

# Records in a changing world

**Joachim Krug**

Institut für Theoretische Physik, Universität zu Köln, 50937 Köln, Germany

E-mail: [krug@thp.uni-koeln.de](mailto:krug@thp.uni-koeln.de)

**Abstract.** In the context of this paper, a record is an entry in a sequence of random variables (RV's) that is larger or smaller than all previous entries. After a brief review of the classic theory of records, which is largely restricted to sequences of independent and identically distributed (i.i.d.) RV's, new results for sequences of independent RV's with distributions that broaden or sharpen with time are presented. In particular, we show that when the width of the distribution grows as a power law in time  $n$ , the mean number of records is asymptotically of order  $\ln n$  for distributions with a power law tail (the *Fréchet class* of extremal value statistics), of order  $(\ln n)^2$  for distributions of exponential type (*Gumbel class*), and of order  $n^{1/(\nu+1)}$  for distributions of bounded support (*Weibull class*), where the exponent  $\nu$  describes the behaviour of the distribution at the upper (or lower) boundary. Simulations are presented which indicate that, in contrast to the i.i.d. case, the sequence of record breaking events is correlated in such a way that the variance of the number of records is asymptotically smaller than the mean.

## 1. Introduction

A record is an entry in a discrete time series that is larger (*upper record*) or smaller (*lower record*) than all previous entries. Thus, records are extreme values that are defined not relative to a fixed threshold, but relative to all preceding events that have occurred since the beginning of the process. Statistical data in areas like meteorology [1, 2, 3, 4], hydrology [5, 6] and athletics [7] are naturally represented in terms of records. Records play an important role in the public perception of issues like anthropogenic climate change and natural disasters such as floods and earthquakes, and they are an integral part of popular culture. Indeed, the *Guinness Book of Records*, first published in 1955, is the world's most sold copyrighted book.

The mathematical theory of records was initiated more than 50 years ago [8], and it is now a mature subfield of probability theory and statistics; see [9, 10, 11, 12] for reviews and [13] for an elementary introduction. Most of this work has been devoted to the case when the time series under consideration consists of independent, identically distributed (i.i.d.) random variables (RV's). For the following discussion, it will be useful to distinguish between the *record times* at which the current record is broken and replaced by a new one, and the associated *record values*. One of the key results of record theory is that the statistical properties of record times for real-valued i.i.d. RV's

are completely independent of the underlying distribution. To illustrate the origin of this universality, we recall the basic observation that the probability  $P_n$  for a record to occur in the  $n$ 'th time step (the *record rate*) is given by

$$P_n = \frac{1}{n} \quad (1)$$

for i.i.d. RV's, because each of the  $n$  first entries  $X_1, \dots, X_n$ , including the last, is equally likely to be the largest or smallest. The mean number of records up to time  $n$ ,  $\overline{R}_n$ , is therefore given by the harmonic series

$$\overline{R}_n = \sum_{k=1}^n P_k = \sum_{k=1}^n \frac{1}{k} \approx \ln n + \gamma + \mathcal{O}(1/n) \quad \text{for } n \rightarrow \infty, \quad (2)$$

with  $\gamma \approx 0.5772156649\dots$ . Further considerations along the same lines lead to a remarkably complete characterization of record times, which will be briefly reviewed below in section 2.1. The universality of record times can be exploited in statistical tests of the i.i.d. property of a given sequence of variables, without the need for any hypothesis about the underlying distribution [9]. By contrast, distributions of record values fall into three distinct universality classes, which are largely analogous to the well-known asymptotic laws of extreme value statistics for distributions with exponential-like tails (*Gumbel*), bounded support (*Weibull*) and power law tails (*Fréchet*), respectively [14, 15].

The decay of the record rate (1) with increasing  $n$  implies that the record breaking events form a non-stationary time series with unusual statistical properties, which will be further discussed below in section 2.1. *Record dynamics* has therefore been proposed as a paradigm for the non-stationary temporal behaviour of diverse complex systems ranging from the low-temperature relaxation of spin glasses to the co-evolution of biological populations [16, 17, 18]. In fact, records appear naturally in the theory of biological adaptation, because any evolutionary innovation that successfully spreads in a population must be a record, in the sense that it accomplishes some task encountered by the organism in a way that is superior to all previously existing solutions. Consequently the statistics of records and extremes has been invoked to understand the distribution of fitness increments in adaptive processes [19, 20] as well as the timing of adaptive events [21, 22, 23, 24, 25, 26]. In the biological context the universality of record time statistics is particularly attractive, because genotypical fitness is a somewhat elusive notion that is hard to quantify in terms of explicit probability distributions.

