\psi^{A\dagger}_{-}\psi^{A\dagger}_{+}\psi^{A\dagger}_{+}\psi^{A\dagger}_{-}\psi^{A}_{-}\psi^{B\dagger}_{-}\psi^{B}_{-} \\ -\psi^{A\dagger}_{-}\psi^{A}_{-}\psi^{B\dagger}_{+}\psi^{B}_{+}\psi^{B\dagger}_{-}\psi^{B}_{-} - \psi^{A\dagger}_{+}\psi^{A}_{+}\psi^{A}_{-}\psi^{A}_{-}\psi^{A}_{-}\psi^{B\dagger}_{+}\psi^{B}_{+}) \\ + H_{2}(\psi^{A\dagger}_{+}\psi^{A}_{+}\psi^{B\dagger}_{+}\psi^{B}_{+}\psi^{A\dagger}_{-}\psi^{A}_{-}\psi^{A}_{-}\psi^{B\dagger}_{-}\psi^{B\dagger}_{-}\psi^{B\dagger}_{-}\psi^{A}_{-}\psi^{B\dagger}_{-}\psi^{B\dagger}_{-}\psi^{B\dagger}_{-}\psi^{A}_{-}).$$
(5.25)

It is interesting to note that

$$\operatorname{Tr}[\Psi^{A\dagger}\sigma_{3}\Psi^{A}\Psi^{A\dagger}\sigma_{3}\Psi^{A}\Psi^{B\dagger}\sigma_{3}\Psi^{B} - \Psi^{B\dagger}\sigma_{3}\Psi^{B}\Psi^{B\dagger}\sigma_{3}\Psi^{B}\Psi^{A\dagger}\sigma_{3}\Psi^{A}] = \frac{1}{2}\operatorname{Tr}[\Psi^{A\dagger}\sigma_{3}\Psi^{A} - \Psi^{B\dagger}\sigma_{3}\Psi^{B}]\operatorname{Tr}[\Psi^{A\dagger}\sigma_{3}\Psi^{A}\Psi^{B\dagger}\sigma_{3}\Psi^{B}].$$
(5.26)

In addition, there is one  $SU(2)_Q$ -breaking (but  $U(1)_Q$ -invariant) 6-fermion term

$$\widetilde{\mathcal{L}}_{0,0,6} = \frac{h}{4} \operatorname{Tr} [\Psi^{A\dagger} \Psi^{A} \sigma_{3} \Psi^{A\dagger} \Psi^{A} \sigma_{3} \Psi^{B\dagger} \Psi^{B} \sigma_{3} + \Psi^{B\dagger} \Psi^{B} \sigma_{3} \Psi^{B\dagger} \Psi^{B} \sigma_{3} \Psi^{A\dagger} \Psi^{A} \sigma_{3}] 
= -h(\psi^{A\dagger}_{+} \psi^{A}_{+} \psi^{A\dagger}_{-} \psi^{A}_{+} \psi^{B\dagger}_{+} \psi^{B}_{+} \psi^{A\dagger}_{+} \psi^{A\dagger}_{+} \psi^{A\dagger}_{-} \psi^{A\dagger}_{-} \psi^{B\dagger}_{-} \psi^{B}_{-} 
+ \psi^{A\dagger}_{+} \psi^{A}_{+} \psi^{B\dagger}_{+} \psi^{B\dagger}_{+} \psi^{B\dagger}_{-} \psi^{B}_{-} + \psi^{A\dagger}_{-} \psi^{A\dagger}_{+} \psi^{A\dagger}_{+} \psi^{B\dagger}_{+} \psi^{B\dagger}_{-} \psi^{B}_{-} \psi^{B}_{-} ).$$
(5.27)

Finally, the only 8-fermion term with no derivatives takes the form

$$\mathcal{L}_{0,0,8} = \frac{I}{24} \operatorname{Tr}[\Psi^{A\dagger}\Psi^{A}\Psi^{B\dagger}\Psi^{B}\Psi^{A\dagger}\Psi^{A}\Psi^{B\dagger}\Psi^{B}\Psi^{B}]$$
  
$$= -I \psi^{A\dagger}_{+}\psi^{A}_{+}\psi^{B\dagger}_{+}\psi^{B}_{+}\psi^{A\dagger}_{-}\psi^{B\dagger}_{-}\psi^{B}_{-}, \qquad (5.28)$$

which is  $SU(2)_Q$ -invariant. It may be noted that

$$\operatorname{Tr}[\Psi^{A\dagger}\Psi^{A}\Psi^{B\dagger}\Psi^{B}\Psi^{A\dagger}\Psi^{A}\Psi^{B\dagger}\Psi^{B}] + \frac{1}{2}(\operatorname{Tr}[\Psi^{A\dagger}\Psi^{A}\Psi^{B\dagger}\Psi^{B}])^{2} = 0.$$
(5.29)

No  $SU(2)_Q$ -breaking 8-fermion term without derivatives exists, such that  $\mathcal{L}_{0,0,8} = 0$ . Terms with more than eight fermion fields vanish due to the Pauli principle, unless one includes derivatives. Since such terms are of higher order than those without derivatives, they will not be constructed here. When one wants to address questions for which the short-distance forces between charge carriers are essential, it will be necessary to consider such terms. While constructing them is a straightforward exercise, it is not very illuminating and will hence be omitted at this stage.

The 4-, 6-, and 8-fermion contact terms parameterize short distance interactions with a large number of undetermined low-energy constants. Since this limits the predictive power of the effective theory at short distances, it is natural to concentrate on long-distance forces between the charge carriers. For example, one-magnon exchange mediates a long-range force that is unambiguously predicted by the effective theory in terms of just a few low-energy parameters.

It should be mentioned that there are many equivalent ways of rewriting the various contributions to the action in terms of traces. Hence, the above choices of terms are to some extent arbitrary. It is important that the selected terms form a maximal linearly independent set. For example, all determinants or products of traces of fermion fields can be written as linear combinations of the traces listed above. To verify the completeness and linear independence of the selected terms is a non-trivial task which was addressed by extensive use of the algebraic manipulation program FORM [90].

It is straightforward to include the fermion chemical potential  $\mu$  in the effective theory. It appears as the temporal component of a purely imaginary  $U(1)_Q$  gauge field and thus manifests itself in an additional contribution to the covariant derivative

$$D_t \Psi^{A,B}(x) = \partial_t \Psi^{A,B}(x) + i v_t^3(x) \sigma_3 \Psi^{A,B}(x) - \mu \Psi^{A,B}(x) \sigma_3.$$
(5.30)

Before one can do consistent loop-calculations in the low-energy effective theory at non-zero Q or at non-zero  $\mu$  one must develop a power-counting scheme, e.g. along the lines of [73]. This will be the subject of a future publication.

#### 5.3 Dispersion Relations of Electrons and Holes

In this subsection, as an application of the effective theory, we consider the dispersion relations of the charge carriers. For this purpose we switch off the magnon field (i.e.  $P(x) = \frac{1}{2}(\mathbf{1} + \sigma_3) \Rightarrow u(x) = \mathbf{1}, v_{\mu}(x) = 0$ ) and consider the propagation of free charge carriers in the antiferromagnetic medium. In the absence of  $SU(2)_Q$ -breaking terms, the Lagrangian (quadratic in the fermion fields) then reduces to

$$\mathcal{L} = \psi_{+}^{A\dagger} \partial_{t} \psi_{+}^{A} + \psi_{-}^{A\dagger} \partial_{t} \psi_{-}^{A} + \psi_{+}^{B\dagger} \partial_{t} \psi_{+}^{B} + \psi_{-}^{B\dagger} \partial_{t} \psi_{-}^{B} \\
+ (\psi_{+}^{A\dagger}, \psi_{+}^{B\dagger}) \begin{pmatrix} M_{2} & M_{1} \\ M_{1} & -M_{2} \end{pmatrix} \begin{pmatrix} \psi_{+}^{A} \\ \psi_{+}^{B} \end{pmatrix} + (\psi_{-}^{A\dagger}, \psi_{-}^{B\dagger}) \begin{pmatrix} -M_{2} & M_{1} \\ M_{1} & M_{2} \end{pmatrix} \begin{pmatrix} \psi_{-}^{A} \\ \psi_{-}^{B} \end{pmatrix} \\
+ (\partial_{i} \psi_{+}^{A\dagger}, \partial_{i} \psi_{+}^{B\dagger}) \begin{pmatrix} \frac{1}{2M'_{2}} & \frac{1}{2M'_{1}} \\ \frac{1}{2M'_{1}} & -\frac{1}{2M'_{2}} \end{pmatrix} \begin{pmatrix} \partial_{i} \psi_{+}^{A} \\ \partial_{i} \psi_{+}^{B} \end{pmatrix} \\
+ (\partial_{i} \psi_{-}^{A\dagger}, \partial_{i} \psi_{-}^{B\dagger}) \begin{pmatrix} -\frac{1}{2M'_{2}} & \frac{1}{2M'_{1}} \\ \frac{1}{2M'_{1}} & \frac{1}{2M'_{2}} \end{pmatrix} \begin{pmatrix} \partial_{i} \psi_{-}^{A} \\ \partial_{i} \psi_{-}^{B} \end{pmatrix}.$$
(5.31)

