# Effect of Strong Magnetic Fields on the Equilibrium of a Degenerate Gas of Nucleons and Electrons

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#### Abstract

We obtain the equations that define the equilibrium of a homogeneous relativistic gas of neutrons, protons and electrons in a constant magnetic field as applied to the conditions that probably occur near the center of neutron stars. We compute the relative densities of the particles at equilibrium and the Fermi momentum of electrons in the strong magnetic field as function of the density of neutrons and the magnetic field induction. Novel features are revealed as to the ratio of the number of protons to the number of neutrons at equilibrium in the presence of large magnetic fields.

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Significant interest in the behaviors of relativistic electrons in a constant strong magnetic field was greatly stimulated by the discovery of giant magnetic fields at the surface of neutron stars [1, 2]. Observations of hard X-rays from pulsar Hercules X-1 allowed one to estimate the magnetic field induction at the pulsar surface to be of the order of  $10^{13}$  G. Such a magnetic field "frozen" in a neutron star must become stronger and stronger reaching the ultrastrong value of the order of  $10^{18}$  G in the central part of the neutron star [3]. Furthermore, the fields of the order of  $10^{23}$  G [4] and, even possibly  $10^{33}$  G [5] may exist at the electroweak phase transition. It is worthy of note that the magnetic field in a neutron star may be considered as a macroscopically uniform field with respect to the characteristic scales, such as the Compton wavelengths, of constituent particles of the neutron star. However, the field is extremely nonuniform at the electroweak phase transition. In any case, such a strong field will affect in an essential way the behaviors of charged particles in the star.

In this paper, we discuss the effects of strong constant magnetic fields on the chemical equilibrium of a degenerate gas of neutrons, protons, and electrons. We assume in what follows that the gas is spatially homogeneous due to the homogeneity of the magnetic field.

In a very strong magnetic field with induction B higher than  $B_0 = m^2 c^3/e\hbar = 4.41 \cdot 10^{13}$ G (*m* and *e* are the electron mass and charge, respectively), electrons in the lowest Landau energy level "move" in the plane transverse to the magnetic field direction in the region with characteristic dimensions of the order of  $\hbar (B_0/B)^{1/2}/mc$  (see, e.g. [6]). We note that each electron energy level is degenerate with an infinite multiplicity. However, for the case under consideration the number of quantum electron states is restricted by both the value of the magnetic field induction and the size of the region in which the magnetic field acts.

The first physical quantity which we must define is the chemical potential  $\mu_{\rm e}$ , or the Fermi energy (if the gas temperature is equal to zero), of relativistic electrons as a function of magnetic field induction (see also [7]). The electron number density  $n_{\rm e}$  in the presence of a magnetic field  $\mathbf{B} = (0, 0, B)$  is defined by

$$n_{\rm e} = \frac{eB}{4\pi^2\hbar^2 c} \sum_{n,\zeta} \int_{-\infty}^{\infty} \frac{dp}{\exp((E_n - \mu_{\rm e})/\theta) + 1} , \qquad (1)$$

where  $E_n = (p^2c^2 + m^2c^4 + 2eB\hbar cn)^{1/2}$  is the energy spectrum of a relativistic electron [8];  $n = 0, 1, 2, \ldots$  enumerates the Landau levels,  $\zeta$  is the quantum number characterizing the spin projection of an electron, p is the electron momentum component parallel to **B**, and  $\theta$  is the temperature. In what follows we shall express values of  $\mu_e$  in dimensionless units:  $\mu = \mu_e/mc^2$ . Since the chemical potential  $\mu_0$  of an electron gas at  $\theta = 0$  and B = 0 is related to  $n_e$  by

$$(\mu_0^2 - 1)^{3/2} = n_e/n_0, \quad n_0 = m^3 c^3 / 3\pi^2 \hbar^3,$$
 (2)

eq. (1) may be considered as a relation between  $\mu$ ,  $\mu_0$  and  $\theta$  at given B. By integrating with respect to p in (1), taking allowance for  $E_{\text{max}} = \mu_e$  at  $\theta = 0$ , it is easy to obtain:

$$\frac{\sqrt{2}n_{\rm e}}{3n_0} \left(\frac{B_0}{B}\right)^{3/2} = \left((\mu^2 - 1)\frac{B_0}{2B}\right)^{1/2} + 2\sum_{n=1}^{n_{\rm max}} \left((\mu^2 - 1)\frac{B_0}{2B} - n\right)^{1/2}.$$
(3)

The value  $n_{\text{max}}$  is determined by the condition  $n_{\text{max}} < (\mu^2 - 1)B_0/2B$ .