Surprisingly few result on record statistics are known that go beyond the standard setting of i.i.d. RV's, and thus consider correlated and/or non-identically distributed RV's. In the present article we focus exclusively on the latter issue, while maintaining the independence among the entries in the sequence. A simple example of this type was introduced by Yang in an attempt to explain the frequency of occurrence of Olympic records, which is much higher than would be expected on the basis of the i.i.d. theory [27]. In his model a specified number of i.i.d. RV's become available simultaneously in each time step, corresponding, in the athletic context, to a variable (growing) population from which the contenders are drawn. Much of the standard theory can be extended to

this case [10, 12] (see section 2.2 for a brief review). In particular, one finds that the record rate becomes asymptotically constant for exponentially growing populations. An application of Yang's model to evolutionary searches in the space of genotypic sequences can be found in [24, 25]. A second line of research has addressed the case of sequences with a linear trend, in which the  $n$ 'th entry is of the form

$$X_n = Y_n + cn \quad (3)$$

with i.i.d. RV's  $Y_n$  and  $c > 0$  [28, 29, 30]. Also in this case the record rate becomes asymptotically constant, see section 2.2 for details.

The effect of trends on the occurrence rate of records is a key issue in the ongoing debate about the observable consequences of global warming [1, 2, 3, 4, 5]. In this context it has been pointed out that climate *variability* is presumably a more important factor in determining the frequency of extreme events than averages [31]. It is therefore of considerable interest to investigate the record statistics of sequences of uncorrelated RV's in which the *shape* of the underlying probability distribution changes systematically with time. To initiate such an investigation is the goal of the present paper. Throughout we assume that the probability density  $p_n(X)$  of the  $n$ 'th entry  $X_n$  is of the form

$$p_n(X) = \lambda_n \Pi(\lambda_n X) \quad (4)$$

where  $\Pi(X)$  is a fixed normalized distribution and the  $\lambda_n$  usually have a power-law time dependence

$$\lambda_n = \lambda_0 n^{-\alpha}, \quad (5)$$

so that  $\alpha > 0$  ( $\alpha < 0$ ) corresponds to a broadening (sharpening) distribution.

After a brief review of a few important classic results of the theory of records in section 2, our new results for non-identically distributed random variables will be presented in section 3. We focus on the asymptotic behaviour of the record rate  $P_n$  and the mean number of records  $\overline{R}_n$ . Preliminary numerical results for the variance of the number of records are reported in section 3.3, but a more complete characterization of record times and record values is left for future work. Finally, some concluding remarks are offered in section 4.

## 2. Brief survey of classic results

Given the distributions  $p_k(X)$  of the entries  $X_k$  in a sequence of independent RV's, the probability  $P_n$  that the  $n$ 'th entry is an upper record is equal to the probability that  $X_n > X_k$  for all  $k < n$ . Hence<sup>‡</sup>

$$P_n = \int dX_n p_n(X_n) \prod_{k=1}^{n-1} q_k(X_n), \quad (6)$$

<sup>‡</sup> Here and in the following limits of integration are omitted whenever the domain of integration is understood to comprise the entire support of the probability distribution.

where

$$q_k(X) = \int^X dx p_k(x) \quad (7)$$

is the cumulative distribution of  $X_k$ . Similarly the probability that  $X_n$  is a lower record reads

$$P_n^* = \int dX_n p_n(X_n) \prod_{k=1}^{n-1} [1 - q_k(X_n)]. \quad (8)$$

Equations (6) and (8) form the basis for most of what follows.

### 2.1. Records from i.i.d. random variables

For i.i.d. RV's the integral (6) can be performed by noting that  $p_k, q_k \equiv p, q$  and  $dq = p dX$ , which yields the universal result (1). To arrive at a characterization of the record time process beyond the mean number of records  $\overline{R_n}$  we introduce the *record indicator variables*  $I_n$ , which take the value  $I_n = 1$  iff  $X_n$  is a record, and  $I_n = 0$  else. It turns out that the  $I_n$  are independent [9, 11], and hence they form a Bernoulli process with success probability  $P_n$ . To see why this is so, consider the two-point correlation function  $\overline{I_i I_j}$  and assume that  $j > i$ . Then the key idea is that the right hand side of

$$\overline{I_i I_j} = \text{Prob}[X_i = \max(X_1, \dots, X_i) \text{ and } X_j = \max(X_1, \dots, X_j)] \quad (9)$$

can be split into independent events according to

$$\begin{aligned} \overline{I_i I_j} &= \text{Prob}[X_i = \max(X_1, \dots, X_i)] \times \text{Prob}[X_j = \max(X_{i+1}, \dots, X_j)] \times \\ &\quad \times \text{Prob}[\max(X_1, \dots, X_i) < \max(X_{i+1}, \dots, X_j)]. \end{aligned} \quad (10)$$