The eigenstates of free particles propagating with a 2-d momentum vector  $\vec{p}$  arise as the eigenvectors of the matrices

$$H_{+}(p^{2}) = \begin{pmatrix} M_{2} + \frac{p^{2}}{2M'_{2}} & M_{1} + \frac{p^{2}}{2M'_{1}} \\ M_{1} + \frac{p^{2}}{2M'_{1}} & -M_{2} - \frac{p^{2}}{2M'_{2}} \end{pmatrix},$$

$$H_{-}(p^{2}) = \begin{pmatrix} -M_{2} - \frac{p^{2}}{2M'_{2}} & M_{1} + \frac{p^{2}}{2M'_{1}} \\ M_{1} + \frac{p^{2}}{2M'_{1}} & M_{2} + \frac{p^{2}}{2M'_{2}} \end{pmatrix}.$$
 (5.32)

Due to the lack of Galilean invariance the eigenvectors depend on  $p^2$ , i.e. the probability for an electron or hole to be found on the A or B sublattice depends on the momentum. As a consequence of the displacement symmetries D and D' the eigenvalues of  $H_+(p^2)$  and  $H_-(p^2)$  are the same. Both matrices have two eigenvalues

$$E_{1,2}(p^2) = \pm \sqrt{\left(M_1 + \frac{p^2}{2M_1'}\right)^2 + \left(M_2 + \frac{p^2}{2M_2'}\right)^2}$$
$$= \pm \left(M + \frac{p^2}{2M_1'} + \mathcal{O}(p^4)\right).$$
(5.33)

The positive energy states correspond to electrons, while the negative energy states correspond to holes. Not surprisingly, due to the  $SU(2)_Q$  symmetry electrons and holes have the same dispersion relation. The rest mass M and the kinetic mass M'are given by

$$M = \sqrt{M_1^2 + M_2^2}, \quad \frac{M}{M'} = \frac{M_1}{M'_1} + \frac{M_2}{M'_2}.$$
 (5.34)

Next we take into account the additional terms that reduce the  $SU(2)_Q$  symmetry to the  $U(1)_Q$  symmetry. Then there are additional contributions to the energy

$$\widetilde{H}_{+}(p^{2}) = \widetilde{H}_{-}(p^{2}) = \begin{pmatrix} m + \frac{p^{2}}{2m'} & 0\\ 0 & m + \frac{p^{2}}{2m'} \end{pmatrix}.$$
(5.35)

and the corresponding eigenvalues now take the form

$$E_{1,2}(p^2) = m + \frac{p^2}{2m'} \pm \left(M + \frac{p^2}{2M'}\right) + \mathcal{O}(p^4).$$
(5.36)

Still, the energies in the + and - sectors are the same. However, the electron and hole dispersion relations now differ.

At this point, we have constructed eigenstates of the free Hamiltonian with definite continuum momentum and with definite spin projection on the direction of the staggered magnetization. However, unlike the eigenstates of the underlying microscopic Hamiltonian, the states of the effective theory do not have a definite lattice momentum. Still, the low-energy effective theory defined in the continuum knows about the underlying lattice structure through the realization of the displacement symmetries D and D'. Since the symmetry D is spontaneously broken, neither the vacuum nor the single particle states are eigenstates of D. Operating twice with Dacts trivially on the fields, i.e.  ${}^{DD}P(x) = P(x), {}^{DD}\Psi^{A,B}_{\pm} = \Psi^{A,B}_{\pm}$ , and hence does not reveal any useful information. It is more useful to operate with the unbroken displacement symmetry D'. In particular, the vacuum state  $P(x) = \frac{1}{2}(\mathbf{1} + \sigma_3)$  is invariant under D'. Still, in the way we constructed them, the electron or hole states of the effective theory are not eigenstates of D'. However, since states with spin parallel and antiparallel to the staggered magnetization are degenerate with each other, one can form appropriate linear combinations that are eigenstates of the displacement symmetry D'. Applying D' twice one obtains

$${}^{D'D'}\psi^{A,B}_{\pm}(x) = \pm {}^{D'}\psi^{B,A}_{\mp}(x) = -\psi^{A,B}_{\pm}(x),$$
 (5.37)

which implies that the corresponding eigenvalue  $\lambda = \exp(ika)$  of D' obeys

$$\lambda^2 = \exp(2ika) = -1 \implies ka = \pm \frac{\pi}{2}.$$
(5.38)

This is reminiscent of the result, mentioned in the introduction, that low-energy hole states are located at lattice momenta  $(\pm \frac{\pi}{2}, \pm \frac{\pi}{2})$  [6,7,17,34,37–40]. However, the comparison with these findings is subtle. In particular, the results of the exact diagonalization study on small [17] and of the Monte Carlo study on larger volumes [34] must be interpreted carefully. In a finite volume (with periodic boundary conditions), in analogy to QCD [91], both the  $SU(2)_s$  spin symmetry and the displacement symmetry D are restored and the staggered magnetization acts as a quantum rotor [48]. As a result, in contrast to the infinite volume limit, the single particle states in a finite volume can be constructed as eigenstates of D. It is interesting to note that the finite volume effects that lead to the restoration of the spontaneously broken symmetries  $SU(2)_s$  and D can be understood in the framework of the effective theory. This requires a nonperturbative quantum mechanical treatment along the lines of [48, 91].

## 6 Systems with Holes only

In this section we consider the t-J model as well as its low-energy effective theory. In the t-J model holes are the only charge carriers which leads to substantial simplifications in the effective theory.

#### 6.1 The t-J Model

The t-J model is defined by the Hamilton operator

$$H = P\left\{-t\sum_{x,i}(c_x^{\dagger}c_{x+\hat{i}} + c_{x+\hat{i}}^{\dagger}c_x) + J\sum_{x,i}\vec{S}_x \cdot \vec{S}_{x+\hat{i}} - \mu\sum_x(n_x - 1)\right\}P,$$
(6.1)

with

$$c_x = \begin{pmatrix} c_{x\uparrow} \\ c_{x\downarrow} \end{pmatrix}, \quad S_x = c_x^{\dagger} \frac{\vec{\sigma}}{2} c_x, \quad n_x = c_x^{\dagger} c_x.$$
(6.2)

In contrast to the Hubbard model, in the t-J model the operators act in a restricted Hilbert space of empty or at most singly occupied sites. In particular, states with doubly occupied sites are exiled from the physical Hilbert space by the projection operator P. Hence, by definition, the t-J model does not allow the addition of electrons to a half-filled state. Consequently, the only charge carriers are holes.

It is straightforward to show that the t-J model has the same symmetries as the Hubbard model. The only exception is the  $SU(2)_Q$  symmetry which relates electrons to holes in the Hubbard model, and which is absent in the t-J model. Still, the Abelian fermion number symmetry  $U(1)_Q$  remains exact in the t-J model.

