Optimizing the SINR operating point of spatial networks

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***Abstract*— This paper addresses the following question, which is of interest in the design and deployment of a multiuser decentralized network. Given a total system bandwidth of W Hz and a fixed data rate constraint of R bps for each transmission, how many frequency slots N of size W/N should the band be partitioned into to maximize the number of simultaneous transmissions in the network? In an interference-limited ad-hoc network, dividing the available spectrum results in two competing effects: on the positive side, it reduces the number of users on each band and therefore decreases the interference level which leads to an increased SINR, while on the negative side the SINR requirement for each transmission is increased because the same information rate must be achieved over a smaller bandwidth. Exploring this tradeoff between bandwidth and SINR and determining the optimum value of N in terms of the system parameters is the focus of the paper. Using stochastic geometry, we analytically derive the optimal SINR threshold (which directly corresponds to the optimal spectral efficiency) on this tradeoff curve and show that it is a function of only the path loss exponent. Furthermore, the optimal SINR point lies between the low-SINR (power-limited) and high-SINR (bandwidth-limited) regimes. In order to operate at this optimal point, the number of frequency bands (i.e., the reuse factor) should be increased until the threshold SINR, which is an increasing function of the reuse factor, is equal to the optimal value.**

I. Introduction

We consider a spatially distributed network, representing either a wireless ad hoc network or unlicensed (and uncoordinated) spectrum usage by many nodes (e.g., WiFi), and consider the tradeoff between bandwidth and SINR. We ask the following question: given a fixed total system bandwidth and a fixed rate requirement for each single-hop transmitter-receiver link in the network, at what point along the bandwidth-SINR tradeoff-curve should the system operate at in order to maximize the spatial density of transmissions subject to an outage constraint? Note that the outage probability is computed with respect to random user locations as well as fading.

For example, given a system-wide bandwidth of 1 Hz and a desired rate of 1 bit/sec, should (a) each transmitter utilize the entire spectrum (e.g., transmit one symbol per second) and thus require an SINR of 1 (utilizing $R = W\log(1+SINR)$ if interference is treated as noise), (b) the band be split into two orthogonal 0.5 Hz sub-bands where each transmitter utilizes one of the sub-bands with the required SINR equal to 3, or (c) the band be split into $N > 2$ orthogonal $\frac{1}{N}$ Hz sub-bands where each transmitter utilizes one of the sub-bands with the required SINR equal to $2^N - 1$? Note that an equivalent formulation is to optimize the bandwidth-SINR operating point such that the outage probability is minimized for some fixed density of transmissions.

We consider a network with the following key characteristics:

- Transmitter node locations are a realization of a homogeneous spatial Poisson process.
- Each transmitter communicates with a single receiver that is a reference distance d meters away.
- All transmissions are constrained to have an absolute rate of R bits/sec regardless of the bandwidth.
- All multi-user interference is treated as noise.
- The channel is frequency-flat, reflects path-loss and possibly fast and/or slow fading, and is constant for the duration of a transmission.
- Transmitters do not have channel state information and no transmission scheduling is performed, i.e., transmissions are independent and random, conceptually like an Aloha system.

The last assumption should make it clear that we are considering only an *off-line* optimization of the frequency band structure, and that no on-line (e.g., channel- and queue-based) transmission or sub-band decisions are considered.

These assumptions are chosen primarily for tractability and their validity will not be assured in all implementations, but generalizations are left to future work.

A. Related Work

The transmission capacity framework introduced in [1] is used to quantify the throughput of such a network, since this metric captures notions of spatial density, data rate, and outage probability, and is more amenable to analysis than the more popular transport capacity [2]. Using tools from stochastic geometry [3], the distribution of interference from other concurrent transmissions at a reference receiving node[1] is characterized as a function of the spatial density of transmitters, the path-loss exponent, and possibly the fading distribution. The distribution of SINR at the receiving node can

[1]The randomness in interference is only due to the random positions of the interfering nodes and fading.

then be computed, and an outage occurs whenever the SINR falls below some threshold β. The outage probability is clearly an increasing function of the density of transmissions, and the transmission capacity is defined to be the maximum density of successful transmissions such that the outage probability is no larger than some prescribed constant ϵ.