Following the symmetry argument used to derive (1), the first two factors are  $1/i$  and  $1/(j-i)$ , respectively, and the third factor can be written as

$$\text{Prob}[\max(X_1, \dots, X_j) \text{ occurs in } \{X_{i+1}, \dots, X_j\}] = \frac{j-i}{j}. \quad (11)$$

We conclude that

$$\overline{I_i I_j} = \frac{1}{i} \frac{1}{j-i} \frac{j-i}{j} = \frac{1}{i} \frac{1}{j} = P_i P_j = (\overline{I_i})(\overline{I_j}). \quad (12)$$

Higher order correlations can be shown to factorize in the same way. The number  $R_n$  of records up to time  $n$  can then be expressed in terms of the indicator variables as

$$R_n = \sum_{k=1}^n I_k, \quad (13)$$

and the variance of  $R_n$  is

$$\overline{(R_n - \overline{R_n})^2} = \sum_{k=1}^n (P_k - P_k^2) = \sum_{k=1}^n \left( \frac{1}{k} - \frac{1}{k^2} \right) \approx \ln n + \gamma - \pi^2/6 + \mathcal{O}(1/n) \quad (14)$$

for  $n \rightarrow \infty$ . The *index of dispersion*  $\rho_n$  of the record time process [19, 32], defined as the ratio of the variance to the mean

$$\rho_n = \frac{\overline{(R_n - \overline{R_n})^2}}{\overline{R_n}} \quad (15)$$

thus tends to unity, and the distribution of the  $R_n$  becomes Poissonian with mean  $\ln n$  for large  $n$ . The record times form a *log-Poisson* process [16, 17, 22].

A second useful observation concerns the ratios between consecutive record times. Let  $t_m$  denote the time of the  $m$ 'th record, with  $t_1 = 1$  by convention. Repeating the symmetry argument used to derive (1), we expect that given  $t_m$ , the preceding  $(m - 1)$ 'th record occurs with equal probability anywhere in the interval  $[1, t_m]$ . This is not quite correct, because the previous  $m - 2$  records also have to be accommodated, but since  $m \sim \ln(t_m)$  this is a small correction which can be neglected for large  $m$ . It is therefore plausible (and can be proved [33]) that the ratio  $t_{m-1}/t_m$  tends to a uniformly distributed RV  $u_m \in [0, 1]$  for large  $m$ . Moreover the  $u_m$  become independent in this limit [34]. This allows us to highlight a peculiar property of the sequence of record breaking events: The expected value of  $t_{m-1}$ , given  $t_m$ , is

$$\overline{t_{m-1}}|_{t_m} = t_m \int_0^1 du u = \frac{1}{2}t_m, \quad (16)$$

but the reverse conditioning yields an *infinite* expectation, because

$$\overline{t_m}|_{t_{m-1}} = t_{m-1} \int_0^1 du u^{-1} = \infty. \quad (17)$$

In this sense, the occurrence of records can be predicted only with hindsight, but not forward in time.

## 2.2. Growing and improving populations

In the model for growing populations introduced by Yang [27] and elaborated by Nevzorov [10], a number  $N_n$  of i.i.d. RV's becomes available simultaneously at time  $n$ . The symmetry argument in section 1 is easily extended to this case: Because of the i.i.d. property, the probability that there is a record among the  $N_n$  newly generated RV's is equal to the ratio of  $N_n$  to the total number of RV's that have appeared up to time  $n$ , and hence

$$P_n = \frac{N_n}{\sum_{k=1}^n N_k}. \quad (18)$$

The independence of the record indicator variables  $Y_n$  introduced above in section 2.1 continues to hold [10, 25], so again the sequence of record breaking events is a Bernoulli process with success probability  $P_n$ .

To give a simple example for the consequences of (18), suppose the  $N_n$  grow exponentially as  $a^n$  with  $a > 1$ . This could model a sequence of athletic competitions in an exponentially growing population, where each athlete is assumed to be able to participate only in one event [27]. Then the evaluation of (18) yields

$$P_n = \frac{a^n(a-1)}{a(a^n-1)} \rightarrow \frac{a-1}{a} \quad \text{for } n \rightarrow \infty, \quad (19)$$

and the distribution of inter-record times  $t_m - t_{m-1}$  is geometric. In his analysis of Olympic records Yang estimated a growth factor of  $a \approx 1.08$  for the four-year period

between two games, and concluded that this growth rate was insufficient to explain the observed high frequency of records.