#### 6.2 Effective Theory for Magnons and Holes

Since, up to the  $SU(2)_Q$  symmetry, the *t-J* model has the same symmetries as the Hubbard model, the effective theory of the previous section also applies in this case. Of course, the values of the low-energy parameters will be different than for the Hubbard model. Still, the absence of electrons beyond half-filling leads to drastic simplifications. In particular, in the effective theory the absence of electrons manifests itself by an infinite electron rest mass. Consequently, with a finite amount of energy these excitations cannot be generated. As discussed in the previous section, the diagonalization of the mass matrices of electrons and holes yields

$$U_{\pm} \begin{pmatrix} m \pm M_2 & M_1 \\ M_1 & m \mp M_2 \end{pmatrix} U_{\pm}^{\dagger} = \begin{pmatrix} m \pm \sqrt{M_1^2 + M_2^2} & 0 \\ 0 & m \mp \sqrt{M_1^2 + M_2^2} \end{pmatrix},$$
$$U_{\pm} = \begin{pmatrix} X & \pm Y \\ \mp Y & X \end{pmatrix}, \quad X, Y \in \mathbb{R}.$$
(6.3)

The eigenvectors corresponding to the eigenvalue  $m + \sqrt{M_1^2 + M_2^2}$  describe electrons, while the ones corresponding to  $m - \sqrt{M_1^2 + M_2^2}$  describe holes. When the electron rest mass  $m + \sqrt{M_1^2 + M_2^2}$  goes to infinity, the corresponding eigenvector fields

$$X\psi_{+}^{A}(x) + Y\psi_{+}^{B}(x) = 0, \quad Y\psi_{-}^{A}(x) + X\psi_{-}^{B}(x) = 0, \tag{6.4}$$

which describe electrons, must be put to zero. The orthogonal combinations

$$\psi_{+}(x) = -Y\psi_{+}^{A}(x) + X\psi_{+}^{B}(x), \quad \psi_{-}(x) = X\psi_{-}^{A}(x) - Y\psi_{-}^{B}(x), \tag{6.5}$$

describe holes and must be kept. As a result, the number of degrees of freedom is reduced by a factor of two. In complete analogy to the discussion in appendix B one can show that the hole field  $\psi_{\pm}(x)$  transforms as follows under the various symmetry operations

$$SU(2)_s: \quad \psi_{\pm}(x)' = \exp(\pm i\alpha(x))\psi_{\pm}(x),$$

$$U(1)_{Q}: \qquad {}^{Q}\psi_{\pm}(x) = \exp(i\omega)\psi_{\pm}(x), D: \qquad {}^{D}\psi_{\pm}(x) = \mp \exp(\mp i\varphi(x))\psi_{\mp}(x), D': \qquad {}^{D'}\psi_{\pm}(x) = \pm\psi_{\mp}(x), O: \qquad {}^{O}\psi_{\pm}(x) = \psi_{\pm}(Ox), R: \qquad {}^{R}\psi_{\pm}(x) = \psi_{\pm}(Rx), T: \qquad {}^{T}\psi_{\pm}(x) = \exp(\mp i\varphi(Tx))\psi_{\pm}^{\dagger}(Tx), \qquad {}^{T}\psi_{\pm}^{\dagger}(x) = -\exp(\pm i\varphi(Tx))\psi_{\pm}(Tx), T': \qquad {}^{T'}\psi_{\pm}(x) = -\psi_{\pm}^{\dagger}(Tx), \qquad {}^{T'}\psi_{\pm}^{\dagger}(x) = \psi_{\pm}(Tx).$$
(6.6)

Hence, except for the  $SU(2)_Q$  symmetry, all symmetries can also be implemented on the hole fields alone. It should be noted that the transformation laws for  $\psi_{\pm}(x)$ result from those for  $\psi_{\pm}^{A,B}(x)$  simply by dropping the sublattice indices A and B.

The absence of electron fields also drastically reduces the number of terms one can write down in the low-energy effective theory. In particular, the leading terms in the effective action now take the form

$$S[\psi_{\pm}^{\dagger},\psi_{\pm},P] = \int d^{2}x \, dt \, \{\rho_{s} \operatorname{Tr}[\partial_{i}P\partial_{i}P + \frac{1}{c^{2}}\partial_{t}P\partial_{t}P] + M(\psi_{\pm}^{\dagger}\psi_{+} + \psi_{\pm}^{\dagger}\psi_{-}) \\ + \psi_{\pm}^{\dagger}D_{t}\psi_{+} + \psi_{-}^{\dagger}D_{t}\psi_{-} + \frac{1}{2M'}(D_{i}\psi_{\pm}^{\dagger}D_{i}\psi_{+} + D_{i}\psi_{-}^{\dagger}D_{i}\psi_{-}) \\ + \Lambda(\psi_{\pm}^{\dagger}v_{t}^{+}\psi_{-} + \psi_{-}^{\dagger}v_{t}^{-}\psi_{+}) \\ + iK(D_{i}\psi_{\pm}^{\dagger}v_{t}^{+}\psi_{-} - \psi_{-}^{\dagger}v_{t}^{-}D_{i}\psi_{+} + D_{i}\psi_{-}^{\dagger}v_{+} - \psi_{\pm}^{\dagger}v_{i}^{+}D_{i}\psi_{-}) \\ + N(\psi_{\pm}^{\dagger}v_{t}^{+}v_{t}^{-}\psi_{+} + \psi_{-}^{\dagger}v_{t}^{-}v_{+}) + G\psi_{\pm}^{\dagger}\psi_{\pm}\psi_{-}^{\dagger}\psi_{-}\}.$$
(6.7)

This form of the effective action is similar to (but not identical with) the ones of [11, 13, 25, 32, 36]. In particular, in some of those works spin-charge separation was invoked and spinless fermions were considered. Also the role of the sublattice indices (which have at this stage disappeared from our description) is different in those approaches. Furthermore, the dynamical role attributed to the composite gauge field in some of those works is different than in our effective theory. It should be pointed out that the above effective Lagrangian correctly describes the low-energy dynamics of holes only if electrons are completely absent beyond half-filling (as it is indeed the case in the t-J model). Otherwise the general effective theory of the previous section with a larger number of low-energy constants (and thus with somewhat reduced predictive power) must be employed.

### 7 Coupling to External Electromagnetic Fields

In the following sections we will couple both microscopic and effective theories for antiferromagnets to external electromagnetic fields. For this purpose, we will make use of an observation by Fröhlich and Studer concerning the Pauli equation [78].

#### 7.1 Local $SU(2)_s$ Symmetry of the Pauli Equation

Up to corrections of order  $1/M_e^3$  (where  $M_e$  is the electron mass) the Pauli equation (i.e. the non-relativistic reduction of the Dirac equation to its upper components) takes the form

$$i(\partial_t - ie\Phi + i\frac{e}{8M_e^2}\vec{\nabla}\cdot\vec{E} + i\frac{e}{2M_e}\vec{B}\cdot\vec{\sigma})\Psi = -\frac{1}{2M_e}(\vec{\nabla} + ie\vec{A} - i\frac{e}{4M_e}\vec{E}\times\vec{\sigma})^2\Psi.$$
 (7.1)

Here  $\Psi(x)$  is a 2-component Pauli spinor at the space-time point  $x = (\vec{x}, t)$ ,  $\vec{\sigma}$  are the Pauli matrices,  $\Phi(x)$  and  $\vec{A}(x)$  are the electromagnetic scalar and vector potentials, and

$$\vec{E}(x) = -\vec{\nabla}\Phi(x) - \partial_t \vec{A}(x), \quad \vec{B}(x) = \vec{\nabla} \times \vec{A}(x), \quad (7.2)$$

are the usual electromagnetic field strengths. The first two terms on the left-hand side of eq.(7.1) form the  $U(1)_Q$  covariant derivative familiar from QED. The third (Darwin) and fourth (Zeeman) term on the left-hand side represent relativistic corrections. The first two terms on the right-hand side again form an ordinary  $U(1)_Q$ covariant derivative, while the third term represents the relativistic spin-orbit coupling. The Pauli equation transforms covariantly under  $U(1)_Q$  gauge transformations

$${}^{Q}\Psi(x) = \exp(i\omega(x))\Psi(x), \quad {}^{Q}\Phi(x) = \Phi(x) + \frac{1}{e}\partial_{t}\omega(x), \quad {}^{Q}\vec{A}(x) = \vec{A}(x) - \frac{1}{e}\vec{\nabla}\omega(x).$$
(7.3)

Obviously, it is also covariant under global spatial rotations

$${}^{\mathcal{O}}\Psi(\vec{x},t) = g\Psi(\mathcal{O}\vec{x},t), \quad {}^{\mathcal{O}}\Phi(\vec{x},t) = \Phi(\mathcal{O}\vec{x},t), \quad {}^{\mathcal{O}}\vec{A}(\vec{x},t) = \mathcal{O}^T\vec{A}(\mathcal{O}\vec{x},t).$$
(7.4)

Here  $\mathcal{O}$  is a general orthogonal  $3 \times 3$  rotation matrix with

$$\mathcal{O}^T \ \vec{\sigma} = g^{\dagger} \vec{\sigma} g, \tag{7.5}$$

where  $g \in SU(2)_s$  represents the rotation  $\mathcal{O} \in SO(3)$  in spinor space.