The problem studied in this work is essentially the optimization of frequency reuse in uncoordinated spatial (ad hoc) networks, which is a well studied problem in the context of cellular networks (see for example [4] and references therein). In both settings the tradeoff is between the bandwidth utilized per cell/transmission, which is inversely proportional to the frequency reuse factor, and the achieved SINR per transmission. A key difference is that in cellular networks, regular frequency reuse patterns can be planned and implemented, whereas in an ad hoc network this is impossible and so the best that can be hoped for is uncoordinated *random* frequency reuse. Another crucial difference is in terms of analytical tractability. Although there has been a tremendous amount of work on optimization of frequency reuse for cellular networks, these efforts do not, to the best of our knowledge, lend themselves to clean analytical results. On the contrary, in this work we are able to derive very simple analytical results in the random network setting that very cleanly show the dependence of the optimal reuse factor on system parameters such as path loss exponent and rate.

Perhaps the most closely related work is [5][6], in which a one-dimensional (i.e., linear), evenly spaced, multi-hop wireless network is studied. In finding the optimal (in terms of total energy minimization) number of intermediate relay nodes in an interference-free network (i.e., each hop is assigned a distinct frequency or time slot), their analysis (rather remarkably) coincides almost exactly with our analysis of an interference-limited, two-dimensional, random network. The issue of frequency reuse in interference-limited 1-D networks is also explicitly considered in [5], and some of the general insights are similar to those derived in this work.

II. Key Insights

The bandwidth-SINR tradeoff reveals itself if the system bandwidth is split into N non-overlapping bands and each transmitter transmits on a randomly chosen band with some fixed power (independent of N). This splitting of the spectrum results in two competing effects. First, the density of transmitters on each band is a factor of N smaller than the overall density of transmitters, which reduces interference and thus increases SINR. Second, the threshold SINR must be increased in order to maintain a fixed rate while transmitting over $\frac{1}{N}$-th of the bandwidth. This requires a reduced network density in order to meet the prescribed outage constraint.

Although intuition from point-to-point AWGN channels – for which capacity is a strictly increasing function of bandwidth if transmission power is fixed – might cause one to think that the optimum solution is trivially to not split the band ($N = 1$), this is generally quite far from the optimum in ad hoc networks. Our analysis shows that N should be chosen such that the required threshold SINR lies between the low-SNR (power-limited) and high-SNR (bandwidth-limited) regimes, for example in the range of 0 - 5 dB for reasonable path loss exponents. This approximately corresponds to the region where the function $\log(1+SINR)$ transitions from linear to logarithmic in SINR.

The intuition behind this result is actually quite simple: if N is such that the threshold SINR is in the wideband regime (roughly speaking, below 0 dB), then doubling N leads to an approximate doubling of the threshold SINR, or equivalently a 3 dB increase. Whenever the path-loss exponent is strictly greater than 2, doubling the threshold SINR reduces the allowable intensity of transmissions on each band by a factor strictly smaller than two. On the other hand, the doubling of N increases the total intensity by exactly a factor of two because the number sub-bands is increased by the a factor of two; the combination of these effects is a net increase in the allowable intensity of transmissions. Therefore, it is beneficial to continue to increase N until the point at which the required SINR threshold begins to increase *exponentially* rather than *linearly* with N.

III. Preliminaries

A. System Model

We consider a set of transmitting nodes at an arbitrary snapshot in time with locations specified by a homogeneous Poisson process of intensity λ on the infinite two-dimensional plane. We consider a reference receiver that is located, without loss of generality, at the origin, and let X_i denote the distance of the i-th transmitting node to the reference receiver. The reference transmitter is placed a fixed distance d away. Received power is modelled by path loss with exponent $\alpha > 2$ and a distance-independent fading coefficient h_i (from the i-th transmitter to the reference receiver). Therefore, the SINR at the reference receiver is:

$$SINR_0 = \frac{\rho d^{-\alpha}|h_0|}{\eta + \sum_{i \in \Pi(\lambda)} \rho X_i^{-\alpha}|h_i|},$$

where $\Pi(\lambda)$ indicates the point process describing the (random) interferer locations, and η is the noise power. If Gaussian signalling is used by all nodes, the mutual information conditioned on the transmitter locations and the fading realizations is:

$$I(X_0; Y_0|\Pi(\lambda), \mathbf{h}) = \log_2(1 + SINR_0),$$

where $\mathbf{h} = (h_0, h_1, \ldots)$. Notice that we assume that all nodes simultaneously transmit with the same power ρ, i.e., power control is not used. Moreover, nodes decide to transmit independently and irrespective of their channel conditions, which corresponds roughly to slotted ALOHA (i.e., no scheduling is performed).