Motivated by this outcome, Ballerini and Resnick [28] considered a model of *improving* populations, where the sequence of RV's displays a linear drift according to (3). They showed that the record rate tends to an asymptotic limit  $P(c)$  given by

$$P(c) = \lim_{n \rightarrow \infty} P_n = \int dy p(y) G_\infty(y), \quad (20)$$

where  $p(Y)$  is the probability density of the i.i.d. RV's  $Y_k$  in (3) and

$$G_\infty(y) = \lim_{n \rightarrow \infty} \text{Prob}[Y_k - ck \leq y \text{ for all } k = 1, \dots, n-1] = \lim_{n \rightarrow \infty} \prod_{k=1}^{n-1} q(y + ck), \quad (21)$$

with  $q(Y) = \int^Y dz p(z)$ . The function  $P(c)$  has the obvious limits  $P(0) = 0$  and  $\lim_{c \rightarrow \infty} P(c) = 1$ , but the explicit evaluation is generally difficult. A simple expression is obtained when  $q(Y)$  is of Gumbel form,  $q(Y) = \exp[-e^{-Y/b}]$ , which yields  $P(c) = 1 - e^{-c/b}$ . For further details on the model (3) and applications to athletic data we refer to [28, 29, 30]. Results for specific distributions and an application to global warming can be found in [4].

### 3. Records in sequences with increasing or decreasing variance

In this section we want to evaluate the record rates (6) and (8) for distributions of the general form (4). Introducing the cumulative distribution corresponding to  $\Pi(X)$ ,

$$Q(X) = \int^X dx \pi(x), \quad (22)$$

the record rates of interest can be written as

$$P_n = \int dz \Pi(z) \prod_{k=1}^{n-1} Q(z\lambda_k/\lambda_n) \quad (23)$$

and

$$P_n^* = \int dz \Pi(z) \prod_{k=1}^{n-1} [1 - Q(z\lambda_k/\lambda_n)], \quad (24)$$

which makes clear the obvious fact that the overall scale of the  $\lambda_n$ 's is without importance.

#### 3.1. Simple cases

In some special cases the record rates can be evaluated exactly for arbitrary choices of the  $\lambda_n$ 's. For example, for the exponential distribution

$$\Pi(X) = e^{-X}, \quad X \geq 0 \quad (25)$$

we have  $Q(X) = 1 - e^{-X}$ , and the evaluation of the lower record rate (24) yields

$$P_n^* = \frac{\lambda_n}{\sum_{k=1}^n \lambda_k}. \quad (26)$$

Inserting the power law behaviour (5) we see that the denominator converges to the Riemann zeta function  $\zeta(\alpha)$  for  $\alpha > 1$ , so that  $P_n^* \rightarrow n^{-\alpha}/\zeta(\alpha)$  for large  $n$ , and the expected number of lower records

$$\overline{R}_n^* = \sum_{k=1}^n P_k^* \quad (27)$$

remains finite for  $n \rightarrow \infty$ . For  $\alpha < 1$  we have instead that  $P_n^* \approx (1 - \alpha)/n$  for large  $n$ , and hence

$$\overline{R}_n^* \approx (1 - \alpha) \ln n \quad (28)$$

asymptotically. As would be intuitively expected, the occurrence of lower records is enhanced for sharpening distributions ( $\alpha < 0$ ) and suppressed for broadening distributions ( $\alpha > 0$ ). Finally, in the borderline case  $\alpha = 1$  we find

$$\overline{R}_n^* \approx \ln(\ln(n)), \quad (29)$$

which is our first example of a nontrivial asymptotic law that differs qualitatively from the i.i.d. result (2).

A simple explicit expression for the upper record rate  $P_n$  can be obtained for the uniform distribution characterized by

$$Q(X) = X \quad \text{for } 0 \leq X \leq 1 \quad (30)$$

when the  $\lambda_k$  are increasing, in the sense that  $\lambda_k/\lambda_n < 1$  for all  $k < n$ , i.e. for the case of a sharpening uniform distribution. Then the arguments of  $Q$  on the right hand side of (23) are all less than unity, and direct integration yields

$$P_n = \frac{1}{n} \prod_{k=1}^{n-1} \frac{\lambda_k}{\lambda_n}. \quad (31)$$

Inserting the power law form (5) with  $\alpha < 0$  one finds that the record rate decays exponentially as  $P_n \sim e^{\alpha n}$ , and hence the asymptotic number of records is finite for all  $\alpha < 0$ .