Fröhlich and Studer noticed that the Pauli equation has a hidden local  $SU(2)_s$ spin symmetry. This symmetry becomes manifest when one writes

$$iD_t\Psi = -\frac{1}{2M_e}D_iD_i\Psi,\tag{7.6}$$

with the  $SU(2)_s \otimes U(1)_Q$  covariant derivative given by

$$D_{\mu} = \partial_{\mu} + W_{\mu}(x) + ieA_{\mu}(x). \tag{7.7}$$

The components of the non-Abelian vector potential

$$W_{\mu}(x) = iW_{\mu}^{a}(x)\frac{\sigma_{a}}{2},$$
(7.8)

can be identified as the electromagnetic field strengths  $\vec{E}(x)$  and  $\vec{B}(x)$ , i.e.

$$W_t^a(x) = \mu_e B^a(x), \quad W_i^a(x) = \frac{\mu_e}{2} \varepsilon_{iab} E^b(x).$$
 (7.9)

The anomalous magnetic moment  $\mu_e = g_e e/2M_e$  of the electron (where, up to QED corrections,  $g_e = 2$ ) appears as a non-Abelian gauge coupling. The Abelian vector potential  $A_{\mu}(x)$  is the usual one, except for a small contribution to the scalar potential due to the Darwin term,

$$A_t(x) = -\Phi(x) + \frac{1}{8M_e^2} \vec{\nabla} \cdot \vec{E}(x).$$
(7.10)

Hence, somewhat unexpected, the Pauli equation also transforms covariantly under local  $SU(2)_s$  transformations

$$\Psi(x)' = g(x)\Psi(x), \quad W_{\mu}(x)' = g(x)(W_{\mu}(x) + \partial_{\mu})g(x)^{\dagger}.$$
(7.11)

It should be pointed out that  $SU(2)_s$  is not a gauge symmetry in the usual sense. In particular, the non-Abelian vector potential  $W_{\mu}(x)$  is not an independent degree of freedom, but just given in terms of the external electromagnetic field strengths  $\vec{E}(x)$  and  $\vec{B}(x)$ . The local  $SU(2)_s$  symmetry is related to the global spatial rotations discussed before. In particular, global  $SU(2)_s$  transformations take the form

$$\Psi(x)' = g\Psi(x), \quad W_{\mu}(x)' = gW_{\mu}(x)g^{\dagger},$$
(7.12)

which, for example, implies

$$\vec{B}(x)' = \mathcal{O}^T \vec{B}(x), \tag{7.13}$$

where the resulting  $3 \times 3$  rotation matrix  $\mathcal{O} \in SO(3)$  is again given by eq.(7.5). In contrast to a full spatial rotation, a global  $SU(2)_s$  transformation does not rotate the argument  $\vec{x}$  of the magnetic field to  $\mathcal{O}\vec{x}$ . Also the potentials  $\Phi(x)$  and  $\vec{A}(x)$  are unaffected by the global  $SU(2)_s$  symmetry. Consequently, the  $SU(2)_s$  symmetry is inconsistent with the relations of eq.(7.2). Despite this, the local  $SU(2)_s$  symmetry of the Pauli equation, which will be inherited by the Hubbard model and by the effective theory, dictates how low-frequency external electromagnetic fields are to be included in those theories. The high-frequency internal electromagnetic fields (for which eq.(7.2) is essential) are integrated out in the effective theory and thus do not spoil the symmetry. The local  $SU(2)_s$  structure implies that in non-relativistic systems spin plays the role of an internal quantum number analogous to flavor in particle physics.

### 7.2 The Hubbard Model in an External Electromagnetic Field

In the next step we want to couple external electromagnetic fields to the Hubbard model. The Fröhlich-Studer  $SU(2)_s$  symmetry of the Pauli equation determines how to do this. One must simply use  $SU(2)_s \otimes U(1)_Q$  covariant derivatives with  $\vec{E}(x)$  and  $\vec{B}(x)$  playing the role of non-Abelian vector potentials for  $SU(2)_s$ . Since the Hubbard model is defined on a spatial lattice, it is natural to construct corresponding  $SU(2)_s \otimes U(1)_Q$  parallel transporters  $\mathcal{U}_{x,i}$  connecting neighboring lattice sites x and  $x + \hat{i}$ ,

$$\mathcal{U}_{x,i} = \mathcal{P} \exp[\int_0^1 ds \ W_i(x+s\hat{i})] \ \exp[ie \int_0^1 ds \ A_i(x+s\hat{i})].$$
(7.14)

Here  $\mathcal{P}$  denotes path ordering along the link. Under local  $SU(2)_s$  transformations the parallel transporter transforms as

$$\mathcal{U}'_{x,i} = g(x) \ \mathcal{U}_{x,i} \ g(x+\hat{i})^{\dagger},$$
 (7.15)

while under  $U(1)_Q$  gauge transformations one has

$${}^{Q}\mathcal{U}_{x,i} = \exp(i\omega(x)) \ \mathcal{U}_{x,i} \ \exp(-i\omega(x+\hat{i})).$$
(7.16)

The Hubbard model Hamiltonian coupled to external electromagnetic fields then reads

$$H[\mathcal{U}] = -t \sum_{x,i} (c_x^{\dagger} \mathcal{U}_{x,i} c_{x+\hat{i}} + c_{x+\hat{i}}^{\dagger} \mathcal{U}_{x,i}^{\dagger} c_x) + \frac{U}{2} \sum_x (c_x^{\dagger} c_x - 1)^2 - \mu \sum_x (c_x^{\dagger} c_x - 1), \quad (7.17)$$

and the corresponding Schrödinger equation takes the form

$$iD_t \Psi = H[\mathcal{U}]\Psi. \tag{7.18}$$

Here  $\Psi$  is the multi-particle wave function and the covariant derivative is given by

$$D_t = \partial_t + i \sum_x [\vec{W}_t(x) \cdot \vec{S}_x + eA_t(x)Q_x].$$
(7.19)

It should be noted that the Zeeman coupling  $\mu_e \vec{B}(x) \cdot \vec{S}_x$  enters the Hubbard model through  $D_t$ , while the spin-orbit coupling appears in the non-Abelian  $SU(2)_s$  part of the parallel transporter  $\mathcal{U}_{x,i}$ .

In the Hilbert space of the theory local  $SU(2)_s \otimes U(1)_Q$  transformations are implemented by unitary operators

$$V = \exp(i\sum_{x}\vec{\eta}(x)\cdot\vec{S}_{x}), \quad W = \exp(i\sum_{x}\omega(x)Q_{x}), \quad (7.20)$$

such that

$$c'_{x} = V^{\dagger}c_{x}V = \exp(i\vec{\eta}(x) \cdot \frac{\sigma}{2})c_{x} = g(x)c_{x}, \quad g(x) \in SU(2)_{s},$$

$${}^{Q}c_{x} = W^{\dagger}c_{x}W = \exp(i\omega(x))c_{x}, \quad \exp(i\omega(x)) \in U(1)_{Q}. \tag{7.21}$$

Together with eqs.(7.15) and (7.16) this implies that under the local transformations the Hamiltonian transforms as

$$H[\mathcal{U}'] = VH[\mathcal{U}]V^{\dagger}, \quad H[^{Q}\mathcal{U}] = WH[\mathcal{U}]W^{\dagger}.$$
(7.22)

Similarly, one obtains

$$D'_t = V D_t V^{\dagger}, \quad {}^Q D_t = W D_t W^{\dagger}, \tag{7.23}$$

such that the Schrödinger equation indeed transforms covariantly when one uses

$$\Psi' = V\Psi, \quad {}^{Q}\Psi = W\Psi. \tag{7.24}$$

## 7.3 External Electromagnetic Fields in the Effective Theory for Magnons and Charge Carriers

The couplings of magnons to external electromagnetic fields have been investigated in detail in [79]. Again, the Fröhlich-Studer symmetry is crucial and one obtains

$$S[\vec{e}, W_{\mu}] = \int d^2x \ dt \ \frac{\rho_s}{2} \left( D_i \vec{e} \cdot D_i \vec{e} + \frac{1}{c^2} D_t \vec{e} \cdot D_t \vec{e} \right), \tag{7.25}$$

with the covariant derivative

$$D_{\mu}\vec{e}(x) = \partial_{\mu}\vec{e}(x) + \vec{e}(x) \times \vec{W}_{\mu}(x).$$
(7.26)

Since magnons are electrically neutral, one may expect that they do not couple directly to the electromagnetic vector potential  $A_{\mu}(x)$ . Still, as discussed in [79] the issue is potentially non-trivial because there is a Goldstone-Wilczek current

$$j_{\mu}^{GW}(x) = \frac{1}{8\pi} \varepsilon_{\mu\nu\rho} \,\vec{e}(x) \cdot [D_{\nu}\vec{e}(x) \times D_{\rho}\vec{e}(x) + \vec{W}_{\nu\rho}(x)], \qquad (7.27)$$

with the non-Abelian field strength given by

$$\vec{W}_{\mu\nu}(x) = \partial_{\mu}\vec{W}_{\nu}(x) - \partial_{\nu}\vec{W}_{\mu}(x) - \vec{W}_{\mu}(x) \times \vec{W}_{\nu}(x).$$
(7.28)

The Goldstone-Wilczek current is an  $SU(2)_s$  gauge-invariant extension of the baby-Skyrmion current of eq.(3.45) and is also topologically conserved, i.e.  $\partial_{\mu} j_{\mu}^{GW} = 0$ . Hence, one may be tempted to add a Goldstone-Wilczek term  $j_{\mu}^{GW}(x)A_{\mu}(x)$  to the Lagrangian. However, just like the Hopf term, the Goldstone-Wilczek term breaks R, T, and T' and is thus forbidden in the present case.