A few comments in justification of the above model are in order. Although the model contains many simplifications to allow for tractability, it contains many of the critical elements of a real ad hoc network. First, the spatial Poisson distribution means that nodes are randomly and independently located; this is reasonable particularly in a network with substantial mobility or indiscriminate node placement (for example a very dense sensor network). The fixed transmission distance of d is clearly not a reasonable assumption; however our prior work [1], [7] has shown rigorously that variable transmit distances do not result in fundamentally different capacity results, so a fixed distance is chosen because it is much simpler analytically and allows for crisper insights. A similar justification can be given for ignoring power control, although power control is often not used in actual ad hoc networks either. Finally, scheduling procedures (e.g., using carrier sensing to intelligently select a sub-band) may significantly affect the results and is definitely of interest, but this opens many more questions and so is left to future work.

B. Transmission Capacity Model

In the outage-based transmission capacity framework, an outage occurs whenever the SINR falls below a prescribed threshold β, or equivalently whenever the instantaneous mutual information falls below $\log_2(1+\beta)$. Therefore, the system-wide outage probability is:

$$P\left(\frac{\rho d^{-\alpha}|h_0|}{\eta + \sum_{i\in\Pi(\lambda)} \rho X_i^{-\alpha}|h_i|} \leq \beta\right).$$

This quantity is computed over the distribution of transmitter positions as well as the iid fading coefficients, and thus corresponds to fading that occurs on a slower time-scale than packet transmission. The outage probability is clearly an increasing function of the intensity λ.

If $\lambda(\epsilon)$ is the maximum intensity of *attempted* transmissions such that the outage probability (for a fixed β) is no larger than ϵ, then the transmission capacity is then defined as $c(\epsilon) = \lambda(\epsilon)(1-\epsilon)b$, which is the maximum density of *successful* transmissions times the spectral efficiency b of each transmission. In other words, transmission capacity is like area spectral efficiency subject to an outage constraint. Using tools from stochastic geometry, in [1] it is shown that the maximum spatial intensity $\lambda(\epsilon)$ for small values of ϵ is:

$$\lambda(\epsilon) = \frac{c}{\pi d^2}\left(\frac{1}{\beta} - \frac{\eta}{\rho d^{-\alpha}}\right)^{\frac{2}{\alpha}} \epsilon + O(\epsilon^2), \tag{1}$$

where c is a constant that depends only on the distribution of the fading coefficients [7]. In the proceeding analysis, the key is the manner in which the transmission capacity varies with the SINR constraint β; for small noise values, which is the case in the interference-limited scenarios we are most interested in, intensity is proportional to $\beta^{-\frac{2}{\alpha}}$. Because fading has only a multiplicative effect on transmission capacity, it does not effect the SINR-bandwidth tradeoff and thus is not considered in the remainder of the paper.

IV. Optimizing Frequency Usage

In this section we consider a network with a fixed total bandwidth of W Hz, and where each link has a rate requirement of R bits/sec and an outage constraint ϵ. Assuming the network operates as described in the previous section, the goal is to determine the optimum number of sub-bands N into which the system bandwidth of W Hz should be divided while meeting these criteria. By optimum, we mean the choice of N that maximizes the intensity of allowable transmissions $\lambda(\epsilon, N)$. As we will see, due to our constraint that the data rate on each link is the same regardless of λ and N, this also corresponds to maximizing transmission capacity.

A. Definitions and Setup

In performing this analysis, we assume that there exist coding schemes that operate at any point along the AWGN capacity curve.[2] To facilitate exposition, we define the *spectral utilization* $\tilde{R}$ to be the ratio of the required rate relative to the total system bandwidth:

$$\tilde{R} \triangleq \frac{R}{W} \quad \text{bps/Hz/user.}$$

Note that we intentionally refer to $\tilde{R}$, which is externally defined, as the spectral utilization; the *spectral efficiency*, on the other hand, is a system design parameter determined by the choice of N.

If the system bandwidth is not split ($N = 1$), each node utilizes the entire bandwidth of W Hz. Therefore, the required SINR β is determined by inverting the standard rate expression:

$$R = W \log_2(1+\beta),$$

which gives $\beta = 2^{\frac{R}{W}} - 1 = 2^{\tilde{R}} - 1$. The maximum intensity of transmissions can be determined by plugging in this value of β into (1), along with the other relevant constants.