### 3.2. Asymptotics of the mean number of records

In this section we focus on broadening distributions,  $\alpha > 0$ , and evaluate the upper record rate (23) asymptotically for representatives of all three universality classes of extreme value statistics. The starting point is to replace the product on the right hand side of (23) by the exponential of a sum of logarithms, and to replace the latter by an integral. It then follows that the asymptotic behaviour of the record rate is given by

$$P_n \approx \int dz \Pi(z) e^{ng_\alpha(z)} = \int_0^1 dQ e^{ng_\alpha(z(Q))}. \quad (32)$$

The second representation will prove to be useful in the final evaluation of  $P_n$ . Note that  $z$  can always be expressed in terms of  $Q$  because  $dQ/dz = \Pi \geq 0$ . The function  $g_\alpha$  is given by

$$g_\alpha(z) = \int_0^1 du \ln Q(z/u^\alpha) \approx - \int_0^1 du (1 - Q(z/u^\alpha)), \quad (33)$$

where in the second step it has been used that the integral in (32) is dominated for large  $n$  by the region where  $g_\alpha \rightarrow 0$  and  $Q \rightarrow 1$ . It is therefore clear that the asymptotic behaviour of the record rate depends only on the tail of  $Q$ , and hence universality in the sense of standard extreme value statistics should apply.

*3.2.1. Fréchet class* The evaluation of (33) is straightforward for the Fréchet class of distributions with power law tails. We set

$$Q(X) = 1 - X^{-\mu}, \quad X \geq 1 \quad (34)$$

and obtain

$$g_\alpha(z) \approx -(1 + \alpha\mu)^{-1} z^{-\mu} = -(1 + \alpha\mu)^{-1} (1 - Q(z)). \quad (35)$$

Inserting this into (32) yields

$$P_n \approx \int_0^1 dQ e^{-n(1+\alpha\mu)^{-1}(1-Q)} \rightarrow \frac{1 + \alpha\mu}{n} \quad (36)$$

for large  $n$ , and hence

$$\overline{R}_n \approx (1 + \alpha\mu) \ln n. \quad (37)$$

This result remains valid for negative  $\alpha$  as long as  $\alpha\mu > -1$ . When  $\alpha\mu < -1$  the evaluation of  $g_\alpha$  shows that  $P_n \sim n^{\alpha\mu}$  and thus the asymptotic number of records remains finite.

*3.2.2. Gumbel class* The Gumbel class comprises unbounded distributions whose tail decays faster than a power law [14, 15]. A typical representative is the exponential distribution (25) with  $Q(X) = 1 - e^{-X}$ . Evaluation of (33) yields

$$g_\alpha(z) \approx -\frac{z^{1/\alpha}}{\alpha} \int_z^\infty dv v^{-(1+1/\alpha)} e^{-v} = -\frac{z^{1/\alpha}}{\alpha} \Gamma(-1/\alpha, z), \quad (38)$$

where  $\Gamma(-1/\alpha, z)$  denotes the incomplete gamma function. For large  $z$  we have [35]

$$\Gamma(-1/\alpha, z) \approx z^{-(1+1/\alpha)} e^{-z}, \quad (39)$$

so that

$$g_\alpha(z) \approx -\frac{e^{-z}}{\alpha z} = \frac{1 - Q(z)}{\alpha \ln(1 - Q(z))}, \quad (40)$$

which yields

$$P_n \approx \int_0^1 dQ \exp\left[\frac{n(1-Q)}{\alpha \ln(1-Q)}\right] = \int_0^1 dv \exp[-nv/(\alpha \ln(1/v))]. \quad (41)$$

To further evaluate the integral we substitute  $w = (n/\ln n)v$  and obtain

$$\begin{aligned} P_n &\approx \frac{\ln n}{n} \int_0^{n/\ln n} dw \exp\left[-\frac{w \ln n}{\alpha(\ln n - \ln(\ln n) - \ln w)}\right] \rightarrow \\ &\rightarrow \frac{\ln n}{n} \int_0^\infty dw e^{-w/\alpha} = \frac{\alpha \ln n}{n} \end{aligned} \quad (42)$$



for  $n \rightarrow \infty$ . Correspondingly the mean number of records grows as

$$\overline{R}_n \approx \frac{\alpha}{2}(\ln n)^2. \quad (43)$$

A second important representative of the Gumbel class is the Gaussian (normal) distribution, for which

$$Q(X) \approx 1 - \frac{1}{2\sqrt{\pi}} \frac{e^{-X^2}}{X} \quad \text{for } X \rightarrow \infty. \quad (44)$$

Proceeding as before, we find

$$g_\alpha(z) \approx -\frac{z^{1/\alpha}}{4\sqrt{\pi}\alpha} \Gamma(-1/2 - 1/2\alpha, z^2) \approx -\frac{1 - Q(z)}{2\alpha z^2} \approx \frac{1 - Q}{2\alpha \ln(1 - Q)}, \quad (45)$$

which becomes identical to (40) upon replacing  $\alpha$  by  $2\alpha$ . We conclude that  $P_n \approx (2\alpha \ln n)/n$  and  $\overline{R}_n \approx 2\alpha(\ln n)^2$  for the Gaussian case. Although this does not constitute a strict proof, it strongly indicates that the behaviour  $\overline{R}_n \sim (\ln n)^2$  is *universal* within this class of probability distributions.