It is seen from (3) that, if the following condition is satisfied:

$$\left(\sqrt{2}n_{\rm e}/3n_0\right)\left(B_0/B\right)^{3/2} < 1 , \qquad (4)$$

then only the lowest (ground) level n = 0 remains populated. For a given  $n_{\rm e}$ , we can find from (3) the relation

$$(\mu^2 - 1)^{1/2} = (2B_0/3B)(n_e/n_0).$$
<sup>(5)</sup>

Let us now consider the conditions for chemical equilibrium of a degenerate gas of protons (p), neutrons (n) and electrons (e) in the presence of large magnetic fields. First we shall consider fields B in the range

$$B_0^* \gg B \gg B_0,\tag{6}$$

where  $B_0^* = m_p^2 c^3/e\hbar = 3.4 \cdot 10^6 B_0$ , and  $m_p$  is the proton mass. Magnetic field *B* in this range is "strong" for the electron but "weak" for the proton. Hence motion of protons in such a magnetic field can be considered as quasiclassical because the spacing between Landau levels for the proton under these conditions is very small. We assume further that the condition (4) is also satisfied. Then only the lowest energy level will be occupied by electrons. We also suppose that the temperature  $\theta$  is equal to zero, since for a typical 100-year old neutron star, its temperature is estimated to be  $10^8$  K (about 10 keV), which can be considered cold as compared to the Fermi energy (about 1000 MeV) of the degenerate relativistic neutrons [9].

We are interested in the reactions in which the total density of baryons  $n_{\rm b} = n_{\rm p} + n_{\rm n}$  is conserved and the electroneutrality condition of a gas  $n_{\rm p} = n_{\rm e}$  is satisfied. These processes are called the direct URCA processes [9, 10, 11, 12]. Since  $n_{\rm b}$  is conserved, the total energy density  $\epsilon$  depends only on  $n_{\rm n}$ :

$$\epsilon = \pi^{-2} \hbar^{-3} \int_{0}^{p_{\rm F}^{\rm n}} p^2 (p^2 c^2 + m_{\rm n}^2 c^4)^{1/2} dp + \pi^{-2} \hbar^{-3} \int_{0}^{p_{\rm F}^{\rm p}} p^2 (p^2 c^2 + m_{\rm p}^2 c^4)^{1/2} dp + \epsilon_{\rm e} (p_{\rm F}^{\rm e}, B), \quad (7)$$

where

$$p_{\rm F}^{\rm n} = (n_{\rm n}/n_{0{\rm n}})^{1/3} m_{\rm n} c, \quad p_{\rm F}^{\rm p} = (n_{\rm p}/n_{0{\rm n}})^{1/3} m_{\rm n} c,$$

$$p_{\rm F}^{\rm e} = (2n_{\rm p}/3n_0)(B_0/B)mc, \quad n_{0{\rm n}} = m_{\rm n}^3 c^3/3\pi^2\hbar^3 , \qquad (8)$$

and  $\epsilon_{\rm e}$  is the energy density of the electrons.

At chemical equilibrium  $d\epsilon/dn_{\rm n} = 0$ , or  $\mu_{\rm n} = \mu_{\rm p} + \mu_{\rm e}$ , where  $\mu_{\rm n}, \mu_{\rm p}$  and  $\mu_{\rm e}$  are the chemical potentials of neutrons, protons and electrons respectively. The chemical potential of a degenerate electron gas in the presence of the magnetic field *B* under the conditions assumed is  $\mu_{\rm e}$  given by formula (5). The quantity  $p_{\rm F}^{\rm e}$  plays the role of the Fermi momentum. More precisely,  $p_{\rm F}^{\rm e}$  here is the "Fermi momentum projection" in the magnetic field direction for electrons.