More generally, if the system bandwidth is split into N orthogonal sub-bands each of width $\frac{W}{N}$, and each transmitter-receiver pair uses only one of these sub-bands at random, the required SINR $\beta(N)$ is determined by inverting the rate expression:

$$R = \frac{W}{N}\log_2(1+\beta(N)),$$

which yields:

$$\beta(N) = 2^{\frac{NR}{W}} - 1 = 2^{N\tilde{R}} - 1.$$

Notice that the spectral efficiency (on each sub-band) is $b = \frac{R}{W/N}$ bps/Hz, which is N times the spectral utilization $\tilde{R}$. The maximum intensity of transmissions *per sub-band* for a particular value of N is determined by plugging $\beta(N)$ into (1) with noise power $\eta = \frac{W}{N}N_0$. Since the N sub-bands are statistically identical, the maximum total intensity of transmissions, denoted by $\lambda(\epsilon, N)$, is the per sub-band intensity multiplied by a factor of N. Dropping the second order term in (1), we have:

$$\lambda(\epsilon, N) \approx N\left(\frac{\epsilon}{\pi d^2}\right)\left(\frac{1}{\beta(N)} - \frac{1}{N \cdot SNR}\right)^{\frac{2}{\alpha}}, \tag{2}$$

[2] In Section V-B we relax this assumption by allowing for operation at a constant coding gap (i.e., power offset) from AWGN capacity, and see that this has no effect on our analysis.

where the constant $SNR \triangleq \frac{\rho d^{-\alpha}}{N_0 W}$ is the signal-to-noise ratio in the absence of interference when the entire band is used.

B. Optimization

Optimizing the number of sub-bands N therefore reduces to the following one-dimensional maximization:

$$N^* = \arg\max_N \lambda(\epsilon, N), \tag{3}$$

which yields a solution that depends only on the path-loss exponent α, the spectral utilization $\tilde{R}$, and the constant SNR.

In general, the interference-free SNR can be ignored because the systems of interest are interference- rather than noise-limited. Assuming SNR is infinite we have:

$$\lambda(\epsilon, N) \approx \left(\frac{\epsilon}{\pi d^2}\right) N \cdot \beta(N)^{-\frac{2}{\alpha}} \tag{4}$$

$$= \left(\frac{\epsilon}{\pi d^2}\right) N (2^{N\tilde{R}} - 1)^{-\frac{2}{\alpha}}. \tag{5}$$

The leading factor of N represents the fact that total transmission intensity is N times the per-band intensity, while the $(2^{N\tilde{R}} - 1)^{-\frac{2}{\alpha}}$ term, which is a decreasing function of N, is the amount by which intensity must be decreased in order to maintain an outage probability no larger than ϵ in light of the monotonically increasing (in N) SINR threshold $\beta(N)$.

Since $\tilde{R}$ is a constant, we can make the substitution $b = N\tilde{R}$ and equivalently maximize the function $b(2^b - 1)^{-\frac{2}{\alpha}}$. Taking the derivative with respect to b we get:

$$\frac{\partial}{\partial b}\left[b(2^b - 1)^{-\frac{2}{\alpha}}\right] = (2^b - 1)^{-\frac{2}{\alpha}}\left[1 - \frac{2}{\alpha} b (1 - 2^{-b})^{-1} \log_e 2\right].$$

Since the first term is strictly positive for $b > 0$, we set the second term to zero to get a fixed point equation for the optimal spectral efficiency b^*:

$$b^* = (\log_2 e)\frac{\alpha}{2}(1 - e^{-b^*}), \tag{6}$$

which has solution

$$b^* = \log_2 e\left[\frac{\alpha}{2} + W\left(-\frac{\alpha}{2} e^{-\frac{\alpha}{2}}\right)\right], \tag{7}$$

where $W(z)$ is the principle branch of the Lambert W function and thus solves $W(z)e^{W(z)} = z$.[3]

Because $1 - \frac{2}{\alpha} b (1-2^{-b})^{-1} \log_e 2$ is strictly decreasing and $(2^b - 1)^{-\frac{2}{\alpha}}$ is strictly positive, the first derivative is strictly positive for $0 < b < b^*$ and is strictly negative for $b > b^*$. Therefore, the b^* in (7) is indeed the unique maximizer. Furthermore, it is easily shown that the optimizing b^* is an increasing function of α, is upper bounded by $\frac{\alpha}{2}\log_2 e$, and that $b^*/(\frac{\alpha}{2}\log_2 e)$ converges to 1 as α grows large.