*3.2.3. Weibull class* As a representative of the Weibull class of distributions with finite support we first consider the uniform distribution (30). The integral on the right hand side of (33) can then be evaluated without approximating  $\ln Q$  by  $-(1 - Q)$ , and one obtains

$$g_\alpha(z) = \int_{z^{1/\alpha}}^1 du \ln\left(\frac{z}{u^\alpha}\right) = \ln z + \alpha(1 - z^{1/\alpha}). \quad (46)$$

This is a negative monotonically increasing function which vanishes quadratically in  $1 - z$  near  $z = 1$ ,

$$g_\alpha(z) \approx -\frac{1}{2\alpha}(1 - z)^2 = -\frac{1}{2\alpha}(1 - Q)^2 \quad \text{for } z, Q \rightarrow 1. \quad (47)$$

The evaluation of the record rate (32) then yields

$$P_n \approx \int_0^1 dQ \exp[-n(1 - Q^2)/2\alpha] \approx \sqrt{\frac{\alpha\pi}{2n}} \quad (48)$$

for large  $n$ , and the number of records grows asymptotically as  $\sqrt{n}$ . The specific power is clearly related to the quadratic behaviour of  $g_\alpha$  near  $z = 1$ , which in turn reflects the behaviour of  $Q(X)$  near the upper boundary  $X = 1$ . More generally we may consider bounded distributions of the form

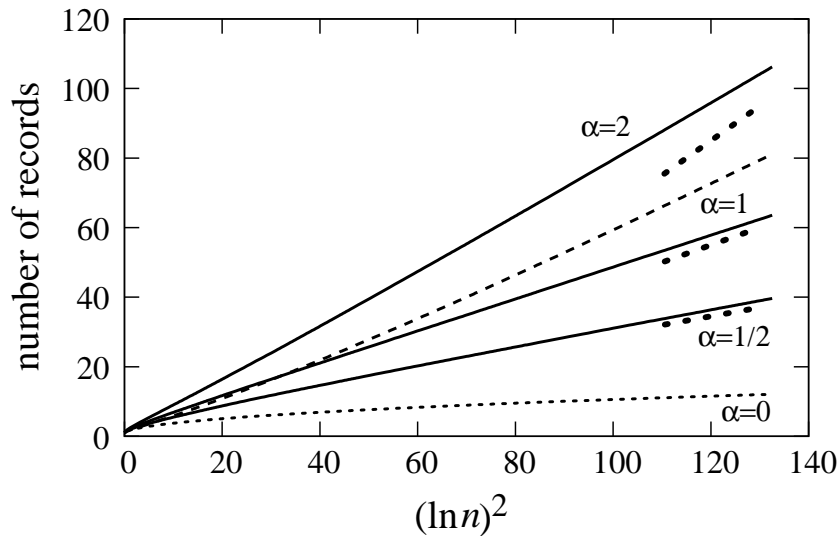
$$Q(X) = 1 - (1 - X)^\nu, \quad 0 \leq X \leq 1, \quad (49)$$

with  $\nu > 0$  and the uniform case corresponding to  $\nu = 1$ . To extract the leading order behaviour of  $g_\alpha$  for  $z \rightarrow 1$  we write

$$\begin{aligned} g_\alpha(z) &\approx -\int_{z^{1/\alpha}}^1 du (1 - z/u^\alpha)^\nu = -\frac{z^{1/\alpha}}{\alpha} \int_z^1 dv v^{-(1+1/\alpha)} (1 - v)^\nu \approx \\ &\approx -\frac{1}{\alpha(\nu + 1)} (1 - z)^{\nu+1} \end{aligned} \quad (50)$$

for  $z \rightarrow 1$ . Hence the record rate decays as  $n^{-\nu/(\nu+1)}$  and the mean number of records grows as

$$\overline{R}_n \approx \frac{\nu\Gamma(\nu/(\nu + 1))}{(\nu + 1)^{1+1/(\nu+1)}} (\alpha^\nu n)^{1/(\nu+1)}. \quad (51)$$