Using the P(x) notation, in the presence of external electromagnetic fields the action of eq.(7.25) is given by

$$S[P, W_{\mu}] = \int d^2x \ dt \ \rho_s \left( \operatorname{Tr}[D_i P D_i P] + \frac{1}{c^2} \operatorname{Tr}[D_t P D_t P] \right), \tag{7.29}$$

where the  $SU(2)_s$  covariant derivative is denoted by

$$D_{\mu}P(x) = \partial_{\mu}P(x) + [W_{\mu}(x), P(x)].$$
(7.30)

As a consequence of the Fröhlich-Studer symmetry, the action of eq.(7.29) is invariant even under local  $SU(2)_s$  transformations

$$P(x)' = g(x)P(x)g(x)^{\dagger}, \quad W_{\mu}(x)' = g(x)(W_{\mu}(x) + \partial_{\mu})g(x)^{\dagger}.$$
(7.31)

Let us now discuss how external electromagnetic fields enter the fermionic part of the effective action. As a rule, ordinary derivatives must be replaced by covariant ones. This is the case also in the construction of the composite vector field which now takes the form

$$v_{\mu}(x) = u(x)D_{\mu}u(x)^{\dagger} = u(x)[\partial_{\mu} + W_{\mu}(x)]u(x)^{\dagger}.$$
 (7.32)

Under the local  $SU(2)_s$  symmetry the field u(x) transforms as

$$u(x)' = h(x)u(x)g(x)^{\dagger},$$
 (7.33)

such that

$$\begin{aligned}
v_{\mu}(x)' &= h(x)u(x)g(x)^{\dagger}[\partial_{\mu} + g(x)(W_{\mu}(x) + \partial_{\mu})g(x)^{\dagger}]g(x)u(x)^{\dagger}h(x)^{\dagger} \\
&= h(x)u(x)[\partial_{\mu} + W_{\mu}(x)]u(x)^{\dagger}h(x)^{\dagger} \\
&= h(x)(v_{\mu}(x) + \partial_{\mu})h(x)^{\dagger}.
\end{aligned}$$
(7.34)

This is exactly the same transformation behavior as for the global  $SU(2)_s$  transformation of eq.(3.26). In particular, this implies that the  $U(1)_s$  covariant derivative  $D_{\mu} = \partial_{\mu} + i v_{\mu}^3(x) \sigma_3$  need not be modified when  $SU(2)_s$  is turned into a local symmetry. Of course, according to eq.(7.32),  $v_{\mu}(x)$  now contains the electromagnetic fields  $\vec{E}(x)$  and  $\vec{B}(x)$  through the non-Abelian "gauge" field  $W_{\mu}(x)$ . Due to the local  $U(1)_Q$  symmetry, the covariant derivatives still need to be extended to

$$D_{\mu}\Psi^{A,B} = \partial_{\mu}\Psi^{A,B} + iv_{\mu}^{3}(x)\sigma_{3}\Psi^{A,B} + \Psi^{A,B}ieA_{\mu}(x)\sigma_{3},$$
  
$$D_{\mu}\Psi^{A,B\dagger} = \partial_{\mu}\Psi^{A,B\dagger} - \Psi^{A,B\dagger}iv_{\mu}^{3}(x)\sigma_{3} - ieA_{\mu}(x)\sigma_{3}\Psi^{A,B\dagger}.$$
 (7.35)

It should also be noted that the low-energy effective theory is not necessarily just minimally coupled. In particular, the field strengths  $F_{\mu\nu}(x) = \partial_{\mu}A_{\nu}(x) - \partial_{\nu}A_{\mu}(x)$  and  $W_{\mu\nu}(x)$  may also directly enter the low-energy effective theory.

## 8 Conclusions

We have constructed a systematic low-energy effective field theory describing the interactions of magnons with charge carriers doped into an antiferromagnet. A key ingredient for constructing the effective theory are symmetry considerations. The effective theory makes model-independent predictions for magnon-magnon, magnon-hole, and magnon-electron scattering. It also determines the long-range magnon-mediated forces between electrons or holes. Although these would be highly non-trivial non-perturbative issues from the point of view of Hubbard-type models, in the framework of the effective theory they can be understood quantitatively by per-turbative analytic calculations. More ambitious non-perturbative questions might also be within reach of the effective theory. Such questions include the quantitative understanding of the Mott insulator state, the reduction of the staggered magnetization upon doping, the formation of a spiral phase, or the systematic investigation of dynamical mechanisms for the preformation of electron or hole pairs in the antiferromagnetic phase. In particular, magnon exchange — the analog of pion exchange in nuclear physics — suggests itself as a relevant mechanism.

Before one can do loop-calculations in the effective theory, one must establish a consistent power-counting scheme. This has originally been done for pion chiral perturbation theory [42], and carries over to magnon chiral perturbation theory in a straightforward manner. When charge carriers are included, the issue must be reconsidered. The same was true for baryon chiral perturbation theory of pions and nucleons. In the baryon number B = 1 sector a consistent power-counting scheme enabling a systematic loop-expansion of the effective theory was established by Becher and Leutwyler [73]. It is to be expected that this scheme can be extended to the low-energy theory of magnons and charge carriers developed here. The systematic power-counting in sectors with  $B \geq 2$  still is a controversial issue in baryon chiral perturbation theory. The Weinberg power-counting scheme [52] seems to work in most (but not necessarily in all) cases. Its relation to the alternative Kaplan-Savage-Wise scheme [53] should be clarified further [59, 61]. In light of the experience with effective theories for the strong interactions, one should hence expect the issue of power-counting to be non-trivial in sectors with two or more charge carriers.

Even when the extra  $SU(2)_Q$  symmetry is imposed, in the fermion sector the effective theory has a large number of low-energy parameters. There are two rest mass parameters  $M_1$  and  $M_2$  as well as two kinetic mass parameters  $M'_1$  and  $M'_2$  for the fermions, four coupling constants  $\Lambda_1$ ,  $\Lambda_2$ ,  $K_1$ , and  $K_2$  for fermion-one-magnon vertices, two coupling constants  $N_1$  and  $N_2$  for fermion-two-magnon vertices, five 4-fermion coupling constants  $G_1, G_2, ..., G_5$ , two 6-fermion couplings  $H_1$  and  $H_2$ , and finally one 8-fermion coupling I. If only the  $U(1)_Q$  symmetry is imposed there are even more parameters. The large number of a priori undetermined low-energy parameters is the price one has to pay for the universality and model-independence of the effective theory. Only in this way the low-energy physics of any arbitrary cuprate antiferromagnet can be captured by the effective theory. Of course, due to the rather large number of parameters, the predictive power of the effective theory is somewhat limited. Still, only a few parameters enter in some relevant physical quantities. For example, the one-magnon exchange potential between charge carriers depends only on certain combinations of the fermion-magnon couplings  $\Lambda_1, \Lambda_2, K_1$ , and  $K_2$ . Also, for example, the details of the short-range 4-, 6-, and 8-fermion couplings are not expected to be essential for identifying potential mechanisms for preforming electron or hole pairs in the antiferromagnetic phase. It is interesting to note that the low-energy effective theory of the t-J model, in which electrons are excluded beyond half-filling and holes are the only charge carriers, has a much smaller number of low-energy parameters. In that case, there are only one rest mass parameter M, one kinetic mass parameter M', two coupling constants  $\Lambda$  and K for hole-one-magnon vertices, one coupling constant N for a hole-two-magnon vertex, and one 4-fermion coupling constant G. It would be interesting to perform numerical simulations of the Hubbard or t-J model in order to determine the values of the corresponding low-energy parameters by comparison with calculations in the effective theory. For example, in the t-J model one can determine the parameters M, M',  $\Lambda$ , K, and N from simulations in the one-hole sector, while the determination of G requires computations in the two-hole sector of the Hilbert space.

It should be pointed out that, as it stands, the effective theory is applicable only at small doping, i.e. for small  $\mu$ . This is sufficient for understanding the long-range forces between electrons or holes in the antiferromagnetic phase. It should also allow a quantitative investigation of the reduction of the staggered magnetization upon doping. However, in order to enter the high-temperature superconducting phase itself, if this is at all possible within the effective field theory presented here, larger values of  $\mu$  will be necessary. Once  $\mu$  becomes large, it sets a new scale which must be taken into account in the power-counting. However, most important, the symmetry considerations of the present paper still apply in that case as well.