[3]Equation (7) is nearly identical, save for a factor of 2, to the expression for the optimal number of hops in an interference-free linear network given in equation (18) of [5]. This similarity is due to the fact that the objective function in equation (17) of [5] coincides almost exactly with (5).

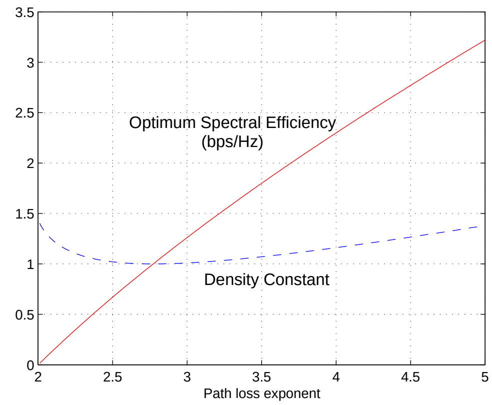

Fig. 1. Optimal Spectral Efficiency vs. Path-Loss Exponent

Recalling that $b = N\tilde{R}$ is the spectral efficiency on each sub-band, the quantity b^*, which is a function of only the path-loss exponent α, is the *optimum spectral efficiency*.[4] Therefore, the optimal value of N (ignoring the integer constraint) is determined by simply dividing the optimal spectrum efficiency b^* by the spectral utilization $\tilde{R}$:

$$N^* = \frac{b^*}{\tilde{R}}. \tag{8}$$

To take care of the integer constraint on N, the nature of the derivative of $b(2^b - 1)^{-\frac{2}{\alpha}}$ makes it sufficient to consider only the integer floor and ceiling of N^* in (8). Note that the optimal number of sub-bands depends only on the spectral utilization $\tilde{R}$ (inversely) and on b^*, which is a function of the path-loss exponent; there is no dependence on either the outage constraint ϵ or on the transmission range d.

If the spectral utilization is larger than the optimum spectral efficiency, i.e., $\tilde{R} \geq b^*$, then choosing $N = 1$ is optimal. On the other hand, if $\tilde{R} \leq \frac{1}{2}b^*$, then the optimal N is strictly larger than 1. In the intermediate regime where $\frac{1}{2}b^* \leq \tilde{R} \leq b^*$, the optimal N is either one or two.

In Fig. 1 the optimal spectral efficiency b^* is plotted (in units of bps/Hz) as a function of the path-loss exponent α, along with the quantity $b^*(2^{b^*} - 1)^{-\frac{2}{\alpha}}$, which is referred to as the density constant because the optimal density $\lambda^*(\epsilon)$ is this quantity multiplied by $\left(\frac{\epsilon}{\tilde{R}\pi d^2}\right)$. The optimal spectral efficiency is very small for α close to 2 but then increases nearly linearly with α; for example, the optimal spectral efficiency for $\alpha = 3$ is 1.26 bps/Hz (corresponding to $\beta = 1.45$ dB).

C. Interpretation

To gain an intuitive understanding of the optimal solution, first consider the behavior of $\lambda(\epsilon, N)$ when the quantity $N\tilde{R}$ is small, i.e. $N\tilde{R} \ll 1$. In this regime, the SINR threshold

[4]An optimal spectral efficiency is derived for interference-free, regularly spaced, 1-D networks in [6]; however, these results differ by approximately a factor of 2 from our results due to the difference in the network dimensionality.

$\beta(N)$ grows approximately linearly with N:

$$\begin{aligned}\beta(N) = 2^{N\tilde{R}} - 1 &= e^{N\tilde{R}\log_e 2} - 1 \\ &= \sum_{k=1}^{\infty} \frac{(N\tilde{R}\log_e 2)^k}{k!} \\ &\approx N\tilde{R}\log_e 2.\end{aligned}$$

Plugging into (5) we have

$$\begin{aligned}\lambda(\epsilon, N) &\approx \left(\frac{\epsilon}{\pi d^2}\right) N (N\tilde{R}\log_e 2)^{-\frac{2}{\alpha}} \\ &= \left(\frac{\epsilon}{\pi d^2}\right) \tilde{R}\log_e 2^{-\frac{2}{\alpha}} N^{\left(1-\frac{2}{\alpha}\right)}.\end{aligned}$$

For any path-loss exponent $\alpha > 2$, the maximum intensity of transmissions monotonically increases with the number of sub-bands N as $N^{\left(1-\frac{2}{\alpha}\right)}$, i.e., *using more sub-bands with higher spectral efficiency leads to an increased transmission capacity*, as long as the linear approximation to $\beta(N)$ remains valid. The key reason for this behavior is the fact that transmission capacity scales with the SINR threshold as $\beta^{-\frac{2}{\alpha}}$, which translates to $N^{-\frac{2}{\alpha}}$ in the low spectral efficiency regime.