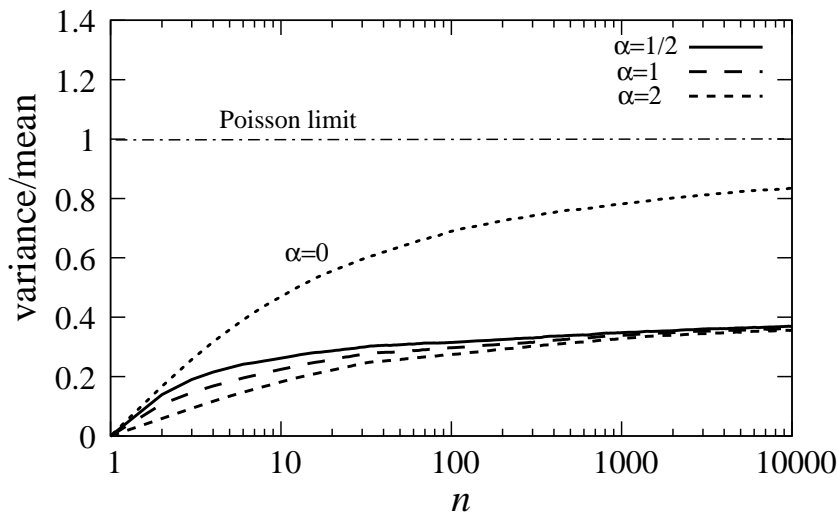


**Figure 1.** Simulation results for the mean number of records for distributions of Gumbel type. Full lines show data obtained for the exponential distribution with  $\alpha = 2$ ,  $\alpha = 1$  and  $\alpha = 1/2$ . The dashed line shows data obtained for the Gaussian distribution and  $\alpha = 1$ . The thin dotted line is the harmonic series (2) which applies universally for  $\alpha = 0$ . The short bold dotted lines show the predicted slope  $\alpha/2$  for the exponential case and  $\alpha$  in the Gaussian case. All data were obtained from  $10^4$  realizations of time series of length  $10^5$ .

### 3.3. Simulations

The asymptotic laws (37, 43, 51) were first discovered in simulations, and they have subsequently been numerically verified for a variety of parameter values. As an example, we show in Figure 1 numerical data for the mean number of records obtained for distributions in the Gumbel class. There are significant corrections to the asymptotic behaviour for the Gaussian distribution as well as for the exponential distribution with  $\alpha = 2$ . This is not surprising in view of the approximations used in the derivation of (43); for example, the last step in (42) requires that  $\ln n \gg \ln(\ln n)$  which is true only for enormously large values of  $n$ .

Simulations have also been used to investigate the occurrence of correlations in the record time process for  $\alpha > 0$ . We have seen in section 2.1 that the Poisson statistics of  $R_n$  is a consequence of the fact that the record indicator variables  $I_n$  are independent in the i.i.d. case. In particular, (14) shows that the variance of  $R_n$  is asymptotically equal to the mean whenever the  $I_n$  are uncorrelated and the record rate  $P_n$  tends to zero for  $n \rightarrow \infty$  in such a way that  $\overline{R_n}$  diverges. As this is true for  $\alpha > 0$  in all cases that we have considered, the index of dispersion (15) can be used as a probe for correlations. The data displayed in Figure 2 clearly show that the asymptotic value of  $\rho_n$  is less than unity and independent of  $\alpha$  for the uniform distribution. Similar results have been obtained for the exponential distribution, whereas we find that  $\rho_n \rightarrow 1$  for the power law case. We conclude that, at least in certain cases, the record time process becomes



**Figure 2.** Simulation results for the ratio of the variance of the number of records to the mean obtained using the uniform distribution with  $\alpha = 0, 1/2, 1$  and  $2$ . While the data for  $\alpha = 0$  approach the asymptotic Poisson limit of unity according to (14), the data for  $\alpha > 0$  converge to a universal sub-Poissonian value. The data were obtained from  $10^5$  realizations of time series of length  $10^4$ .

more regular than the log-Poisson process when the underlying distribution broadens with time.

#### 4. Summary and discussion

The main results of this paper are the asymptotic laws (37, 43, 51) for the mean number of records in sequences of random variables drawn from broadening distributions. In all three cases the exponent  $\alpha$  governing the time dependence of the width of the distribution enters only in the prefactors and does not affect the functional form of the result. Comparing the three cases, we see that the effect of the broadening on  $\overline{R}_n$  is stronger the faster the underlying distribution  $\Pi(X)$  decays for large arguments: For fat-tailed power law distributions the number of records remains logarithmic, for exponential-like distributions it changes from  $\ln n$  to  $(\ln n)^2$ , while for distributions with bounded support the logarithm speeds up to a power law in time.