The chemical potentials of relativistic nucleons are given by

$$\mu_{\rm n} = m_{\rm n} c^2 \left[ 1 + (n_{\rm n}/n_{0{\rm n}})^{2/3} \right]^{1/2},$$
  
$$\mu_{\rm p} = m_{\rm p} c^2 \left[ 1 + (n_{\rm p}/n_{0{\rm n}})^{2/3} (m_{\rm n}/m_{\rm p})^2 \right]^{1/2},$$
(9)

for B satisfying (6). The proton number density  $n_{\rm p}$  as a function of  $n_{\rm n}$  at chemical equilibrium can be found from the following equation:

$$m_{\rm n} \left[ 1 + (n_{\rm n}/n_{0{\rm n}})^{2/3} \right]^{1/2} = m_{\rm p} \left[ 1 + (n_{\rm p}/n_{0{\rm n}})^{2/3} (m_{\rm n}/m_{\rm p})^2 \right]^{1/2} + m \left[ 1 + (2n_{\rm p}m_{\rm n}^3B_0)^2 / (3n_{0{\rm n}}m^3B)^2 \right]^{1/2}.$$
(10)

Writing inequality (4) in the form

$$n_{\rm p}/n_{\rm 0n} < (3/\sqrt{2}) (m/m_{\rm n})^3 (B/B_0)^{3/2} \approx 3.4 \cdot 10^{-10} (B/B_0)^{3/2},$$
 (11)

we see that, for  $B_0^* \gg B \gg B_0$ , the inequality  $n_p/n_{0n} \ll 1$  must be satisfied.

We now turn to discuss equilibrium condition for the case of ultrastrong magnetic fields (with  $B > B_0^*$ ) and very high densities (with  $n_p$ ,  $n_n > n_{0n}$ ). The existence of such ultrastrong magnetic fields near the center of neutron stars is allowed, or, at least, is not forbidden in principle. In this case formula (10) needs to be modified because now the protons, just as the electrons, must occupy their lowest Landau energy levels. The limit of the proton density under which all protons, and consequently all electrons, populate only their respective ground levels is again given by (11). But now one could have  $n_p/n_{0n} > 1$ . As in the case of electrons, the Fermi energy for protons is given by

$$\mu_{\rm p} = m_{\rm p} c^2 \left[ 1 + (2n_{\rm p} B_0^*)^2 / (3n_{0\rm p} B)^2 \right]^{1/2} .$$
(12)

Eq. (10) is then replaced by

$$m_{\rm n} \left[ 1 + (n_{\rm n}/n_{0{\rm n}})^{2/3} \right]^{1/2} = m_{\rm p} \left[ 1 + (2n_{\rm p}B_0^*)^2 / (3n_{0{\rm p}}B)^2 \right]^{1/2} + m \left[ 1 + (2n_{\rm p}m_{\rm n}^3B_0)^2 / (3n_{0{\rm n}}m^3B)^2 \right]^{1/2}.$$
(13)

Numerical solutions of eq. (10) and (13) subject to the constraint (11) are given in Fig. 1  $(B \leq 10^5 B_0 \text{ for eq. (10)}, \text{ and } B \geq 10^7 B_0 \text{ for (13)})$ . These curves represent the normalized proton density number  $n_{\rm p}/n_{0\rm n}$  as a function of the normalized neutron density number  $n_{\rm n}/n_{0\rm n}$  at equilibrium at various values of magnetic field induction. The dotted curve gives the corresponding values in the absence of magnetic fields. The normalized Fermi momenta of electrons  $p_{\rm F}^{\rm e}/mc$  as a function of the normalized neutron density number for these values of magnetic field are given in Fig. 2.

One can see from Fig. 1 that results obtained here differ from analogous results with zero magnetic field. Such a difference takes place at various values of magnetic field induction for some ranges both of low and high particles densities. Under the conditions assumed in this paper, for a given value of  $n_n/n_{0n}$ , the values of  $n_p/n_{0n}$  in the presence of finite B's are for the most part higher than the corresponding value when the field is absent, until the proton density reaches a value  $(n_p/n_{0n})_{cross} = (3B/2B_0)^{3/2} (m/m_n)^3$ , which is the point of intersection between the curves with  $B \neq 0$  and B = 0. The value of  $(n_p/n_{0n})_{cross}$  is slightly less than the upper limit given in (11) for a given B. Also, in the presence of ultrastrong magnetic field, there appear values of densities for which  $n_p > n_n$ . These ranges of particle densities are of great interest since in zero magnetic field the ratio  $n_p/n_n$  is always less than unity, with a maximum equals 1/8 [10]. The functional form of (13) implies that  $n_p/n_{0n}$  is directly proportional to B. This is evident in Fig. 1 from the fact that curves corresponding to  $B \geq 10^7 B_0$  are parallel to one another, and are seperated by equal distance.