As $N\tilde{R}$ increases, the linear approximation to $\beta(N)$ becomes increasingly inaccurate because $\beta(N)$ begins to grow *exponentially* rather than linearly with N. In this regime, the SINR cost of increasing spectral efficiency is extremely large. For example, doubling spectral efficiency requires doubling the SINR *in dB units* rather than in linear units. Clearly, the benefit of further increasing the number of sub-bands is strongly outweighed by the SINR cost.

Thus, when N is such that the spectral efficiency $N\tilde{R}$ is relatively small (i.e., less than one), N should be increased because the benefit of reduced interference outweighs the cost of the increasing SINR threshold. However, as $N\tilde{R}$ increases, the cost of the (exponentially) increasing the SINR threshold eventually outweighs the benefit of reduced interference. Since transmission capacity depends on the SINR threshold raised to the power $-\frac{2}{\alpha}$, a larger path loss exponent corresponds to weaker dependence on the SINR threshold and thus a larger optimum spectral efficiency b^*.

V. Numerical Results and Discussion

In Figure 2, the maximum density of transmissions is plotted as a function of N for two different spectrum utilizations $\tilde{R}$ for a network with $\alpha = 4$, $d = 10$ m, and an outage constraint of $\epsilon = 0.1$. The bottom set of curves correspond to a relatively high utilization of $\tilde{R} = 0.5$ bps/Hz, while the top set corresponds to $\tilde{R} = 0.25$ bps/Hz. Each set of three curves correspond to the approximation from (2): $\lambda(\epsilon, N) \approx N\left(\frac{\epsilon}{\pi d^2}\right)\beta(N)^{-\frac{2}{\alpha}}$, numerically computed values of $\lambda(\epsilon, N)$ for $SNR = \infty$, and numerically computed values for $SNR = 20$ dB. For both sets of curves, notice that the approximation, based on which the optimal value of N was derived, matches almost exactly with the numerically computed values. Furthermore, introducing noise into the network has a minimal effect on the density of transmissions.

For a path loss exponent of 4, evaluation of (7) yields an optimal spectral efficiency of 2.3 bps/Hz. When $\tilde{R} = 0.25$,

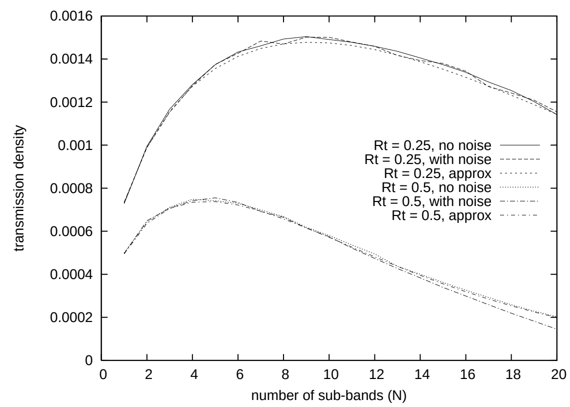

Fig. 2. Optimal Spectral Efficiency vs. Path-Loss Exponent

this corresponds to $N^* = \frac{2.3}{0.25} = 9.2$ and $N = 9$ is seen to be the maximizing integer value. When $\tilde{R} = 0.5$, we have $N^* = 4.6$ and $N = 5$ is the optimal integer choice. Note that there is a significant penalty to naively choosing $N = 1$: for $\tilde{R} = 0.25$ this leads to a factor of 2 decrease in density, while for $\tilde{R} = 0.5$ this leads to loss of a factor of 1.5.