Apart from the presentation of new results, a secondary purpose of this paper has been to advertise record dynamics as a paradigm of non-stationary point processes with interesting mathematical properties and wide-spread applications ranging from fundamental issues in the dynamics of complex systems to the consequences of climatic change. In the present work we have combined the intrinsic non-stationarity of record dynamics with an explicit non-stationarity of the underlying sequence of random variables. This turns out to be a relevant modification which may alter the basic logarithmic time-dependence of the mean number of records, and it can induce correlations among the record times, as detected in deviations of the index of dispersion (15) from unity. It is worth noting that evidence for such correlations can also be found

in recent applications of record dynamics in simulations of complex systems [17, 18]. An analytic understanding of the origin of correlations in the models presented here is clearly an important goal for the near future.

## Acknowledgments

I am grateful to Kavita Jain for her contributions in the early stages of this project, and to Sid Redner for useful correspondence and discussions. This work was supported by DFG within SFB-TR 12 *Symmetries and universality in mesoscopic systems*.

## References

- [1] Hoyt D V 1981 *Climatic Change* **3** 243
- [2] Bassett Jr. GW 1992 *Climatic Change* **21** 303
- [3] Benestad RE 2003, *Climate Research* **25** 3
- [4] Redner S and Petersen MR 2006 *Phys. Rev. E* **74** 061114
- [5] Matalas NC 1997 *Climatic Change* **37** 89
- [6] Vogel RM, Zafirakou-Koulouris A and Matalas NC 2001 *Water Res. Research* **37** 1723
- [7] Gembris D, Taylor JG and Suter D (2002) *Nature* **417** 506
- [8] Chandler KN 1952 *J. Roy. Stat. Soc. Ser. B* **14** 220
- [9] Glick N 1978 *Amer. Math. Monthly* **85** 2
- [10] Nevzorov VB 1987 *Theory Probab. Appl.* **32** 201
- [11] Arnold BC, Balakrishnan N and Nagaraja HN 1998 *Records* (New York: Wiley)
- [12] Nevzorov VB 2001 *Records: Mathematical Theory* (Providence: American Mathematical Society)
- [13] Schmittmann B and Zia RKP 1999 *Am. J. Phys.* **67** 1269
- [14] Galambos J 1987 *The Asymptotic Theory of Extreme Order Statistics* (Malabar: R.E. Krieger)
- [15] Sornette D 2000 *Critical Phenomena in Natural Sciences* (Berlin: Springer)
- [16] Sibani P and Littlewood P 1993 *Phys. Rev. Lett.* **71** 1482
- [17] Sibani P and Dall J 2003 *Europhys. Lett.* **64** 8
- [18] Anderson PE, Jensen HJ, Oliveira LP and Sibani P 2004 *Complexity* **10** 49
- [19] Gillespie JH 1991 *The Causes of Molecular Evolution* (New York: Oxford University Press)
- [20] Orr HA 2005 *Nature Rev. Gen.* **6** 119
- [21] Kauffman SA and Levin S 1987 *J. theor. Biol.* **128** 11
- [22] Sibani P, Brandt M and Alstrøm P 1991 *Int. J. Mod. Phys. B* **12** 361
- [23] Krug J and Karl C 2003 *Physica A* **318** 137
- [24] Krug J and Jain K 2005 *Physica A* **358** 1
- [25] Jain K and Krug J 2005 *J. Stat. Mech.: Theory and Experiment* P04008
- [26] Sire C, Majumdar SN and Dean DS 2006 *J. Stat. Mech.: Theory and Experiment* L07001
- [27] Yang MCK 1975 *J. Appl. Prob.* **12**, 148
- [28] Ballerini R and Resnick S 1985 *J. Appl. Prob.* **22** 487
- [29] Ballerini R and Resnick S 1987 *Adv. Appl. Prob.* **19** 801
- [30] Borovkov K 1999 *J. Appl. Prob.* **36** 669
- [31] Katz RW and Brown BG 1992 *Climatic Change* **21** 289
- [32] Cox DR and Isham V 1980 *Point processes* (London: Chapman and Hall)
- [33] Tata MN 1969 *Z. Warsch. verw. Geb.* **12** 9
- [34] Shorrock RW 1972 *J. Appl. Prob.* **9** 316
- [35] Gradshteyn IS and Ryzhik IM 2000 *Table of Integrals, Series and Products* (San Diego: Academic Press)