One may understand the behavior of these curves as follows. Eq. (8) shows that higher magnetic field B tends to lower the Fermi momentum  $p_{\rm F}^{\rm e}$  of the electrons, and hence the chemical potential of electrons (chemical potential of protons as well, when  $B > B_0^*$ , see (12)). To maintain chemical equilibrium among the particles at fixed value of  $n_{\rm n}/n_{\rm 0n}$ , the chemical potentials of electrons and protons have to be raised so that eq.(10) or (13) is still satisfied. This is achieved through the increase in the density of electrons, and hence the density of protons by neutrality condition assumed here. The shape of the curves, which consists mainly of segments of straight lines in the log-log plot, indicates power-law behavior between the quantities  $n_{\rm p}/n_{0\rm n}$  and  $n_{\rm n}/n_{0\rm n}$ . This can also be easily understood. Consider first the situation in which  $B > B_0^*$ . In the high density region where  $n_{\rm n}/n_{0\rm n} \gg 1$ , the densities of electrons and protons are high enough that their Fermi momenta satisfy  $p_{\rm F}^{\rm e} \gg mc$  and  $p_{\rm F}^{\rm p} \gg m_{\rm p}c$ . Hence only the second term within each square-root in (13) dominates. After some algebra we get

$$\frac{n_{\rm p}}{n_{\rm 0n}} = \frac{3}{4} \left(\frac{m}{m_{\rm n}}\right)^2 \frac{B}{B_0} \left(\frac{n_{\rm n}}{n_{\rm 0n}}\right)^{1/3} \,. \tag{14}$$

This shows that the straight lines on the upper right part of Fig.1 have slope 1/3. For the region where  $n_{\rm n}/n_{0{\rm n}} \ll 1$ , the densities of electrons and protons are lower such that  $p_{\rm F}^{\rm p} \ll m_{\rm p}c$ , but still  $p_{\rm F}^{\rm e} \gg mc$ . Eq.(13) then reduces approximately to

$$m_{\rm n} \left[ 1 + \frac{1}{2} \left( \frac{n_{\rm n}}{n_{0{\rm n}}} \right)^{2/3} \right] \approx m_{\rm p} + \frac{2}{3} \frac{n_{\rm p}}{n_{0{\rm n}}} \frac{m_{\rm n}^3}{m^2} \frac{B_0}{B} .$$
 (15)

From this one has, after setting  $m_{\rm n} \approx m_{\rm p}$ ,

$$\frac{n_{\rm p}}{n_{\rm 0n}} = \frac{3}{4} \left(\frac{m}{m_{\rm n}}\right)^2 \frac{B}{B_0} \left(\frac{n_{\rm n}}{n_{\rm 0n}}\right)^{2/3} \,. \tag{16}$$

The slopes of the corresponding line segments in Fig.1 are 2/3. We note here that (14) and (16) have the same coefficients in front of the factor  $n_n/n_{0n}$ , which mean the two line segments in Fig.1 have the same intercept. Similar consideration, when applied to (10) for the case  $B_0^* \gg B \gg B_0$ , leads also to the power law (16). When B = 0, we get the power laws  $\frac{n_p}{n_{0n}} = \frac{1}{8} \frac{n_n}{n_{0n}}$  and  $\frac{n_p}{n_{0n}} = \frac{1}{8} (\frac{n_n}{n_{0n}})^2$  for high and low neutron density, respectively.

Fig. 2(a) shows that, for a given neutron density, the Fermi momentum of electrons in the presence of field  $B < B_0^*$  decreases as B increases. It is not higher than the corresponding value when B = 0 until  $(n_p/n_{0n})_{cross}$  is reached. On the other hand, one sees from Fig. 2(b)

that electron Fermi momenta in the presence of ultrastrong magnetic fields are independent of the magnetic field strength. Furthermore, they equal the corresponding value in the absence of magnetic field. The former situation observed in Fig. 2(b) is just the manifestation of the fact that, for a fixed  $n_n/n_{0n}$  value,  $n_p/n_{0n}$  is directly proportional to B in this case, as mentioned before, and the fact that  $p_F^e$  depends only on their ratio. To understand the latter result, we simply observe that the equilibrium equation (13), expressed in terms of the Fermi momenta of the nucleons, and the electrons, is identical to the corresponding equation without magnetic field. This is because  $p_F^n$  has the same form in both cases, and while the corresponding expressions of  $p_F^e$  and  $p_F^p$  differ in the two cases, the relation  $p_F^e = p_F^p$  holds well. So for a fixed neutron density (*i.e.* fixed  $p_F^n$ ), both equations give the same solution for the value of  $p_F^e$ , solved by different  $n_p/n_{0n}$  at different B when  $B \neq 0$ , of course.