Some of the most interesting questions one can address in the framework of the effective theory may require non-perturbative calculations. While in some cases such calculations can be performed in the continuum, in others they may require a non-perturbative regularization of the effective theory. In [92] the effective theory of pions and nucleons was regularized on a space-time lattice in order to address non-perturbative questions concerning the strong interactions. It may also be useful to formulate the effective theory of magnons and charge carriers on the lattice. For example, it would be interesting to investigate if the effective theory is more easily solvable by numerical simulation than the standard Hubbard-type models.

We like to emphasize again that effective field theory also allows us to include phonons in addition to magnons. This may shed light on more complicated potential mechanisms for Cooper pair preformation which involve both magnon and phonon exchange. It is interesting to construct such an effective theory. In particular, the Galilean (or even Poincaré) symmetry is then non-linearly realized.

To summarize, low-energy effective field theory is a powerful tool that has several advantages compared to the direct use of microscopic models. First, it is modelindependent and provides universal predictions. Material-specific details of the underlying microscopic system enter the effective theory only through low-energy parameters whose values can be determined by comparison with experiments or with numerical simulations. Second, and most important, the electrons or holes of the effective theory are quasi-particles whose long-range forces are weak and calculable in perturbation theory. This is a significant advantage compared to calculations in microscopic models of strongly correlated electrons which are necessarily nonperturbative. While it is practically impossible to reliably determine the long-range forces between charge carriers from Hubbard-type models, in the effective theory the calculation of the one-magnon exchange forces is straightforward and presently in progress. It is very interesting to ask if these forces will provide a potential mechanism for the preformation of electron or hole pairs. In any case, we propose the systematic low-energy effective field theory approach as a better compromise between calculability and predictive power than the one offered by Hubbard-type models. Effective field theory sheds new light on the dynamics of charge carriers in antiferromagnets, and there is hope that it may even be applicable to the hightemperature superconductors themselves.

## Acknowledgements

We have benefitted from discussions with M. Bissegger, S. Chandrasekharan, G. Colangelo, J. Gasser, P. Hasenfratz, H. Leutwyler, P. Minkowski, and F. Niedermayer. This work is supported by funds provided by the Schweizerischer National-fonds.

## A Electron-Hole Representation of the Hubbard Model Operators

For  $U \gg |t|$  the Hubbard model at half-filling reduces to the antiferromagnetic quantum Heisenberg model. In contrast to the Heisenberg ferromagnet, the ground state of the antiferromagnet is not known analytically. In particular, the naive Néel state

$$|N\rangle = \prod_{x \in A} c_{x\downarrow}^{\dagger} \prod_{x \in B} c_{x\uparrow}^{\dagger} |0\rangle, \qquad (A.1)$$

with all spins down on the even sublattice A and all spins up on the odd sublattice B is not an eigenstate of the Hubbard Hamiltonian. Still, we use this state in order to define electron and hole operators. For even sites we then find

$$c_{x\uparrow}|N\rangle = 0, \quad c_{x\downarrow}^{\dagger}|N\rangle = 0, \quad x \in A.$$
 (A.2)

Correspondingly,  $c_{x\uparrow}^{\dagger}$  creates an electron, while  $c_{x\downarrow}$  creates a hole. Hence, just like a relativistic Dirac spinor, the  $SU(2)_s$  spinor

$$c_x = \begin{pmatrix} c_{x\uparrow} \\ c_{x\downarrow} \end{pmatrix} = \begin{pmatrix} a_{x\uparrow} \\ b_{x\uparrow}^{\dagger} \end{pmatrix}, \quad x \in A,$$
(A.3)

consists of a particle annihilation operator  $a_{x\uparrow}$  in the upper component and a hole creation operator  $b_{x\uparrow}^{\dagger}$  in the lower component. Note that the annihilation of an electron with spin down via  $c_{x\downarrow}$  corresponds to the creation of a hole with spin up via  $b_{x\uparrow}^{\dagger}$ . Similarly, on the odd sites one has

$$c_{x\downarrow}|N\rangle = 0, \quad c^{\dagger}_{x\uparrow}|N\rangle = 0, \quad x \in B.$$
 (A.4)

In this case,  $c_{x\downarrow}^{\dagger}$  creates a particle, while  $c_{x\uparrow}$  creates a hole and we write

$$c_x = \begin{pmatrix} c_{x\uparrow} \\ c_{x\downarrow} \end{pmatrix} = \begin{pmatrix} b_{x\downarrow}^{\dagger} \\ a_{x\downarrow} \end{pmatrix}, \quad x \in B.$$
(A.5)

## B Removal of Non-Canonical Terms by a Field Redefinition

The most general  $SU(2)_Q$ -breaking but  $U(1)_Q$ -symmetric terms containing one covariant time-derivative are given by

$$\frac{a}{2} \operatorname{Tr}[\Psi^{A\dagger} D_t \Psi^A + \Psi^{B\dagger} D_t \Psi^B] + \frac{b}{2} \operatorname{Tr}[\Psi^{A\dagger} \sigma_3 D_t \Psi^A \sigma_3 - \Psi^{B\dagger} \sigma_3 D_t \Psi^B \sigma_3] 
+ \frac{c}{2} \operatorname{Tr}[\Psi^{A\dagger} D_t \Psi^B \sigma_3 + \Psi^{B\dagger} D_t \Psi^A \sigma_3] 
= (\psi_+^{A\dagger}, \psi_+^{B\dagger}) \begin{pmatrix} a+b & c \\ c & a-b \end{pmatrix} \begin{pmatrix} D_t \psi_+^A \\ D_t \psi_+^B \end{pmatrix} 
+ (\psi_-^{A\dagger}, \psi_-^{B\dagger}) \begin{pmatrix} a-b & c \\ c & a+b \end{pmatrix} \begin{pmatrix} D_t \psi_-^A \\ D_t \psi_-^B \end{pmatrix} 
= (\tilde{\psi}_+^{A\dagger}, \tilde{\psi}_+^{B\dagger}) \begin{pmatrix} D_t \tilde{\psi}_+^A \\ D_t \tilde{\psi}_+^B \end{pmatrix} + (\tilde{\psi}_-^{A\dagger}, \tilde{\psi}_-^{B\dagger}) \begin{pmatrix} D_t \tilde{\psi}_-^A \\ D_t \tilde{\psi}_-^B \end{pmatrix}.$$
(B.1)

Here  $\tilde{\psi}^{A,B}_{\pm}(x)$  results from a field redefinition that diagonalizes the matrices in the previous expression. Only the term proportional to a contains the standard form  $\psi^{A\dagger}_{+}\partial_t\psi^A_{+} + \psi^{A\dagger}_{-}\partial_t\psi^B_{-} + \psi^{B\dagger}_{+}\partial_t\psi^B_{+} + \psi^{B\dagger}_{-}\partial_t\psi^B_{-}$  which implies canonical anticommutation relations between fermionic creation and annihilation operators in the Hamiltonian formulation. The non-canonical terms (proportional to b and c) can be removed by an appropriate field redefinition

$$\begin{pmatrix} \tilde{\psi}_{\pm}^{A}(x) \\ \tilde{\psi}_{\pm}^{B}(x) \end{pmatrix} = \begin{pmatrix} \sqrt{\lambda_{\pm}} & 0 \\ 0 & \sqrt{\lambda_{\mp}} \end{pmatrix} U_{\pm} \begin{pmatrix} \psi_{\pm}^{A}(x) \\ \psi_{\pm}^{B}(x) \end{pmatrix},$$

$$\lambda_{\pm} = a \pm \sqrt{b^2 + c^2}, \quad U_{\pm} = \begin{pmatrix} X & \pm Y \\ \mp Y & X \end{pmatrix}.$$
(B.2)

Here  $U_{\pm}$  are unitary matrices with  $X, Y \in \mathbb{R}$  which obey

$$U_{\pm} \begin{pmatrix} a \pm b & c \\ c & a \mp b \end{pmatrix} U_{\pm}^{\dagger} = \begin{pmatrix} \lambda_{\pm} & 0 \\ 0 & \lambda_{\mp} \end{pmatrix}.$$
 (B.3)