A. Direct Sequence Spread-Spectrum

Another method of utilizing the bandwidth is to use direct-sequence spread spectrum with a spreading gain of N and an information bandwidth of $\frac{1}{N}$ (i.e., a symbol rate of $\frac{1}{N}$). However, the results of [1] show that direct-sequence is (rather significantly) inferior to splitting the frequency band (FDMA) because it is preferable to avoid interference (FDMA) rather than to suppress it (DS). More concretely, if DS-CDMA is used with completely separate despreading (assuming a spreading gain of $\frac{1}{N}$) and decoding, the maximum density of transmissions is approximately equal to:

$$\lambda(\epsilon, N)^{DS} \approx \left(\frac{\epsilon}{\pi d^2}\right) N^{\frac{2}{\alpha}} \beta(N)^{-\frac{2}{\alpha}} \tag{9}$$

Comparing this with the analogous expression in (5), we see that the key difference is that DS results in a leading term of $N^{\frac{2}{\alpha}}$ rather than N. As a result, the density remains constant as N is increased in the regime where $\beta(N)$ is approximately linear, and decreases with N once $\beta(N)$ begins to behave exponentially. As a result, using DS can lead to a considerable performance loss relative to spreading via frequency orthogonalization.

Another way to understand the inferiority of direct-sequence is the following: using direct-sequence with a spreading gain of $\frac{1}{N}$ reduces interference power by a factor of N and thereby increases the SINR roughly by a factor of N. In the wideband regime, the SINR threshold $\beta(N)$ increases approximately linearly with N and thus completely offsets the value of spreading; as a result the maximum density does not depend on N in this regime. Beyond the wideband regime, the SINR threshold $\beta(N)$ increases exponentially with N, which clearly outweighs the linearly increasing SINR provided by the spreading gain; the maximum density decreases with N in this regime.

B. Below-Capacity Transmission

In practical systems, it is not generally possible to signal precisely at capacity. One very useful approximation is the capacity gap metric, where $R = \log_2(1 + \Gamma \cdot SINR)$ and $\Gamma \leq 1$ is the (power) gap between the signaling rate and Shannon capacity. It is straightforward to see that the gap only increases the SINR threshold by a multiplicative constant: $\beta(N) = \frac{1}{\Gamma}\left(2^{N\tilde{R}} - 1\right)$. As a result, the earlier analysis remains unchanged and the optimum spectral efficiency as well as the optimum number of sub-bands are independent of the gap Γ. Indeed, the effect of the coding gap is simply to reduce the density of transmissions by a factor $\Gamma^{-\frac{2}{\alpha}}$.

C. Fixed vs. Random Networks

Our analysis holds for networks in which nodes are *randomly* located according to a homogeneous 2-D Poisson process. It would be interesting to know how this compares with the transmission capacity for any arbitrary, deterministic, placement of nodes (with zero outage). By comparing the two, we can determine the penalty that is paid for by having randomly rather than regularly placed nodes.

To allow for a fair comparison, we develop bounds on a network in which $\tilde{R} = b^*$ and thus $N = 1$ is optimal. A simple upper bound on the transmission density can be developed by considering only the interference contribution of the nearest interferer. The received SIR, again ignoring thermal noise, is upper bounded by considering the contribution of only the nearest interferer, assumed to be a distance s away. The SIR upper bound is thus given by $\frac{\rho d^{-\alpha}}{\rho s^{-\alpha}} = \left(\frac{s}{d}\right)^{\alpha}$. The upper bound must be above the threshold β if the actual SINR is above β, and thus the following is a necessary condition:

$$\left(\frac{s}{d}\right)^{\alpha} \geq \beta \quad \rightarrow \quad s \geq d\beta^{\frac{1}{\alpha}}.$$

Therefore, a necessary but not sufficient condition for meeting the SINR threshold is that there is no interferer within $d\beta^{\frac{1}{\alpha}}$ meters of a receiver. As a result, it is necessary that an area of $\pi d^2 \beta^{\frac{2}{\alpha}}$ meters^2 around each receiver be clear of interferers, which translates into a density upper bound of $\frac{1}{\pi d^2}\beta^{-\frac{2}{\alpha}}$. Since $\beta = 2^{b^*} - 1$, this gives

$$\lambda^{\text{det}} \leq \frac{1}{\pi d^2}(2^{b^*} - 1)^{-\frac{2}{\alpha}}. \tag{10}$$

A lower bound to the optimal density is derived by actually designing a (infinite) placement of transmitters and receivers. Indeed, by placing transmitters according to a standard square lattice and placing receivers on a horizontally shifted version of this lattice, one can achieve a density within about a factor of two of the upper bound.