The results obtained here may serve to give us hints as to the chemical composition of the central region of neutron stars, if we suppose that this region is composed of "cold" degenerate neutrons, protons and electrons in the superstrong magnetic field. The assumption was also made that all the electrons occupied only the lowest energy level when  $B \gg B_0$ . If the gas temperature in this region is not equal to zero, then we must introduce definite restrictions not only on the quantity  $n_e$  but also on the gas temperature. There are several aspects need to be considered. Firstly, it is clear that at  $\theta \neq 0$  the electrons can occupy the higher energy levels of transverse (to the magnetic induction vector) motion of electrons due to thermal excitations. Secondly, the electron-positron pair production due to electron collisions in the ultradense hot medium can become noticeable. Finally, the chemical potential of the electrons depends on the gas temperature.

It is easy to see that the electrons will occupy the lowest energy level of transverse motion at  $B \gg B_0$  if

$$\frac{k_B\theta}{mc^2} \ll \left(\frac{2B}{B_0}\right)^{1/2} , \qquad (17)$$

where  $k_B$  is the Boltzmann constant. It follows from this inequality that, even at a very high temperature of the order m, electrons are still in the lowest energy state. One must also estimate the effect of electron-positron pair production by electron collisions in the presence of a magnetic field. As a result of electron-positron pair production, the equality  $n_p = n_e$  will be violated but the electroneutrality condition as a whole must be conserved. We estimate that, even when  $B \gg B_0$ , the densities of electrons and positrons produced will be much less than the initial density of electrons  $n_e$  if the temperature satisfies the strong inequality  $k_B\theta/mc^2 \ll (n_e B_0/3n_{0n}B \ln 2)(m_n/m)^3$ . Thus the thermal effects of the electron-positron pair production (even when  $k_B\theta$  is of order  $mc^2$ ) will not substantially affect the chemical equilibrium of neutrons, protons and electrons over a sufficiently wide interval of values  $n_e$ . We have also made an estimate of the chemical potential of electrons which occupy the lowest Landau energy level as a function of temperature and found that, if inequality (17) is satisfied, then the chemical potential in the form (5) can still be employed.

Finally we remark that in this paper we consider only the case of ideal gas model, as are usually done in the general analyses of basic properties of compact stars [9, 10]. We see, however, that even such simple model gives nontrivial and unexpected results. So a careful analysis of more realistic models in the presence of strong magnetic fields, and at finite temperatures, is extremely needed to answer the question about the actual behavior of such models. Many factors need be considered. For instance, the density of matter near the center of a neutron star is probably an inhomogeneous function of distance [11, 12], and the problem of equilibrium in this case is significantly more complicated. Also, one must consider the effect on the chemical equilibrium by the coulomb interaction and the quantum exchange effect among the particles. These consideration will be reserved for future investigations.

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#### **Figures Captions**

Fig. 1. Plot of  $\log(n_{\rm p}/n_{0{\rm n}})$  verses  $\log(n_{\rm n}/n_{0{\rm n}})$  (continuous curves) for values of  $B/B_0$ at (from bottom to top): 10<sup>2</sup>, 10<sup>3</sup>, 10<sup>4</sup>, 10<sup>5</sup>, 10<sup>7</sup>, 10<sup>9</sup>, 10<sup>11</sup> and 10<sup>13</sup>. The dotted line corresponds to the curve with B = 0. Equation  $\log(n_{\rm p}/n_{0{\rm n}}) = \log(n_{\rm n}/n_{0{\rm n}})$  is indicated by the dashed line.

Fig. 2 (a). Plot of the normalized Fermi momenta of electrons  $p_{\rm F}^{\rm e}/mc$  verses  $n_{\rm n}/n_{0{\rm n}}$ for  $B/B_0 = 10^4$  (upper curve) and  $10^5$  (lower curve). Dotted curve corresponds to B = 0.

Fig. 2 (b). Plot of the normalized Fermi momenta of electrons for  $p_{\rm F}^{\rm e}/mc$  verses  $n_{\rm n}/n_{0{\rm n}}$ for  $B/B_0 = 0, \ 10^7, \ 10^9, \ 10^{11}$  and  $10^{13}$ .