It is straightforward to show that the redefined fields  $\tilde{\psi}^{A,B}_{\pm}(x)$  have the same symmetry properties of eqs.(5.4) and (5.6) as the original fields  $\psi^{A,B}_{\pm}(x)$ . Under the  $SU(2)_s$  symmetry the original fields transform as

$$\psi_{\pm}^{A,B}(x)' = \exp(\pm i\alpha(x))\psi_{\pm}^{A,B}(x),$$
 (B.4)

and after the field redefinition again

$$\begin{split} \widetilde{\psi}^{A}_{\pm}(x)' &= \sqrt{\lambda_{\pm}} [X\psi^{A}_{\pm}(x)' \pm Y\psi^{B}_{\pm}(x)'] \\ &= \exp(\pm i\alpha(x))\sqrt{\lambda_{\pm}} [X\psi^{A}_{\pm}(x) \pm Y\psi^{B}_{\pm}(x)] = \exp(\pm i\alpha(x))\widetilde{\psi}^{A}_{\pm}(x), \\ \widetilde{\psi}^{B}_{\pm}(x)' &= \sqrt{\lambda_{\mp}} [\mp Y\psi^{A}_{\pm}(x)' + X\psi^{B}_{\pm}(x)'] \\ &= \exp(\pm i\alpha(x))\sqrt{\lambda_{\mp}} [\mp Y\psi^{A}_{\pm}(x) + X\psi^{B}_{\pm}(x)] = \exp(\pm i\alpha(x))\widetilde{\psi}^{B}_{\pm}(x). \quad (B.5) \end{split}$$

Similarly, under the  $U(1)_Q$  symmetry the original fields transform as

$${}^{Q}\psi^{A,B}_{\pm}(x) = \exp(i\omega)\psi^{A,B}_{\pm}(x), \qquad (B.6)$$

and again

$${}^{Q}\widetilde{\psi}^{A}_{\pm}(x) = \sqrt{\lambda_{\pm}} [X {}^{Q}\psi^{A}_{\pm}(x) \pm Y {}^{Q}\psi^{B}_{\pm}(x)]$$

$$= \exp(i\omega)\sqrt{\lambda_{\pm}} [X\psi^{A}_{\pm}(x) \pm Y\psi^{B}_{\pm}(x)] = \exp(i\omega)\widetilde{\psi}^{A}_{\pm}(x),$$

$${}^{Q}\widetilde{\psi}^{B}_{\pm}(x) = \sqrt{\lambda_{\mp}} [\mp Y {}^{Q}\psi^{A}_{\pm}(x) + X {}^{Q}\psi^{B}_{\pm}(x)]$$

$$= \exp(i\omega)\sqrt{\lambda_{\mp}} [\mp Y\psi^{A}_{\pm}(x) + X\psi^{B}_{\pm}(x)] = \exp(i\omega)\widetilde{\psi}^{B}_{\pm}(x). \quad (B.7)$$

Under the modified displacement symmetry D' one has

$${}^{D'}\psi^{A,B}_{\pm}(x) = \pm \psi^{B,A}_{\mp}(x),$$
 (B.8)

and after the field redefinition one again obtains

$$D' \tilde{\psi}^{A}_{\pm}(x) = \sqrt{\lambda_{\pm}} [X \ ^{D'} \psi^{A}_{\pm}(x) \pm Y \ ^{D'} \psi^{B}_{\pm}(x)]$$

$$= \pm \sqrt{\lambda_{\pm}} [X \psi^{B}_{\mp}(x) \pm Y \psi^{A}_{\mp}(x)] = \pm \tilde{\psi}^{B}_{\mp}(x),$$

$$D' \tilde{\psi}^{B}_{\pm}(x) = \sqrt{\lambda_{\mp}} [\mp Y \ ^{D'} \psi^{A}_{\pm}(x) + X \ ^{D'} \psi^{B}_{\pm}(x)]$$

$$= \pm \sqrt{\lambda_{\mp}} [\mp Y \psi^{B}_{\mp}(x) + X \psi^{A}_{\mp}(x)] = \pm \tilde{\psi}^{A}_{\mp}(x). \quad (B.9)$$

Since the displacement symmetry D is a combination of D' and  $SU(2)_s$  it also maintains its original form. The same is true for the discrete symmetries O and R. Finally, under the modified time-reversal T' the original fields transform as

$$T'\psi_{\pm}^{A,B}(x) = -\psi_{\pm}^{A,B\dagger}(Tx),$$
 (B.10)

such that

$${}^{T'}\tilde{\psi}_{\pm}^{A}(x) = \sqrt{\lambda_{\pm}} [X {}^{T'}\psi_{\pm}^{A}(x) \pm Y {}^{T'}\psi_{\pm}^{B}(x)]$$
  
=  $-\sqrt{\lambda_{\pm}} [X\psi_{\pm}^{A\dagger}(Tx) \pm Y\psi_{\pm}^{B\dagger}(Tx)] = -\tilde{\psi}_{\pm}^{A\dagger}(Tx),$   
 ${}^{T'}\tilde{\psi}_{\pm}^{B}(x) = \sqrt{\lambda_{\mp}} [\mp Y {}^{T'}\psi_{\pm}^{A}(x) + X {}^{T'}\psi_{\pm}^{B}(x)]$   
=  $-\sqrt{\lambda_{\mp}} [\mp Y\psi_{\pm}^{A\dagger}(Tx) + X\psi_{\pm}^{B\dagger}(Tx)] = -\tilde{\psi}_{\pm}^{B\dagger}(Tx).$  (B.11)

As a combination of T' and  $SU(2)_s$  the time-reversal symmetry T also maintains its original form after the field redefinition. The only symmetry that does not maintain its original form is  $SU(2)_Q$ . This is no problem since the non-canonical terms can arise only when the  $SU(2)_Q$  symmetry is explicitly broken down to  $U(1)_Q$  and is hence no longer a symmetry of the theory.

Since the redefined fields transform exactly like the original ones, the terms in the effective Lagrangian take exactly the same form as before. Hence, it is indeed justified not to include the non-canonical terms in the effective Lagrangian.

## References

- [1] J. C. Bednorz and K. A. Müller, Z. Phys. B64 (1986) 189.
- [2] W. F. Brinkman and T. M. Rice, Phys. Rev. B2 (1970) 1324.
- [3] J. E. Hirsch, Phys. Rev. Lett. 54 (1985) 1317.
- [4] P. W. Anderson, Science 235 (1987) 1196.
- [5] C. Gros, R. Joynt, and T. M. Rice, Phys. Rev. B36 (1987) 381.
- [6] B. I. Shraiman and E. D. Siggia, Phys. Rev. Lett. 60 (1988) 740; Phys. Rev. Lett. 61 (1988) 467; Phys. Rev. Lett. 62 (1989) 1564; Phys. Rev. B46 (1992) 8305.
- [7] S. A. Trugman, Phys. Rev. B37 (1988) 1597.
- [8] J. R. Schrieffer, X. G. Wen, and S. C. Zhang, Phys. Rev. Lett. 60 (1988) 944; Phys. Rev. B39 (1989) 11663.
- [9] C. L. Kane, P. A. Lee, and N. Read, Phys. Rev. B39 (1989) 6880.

- [10] S. Sachdev, Phys. Rev. B39 (1989) 12232.
- [11] X. G. Wen, Phys. Rev. B39 (1989) 7223.
- [12] A. Krasnitz, E. G. Klepfish, and A. Kovner, Phys. Rev. B39 (1989) 9147.
- [13] R. Shankar, Phys. Rev. Lett. 63 (1989) 203; Nucl. Phys. B330 (1990) 433.
- [14] P. W. Anderson, Phys. Rev. Lett. 64 (1990) 1839.
- [15] A. Singh and Z. Tesanovic, Phys. Rev. B41 (1990) 614.
- [16] S. A. Trugman, Phys. Rev. B41 (1990) 892.
- [17] V. Elser, D. A. Huse, B. I. Shraiman, and E. D. Siggia, Phys. Rev. B41 (1990) 6715.
- [18] E. Dagotto, R. Joynt, A. Moreo, S. Bacci, and E. Gagliano, Phys. Rev. B41 (1990) 9049.
- [19] G. Vignale and M. R. Hedayati, Phys. Rev. B42 (1990) 786.
- [20] P. Kopietz, Phys. Rev. B42 (1990) 1029.
- [21] J. A. Verges, E. Louis, P. S. Lomdahl, F. Guinea, and A. R. Bishop, Phys. Rev. B43 (1991) 6099.
- [22] F. Marsiglio, A. E. Ruckenstein, S. Schmitt-Rink, and C. M. Varma, Phys. Rev. B43 (1991) 10882.
- [23] P. Monthoux, A. V. Balatsky, and D. Pines, Phys. Rev. Lett. 67 (1991) 3448.
- [24] Z. Liu and E. Manousakis, Phys. Rev. B45 (1992) 2425.
- [25] C. Kübert and A. Muramatsu, Phys. Rev. B47 (1993) 787; cond-mat/9505105.
- [26] T. Dahm, J. Erdmenger, K. Scharnberg, and C. T. Rieck, Phys. Rev. B48 (1993) 3896.
- [27] O. P. Sushkov, Phys. Rev. B49 (1994) 1250.
- [28] E. Dagotto, Rev. Mod. Phys. 66 (1994) 763.
- [29] J. Altmann, W. Brenig, A. P. Kampf, and E. Müller-Hartmann, Phys. Rev. B52 (1995) 7395.
- [30] B. Kyung and S. I. Mukhin, Phys. Rev. B55 (1997) 3886.
- [31] A. V. Chubukov and D. K. Morr, Phys. Rev. B57 (1998) 5298.
- [32] N. Karchev, Phys. Rev. B57 (1998) 10913.