The optimal density bounds should be compared to the density of a random network with $\tilde{R} = b^*$ found from (5):

$$\lambda^{\text{ran}} \approx \frac{\epsilon}{\pi d^2}(2^{b^*} - 1)^{-\frac{2}{\alpha}}. \tag{11}$$

Note that the random density is a factor ϵ smaller than the upper bound to the deterministic density. Thus, when ϵ is small, e.g., $\epsilon = 0.1$, there is a rather large penalty associated with random placement of nodes. This indicates that there potentially is a very significant benefit to performing localized transmission scheduling in random networks, assuming that the associated overhead is not too costly.

VI. Information Density

An interesting *information density* interpretation can be arrived at by plugging in the appropriate expressions for the maximum density of transmissions when the number of sub-bands is optimized. By plugging in the optimal value of N (and ignoring the integer constraint on N, which is reasonable when $\tilde{R}$ is considerably smaller than one) we have:

$$\lambda^*(\epsilon) = \max_N \lambda(\epsilon, N) \tag{12}$$

$$\approx \left(\frac{\epsilon}{\pi d^2}\right)\frac{1}{\tilde{R}} b^*(2^{b^*} - 1)^{-\frac{2}{\alpha}} \tag{13}$$

where b^* is defined in (7) and the quantity $b^*(2^{b^*} - 1)^{-\frac{2}{\alpha}}$ is denoted as the density constant in Fig. 1. The quantity $\lambda^*(\epsilon)$ is the maximum allowable spatial density of attempted transmissions per m^2 assuming each transmission occurs over a distance of d meters at spectral utilization $\tilde{R}$ (i.e., with rate equal to $W\tilde{R}$) and that an outage constraint of ϵ must be maintained.

From this expression we can make a number of observations regarding the tradeoffs between the various parameters of interest. First note that density is directly proportional to outage ϵ and to the inverse of the square of the range d^{-2}. Thus, doubling the outage constraint leads to a doubling of density, or inversely tightening the outage constraint by a factor of two leads to a factor of two reduction in density. The quadratic nature of the range dependence implies that doubling transmission distance leads to a factor of four reduction in density; this is not surprising given that the area of the circle centered at the receiver with radius d is πd^2. Perhaps one of the most interesting tradeoffs is between density and rate: since the two quantities are inversely proportional, doubling the rate leads to halving the density, and vice versa. Note that this relationship is directly attributable to the fact that N^* is inversely proportional to $\tilde{R}$: doubling rate leads to reducing N^* by a factor of two, which reduces total density (across all sub-bands) by a factor of two.

If we consider the product of density and spectral utilization, we get a quantity that has units bps/Hz/m^2:

$$\lambda^*(\epsilon)\tilde{R} \approx \left(\frac{\epsilon}{\pi d^2}\right) b^*(2^{b^*} - 1)^{-\frac{2}{\alpha}} \tag{14}$$

This quantity is very similar to the *area spectral efficiency* (ASE) defined in [8]. In our random network setting, the ASE is inversely proportional to the square of the transmission distance, which is somewhat analogous to cell radius in a cellular network, and is directly proportional to the outage constraint. Since the quantity $b^*(2^{b^*} - 1)^{-\frac{2}{\alpha}}$ does not vary too significantly with the path-loss exponent (see Fig. 1) for α between 2 and 5, we see that ASE and path-loss exponent are not very strongly dependent. Perhaps most interesting is the fact that the ASE does not depend on the desired rate (assuming N is optimized for rate). A random network can support a low density of high rate transmissions or a high density of low rate transmissions, or any intermediate point between these extremes.

VII. Conclusion

In this work we studied bandwidth-SINR tradeoffs in ad-hoc networks and derived the optimal operating spectral efficiency, assuming that multi-user interference is treated as noise and that no transmission scheduling is performed. A network can operate at this optimal point by dividing the total available bandwidth into sub-bands sized such that each transmission occurs on one of the sub-bands at precisely the optimal spectral efficiency. As a result, the optimal number of sub-bands is simply the optimal spectral efficiency (which is a deterministic function of the path loss exponent) divided by the normalized (by total bandwidth) rate.

The key takeaway of this work is that an interference-limited ad-hoc network should operate in neither the wideband (power-limited) nor high-SNR (bandwidth-limited) regimes, but rather at a point between the two extremes because this is where the optimal balance between multi-user interference and bandwidth is achieved. Although we considered a rather simple network model, we believe that many of the insights developed here will also apply to more complicated scenarios, e.g., wideband fading channels and networks in which some degree of local transmission scheduling is performed.

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