- [33] W. Apel, H.-U. Everts, and U. Körner, Eur. Phys. J. B5 (1998) 317.
- [34] M. Brunner, F. F. Assaad, and A. Muramatsu, Phys. Rev. B62 (2000) 15480.
- [35] S. Sachdev, Rev. Mod. Phys. 75 (2003) 913; Annals Phys. 303 (2003) 226.
- [36] W. V. Liu and F. Wilczek, cond-mat/0312685.
- [37] B. O. Wells, Z.-X. Shen, D. M. King, M. H. Kastner, M. Greven, and R. J. Birgeneau, Phys. Rev. Lett. 74 (1995) 964.
- [38] S. La Rosa, I. Vobornik, F. Zwick, H. Berger, M. Grioni, G. Margaritondo, R. J. Kelley, M. Onellion, and A. Chubukov, Phys. Rev. B56 (1997) R525.
- [39] C. Kim, P. J. White, Z.-X. Shen, T. Tohyama, Y. Shibata, S. Maekawa, B. O. Wells, Y. J. Kim, R. J. Birgeneau, and M. A. Kastner, Phys. Rev. Lett. 80 (1998) 4245.
- [40] F. Ronning, C. Kim, D. L. Feng, D. S. Marshall, A. G. Loeser, L. L. Miller, J. N. Eckstein, I. Borovic, and Z. X. Shen, Science 282 (1998) 2067.
- [41] S. Weinberg, Physica 96 A (1979) 327.
- [42] J. Gasser and H. Leutwyler, Nucl. Phys. B250 (1985) 465.
- [43] H. Leutwyler, Phys. Rev. D49 (1994) 3033.
- [44] C. P. Hofmann, Phys. Rev. B60 (1999) 388; Phys. Rev. B60 (1999) 406; condmat/0106492; cond-mat/0202153.
- [45] H. Neuberger and T. Ziman, Phys. Rev. B39 (1989) 2608.
- [46] D. S. Fisher, Phys. Rev. B39 (1989) 11783.
- [47] P. Hasenfratz and H. Leutwyler, Nucl. Phys. B343 (1990) 241.
- [48] P. Hasenfratz and F. Niedermayer, Z. Phys. B92 (1993) 91.
- [49] A. Chubukov, T. Senthil, and S. Sachdev, Phys. Rev. Lett. 72 (1994) 2089;
   Nucl. Phys. B426 (1994) 601.
- [50] J. M. Roman and J. Soto, Int. J. Mod. Phys. B13 (1999) 755; Ann. Phys. 273 (1999) 37; Phys. Rev. B59 (1999) 11418; Phys. Rev. B62 (2000) 3300.
- [51] K. Rajagopal and F. Wilczek, At the frontier of Particle Physics/ Handbook of QCD, edited by M. Shifman, hep-ph/0011333.
- [52] S. Weinberg, Phys. Lett. B251 (1990) 288; Nucl. Phys. B363 (1991) 3; Phys. Lett. B295 (1992) 114.

- [53] D. B. Kaplan, M. J. Savage, and M. B. Wise, Phys. Lett. B424 (1998) 390; Nucl. Phys. B534 (1998) 329.
- [54] E. Epelbaum, W. Glöckle, and U.-G. Meissner, Nucl. Phys. A637 (1998) 107;
   Nucl. Phys. A684 (2001) 371; Nucl. Phys. A714 (2003) 535.
- [55] P. F. Bedaque, H.-W. Hammer, and U. van Kolck, Phys. Rev. C58 (1998) 641;
   Phys. Rev. Lett. 82 (1999) 463; Nucl. Phys. A676 (2000) 357.
- [56] U. van Kolck, Prog. Part. Nucl. Phys. 43 (1999) 337.
- [57] T.-S. Park, K. Kubodera, D.-P. Min, and M. Rho, Nucl. Phys. A646 (1999) 83.
- [58] Epelbaum, H. Kamada, A. Nogga, H. Witali, W. Glöckle, and U.-G. Meissner, Phys. Rev. Lett. 86 (2001) 4787.
- [59] S. Beane, P. F. Bedaque, M. J. Savage, and U. van Kolck, Nucl. Phys. A700 (2002) 377.
- [60] P. F. Bedaque and U. van Kolck, Ann. Rev. Nucl. Part. Sci. 52 (2002) 339.
- [61] A. Nogga, R. G. E. Timmermans, and U. van Kolck, nucl-th/0506005.
- [62] M. Rho, A. Wirzba, and I. Zahed, Phys. Lett. B473 (2000) 126.
- [63] B.-Y. Park, M. Rho, A. Wirzba, and I. Zahed, Phys. Rev. D62 (2000) 034015.
- [64] H. W. Müller, S. E. Koonin, R. Seki, and U. van Kolck, Phys. Rev. C61 (2000) 044320.
- [65] U.-G. Meissner, J. A. Oller, and A. Wirzba, Annals Phys. 297 (2002) 27.
- [66] A. Wirzba, Prog. Part. Nucl. Phys. 50 (2003) 217.
- [67] S. Chakravarty, B. I. Halperin, and D. R. Nelson, Phys. Rev. B39 (1989) 2344.
- [68] P. Hasenfratz and F. Niedermayer, Phys. Lett. B268 (1991) 231.
- [69] H. Georgi, Weak Interactions and Modern Particle Theory, Benjamin-Cummings Publishing Company, 1984.
- [70] J. Gasser, M. E. Sainio, and A. Svarc, Nucl. Phys. B307 (1988) 779.
- [71] E. Jenkins and A. Manohar, Phys. Lett. B255 (1991) 558.
- [72] V. Bernard, N. Kaiser, J. Kambor, and U.-G. Meissner, Nucl. Phys. B388 (1992) 315.
- [73] T. Becher and H. Leutwyler, Eur. Phys. J. C9 (1999) 643.
- [74] S. Coleman, J. Wess, and B. Zumino, Phys. Rev. 177 (1969) 2239.

- [75] C. G. Callan, S. Coleman, J. Wess, and B. Zumino, Phys. Rev. 177 (1969) 2247.
- [76] S. Zhang, Phys. Rev. Lett. 65 (1990) 120.
- [77] C. N. Yang and S. Zhang, Mod. Phys. Lett. B4 (1990) 759.
- [78] J. Fröhlich and U. M. Studer, Commun. Math. Phys. 148 (1992) 553; Int. J. Mod. Phys. B6 (1992) 2201; Rev. Mod. Phys. 63 (1993) 733.
- [79] O. Bär, M. Imboden, and U.-J. Wiese, Nucl. Phys. B686 (2004) 347.
- [80] S. Chandrasekharan, Pramana 61 (2003) 901.
- [81] U.-J. Wiese and H.-P. Ying, Z. Phys. B93 (1994) 147.
- [82] B. B. Beard and U.-J. Wiese, Phys. Rev. Lett. 77 (1996) 5130.
- [83] T. Skyrme, Proc. Roy. Soc. A260 (1961) 127; Nucl. Phys. 31 (1962) 556.
- [84] F. Wilczek and A. Zee, Phys. Rev. Lett. 51 (1983) 2250.
- [85] X. G. Wen and A. Zee, Phys. Rev. Lett. 61 (1988) 1025.
- [86] F. D. M. Haldane, Phys. Rev. Lett. 61 (1988) 1029.
- [87] T. Dombre and N. Read, Phys. Rev. B38 (1988) 7181.
- [88] E. Fradkin and M. Stone, Phys. Rev. B38 (1988) 7215.
- [89] N. Read and S. Sachdev, Phys. Rev. Lett. 62 (1989) 1694; Nucl. Phys. B316 (1989) 609.
- [90] J. A. M. Vermaseren, "New Features of FORM", math-ph/0010025.
- [91] H. Leutwyler, Phys. Lett. B189 (1987) 197.
- [92] S. Chandrasekharan, M. Pepe, F. Steffen, and U.-J. Wiese, JHEP 0312 (2003) 207.