

# Relativistic hydrodynamics with sources for cosmological K-fluids

Alberto Díez-Tejedor\* and Alexander Feinstein†

*Dpto. de Física Teórica, Universidad del País Vasco, Apdo. 644, 48080, Bilbao, Spain.*

We consider hydrodynamics with non conserved number of particles and show that it can be modeled with effective fluid Lagrangians which explicitly depend on the velocity potentials. For such theories, the “shift symmetry”  $\phi \rightarrow \phi + \text{const.}$  leading to the conserved number of fluid particles in conventional hydrodynamics is globally broken and, as a result, the non conservation of particle number appears as a source term in the continuity equation. The particle number non-conservation is balanced by the entropy change, with both the entropy and the source term expressed in terms of the fluid velocity potential. Equations of hydrodynamics are derived using a modified version of Schutz’s variational principle method. Examples of fluids described by such Lagrangians (tachyon condensate, k-essence) in spatially flat isotropic universe are briefly discussed.

## I. INTRODUCTION

It is well known that complex physical phenomena can be often modeled with good accuracy by an effective theory. One such effective macroscopic model, for example, is the hydrodynamical model of Landau [1], which has had a considerable success in explaining certain features of the collisions of highly relativistic nuclei [2, 3]. The universe, the most complex of all the physical systems, is in general successfully modeled by an isentropic perfect fluid. Hydrodynamic language, back in high regard, is now invoked to describe non-trivial field theories [4].

In cosmology, as mentioned above, the perfect fluid description, despite the generic complexity of the system, works fine. One of the usual assumptions in the conventional hydrodynamical description of the universe is that the universe expands adiabatically. Closely related to it is the assertion that the so-called mass, or particle number conservation, holds. Yet, one can imagine a universe where creation or destruction of particles takes place. This may happen due to the time variation and inhomogeneities of the gravitational field itself, not to discard a more speculative possibility of a universe filled with white and black holes where particles suddenly appear or disappear. What kind of an effective hydrodynamics would then describe such a universe?

There are several ways to approach the problem of the universe where particles are created or annihilated. If this happens due to quantum processes, then presumably the most direct approach would be to consider the quantization of the matter fields on a curved background using the machinery of the quantum field theory [5], and then evaluating the back-reaction of the created fields on the classical geometry. The promising direction within this approach is the study of stochastic gravity [6].

It is possible, though, that for some reason, one is not interested in the detailed description of the particle creation (destruction) mechanism. Then one would

be trying to model the effects of the microscopic processes by a kind of an effective macroscopic model. In hadron-hadron collision theory [2, 3], such an effective macroscopic model is the Landau’s hydrodynamical description.

In the framework of cosmology with non conserved number of particles, a possible macroscopic description was put forward by Prigogine et al [7], and later generalised by Calvao et al [8] some years ago. In this approach, the creation of particles is considered in the context of thermodynamics of open systems. What follows then, roughly speaking, is that an extra negative “viscous” pressure term appears in the energy-momentum tensor to account for the created particles.

Yet, there exists a more “economic” and elegant way to describe particle creation (annihilation) without a change in the form of the energy-momentum tensor, and without introducing an extra pressure term. To introduce a source term into the particle number conservation equation it is sufficient to allow entropy flow. The change in the particle number allowed by the continuity equation will then come at the expense of the entropy change.

In this paper we are interested in exploring a Lagrangian formulation of relativistic hydrodynamics with non conserved number of particles. In the conventional variational approach to relativistic hydrodynamics developed by Schutz [9], the action does not depend on the velocity potential, but rather is a functional of its derivatives. This, in turn, maintains the symmetry  $\phi \rightarrow \phi + \text{const.}$  which allows particle number conservation. Here, we propose a Lagrangian formulation for the equations of hydrodynamics, where the Lagrangian, to break globally the symmetry leading to the particle number conservation, depends not only on the derivatives, but on the velocity potential itself.

We propose to modify Schutz’s original Lagrangian [9, 10, 11], by introducing sources and sinks in the continuity equation, modeled by a velocity potential dependent function. Our formulation is mathematically self-consistent, in that it gives the right set of hydrodynamical equations. Physically, the fluid Lagrangians we consider have connection to matter described by the rolling tachyon condensate [13] or by the K-essence [14, 15].

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\*wtbditea@lg.ehu.es

†wtpfexxa@lg.ehu.es

## II. THE HYDRODYNAMICS WITH PARTICLE NUMBER VARIATION

We start by assuming that we deal with simple thermodynamical systems (fluids), which are characterised by a fundamental equation of the form  $U = U[S, V, N]$ , where all the variables have their usual meanings, and we use  $k_B = c = 8\pi G = 1$ . Assuming the standard thermodynamic relations, one may show that such a system is completely specified by the energy density function  $\rho(n, s)$  and the system's size  $V$ :

$$U[S, V, N] = V\rho(n, s),$$

where  $n$  is the particle number density and  $s$  is the entropy per particle. We can write the first law of thermodynamics as

$$d\rho = hdn + nTds, \quad (1)$$

where  $h$  is the enthalpy per particle. Assuming further that the particle number in the system is not conserved, the equations of hydrodynamics take the following form:

$$T^{\mu\nu}{}_{;\mu} = 0, \quad (2)$$

$$(nu^\mu)_{;\mu} = \psi, \quad (3)$$

where  $T^{\mu\nu}$  stands for the usual stress-energy tensor of a perfect fluid:

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}. \quad (4)$$

Here  $p$  and  $\rho$  are the pressure and the energy density of the fluid respectively, and  $u^\mu$  is the four-velocity field ( $u_\mu u^\mu = -1$ ). The equation (2) represents the conservation of the energy-momentum tensor, whereas (3) is the continuity equation with the source ( $\psi > 0$ ) or the sink ( $\psi < 0$ ) term for the particles. We must further specify the equation of state  $\rho = \rho(n, s)$ , along with the source term  $\psi$ , which we take to have the form  $\psi = \psi(n, s)$ . To close this system of equations we add the first law of thermodynamics (1). The equations (2), (3) and (1) form a self consistent field theory describing a fluid with particle number variation in terms of five macroscopic (or Eulerian) variables  $(n, s, u^\mu)$ .

To obtain a more intuitive form of these equations it is convenient to project the energy conservation equation (2) along and, in the direction perpendicular, to the four-velocity. The parallel projection ( $u_\mu T^{\mu\nu}{}_{;\nu} = 0$ ), after the balance equation (3) and the thermodynamical relations have been substituted, gives the following continuity equation:

$$s_{,\mu}u^\mu = -\frac{h\psi}{nT}. \quad (5)$$

This equation was first given, in a somewhat different form, by Schutz and Sorkin [10]. One can appreciate how the change in the number of particles ( $\psi$ ) is accompanied

by a change in the entropy per particle ( $u^\mu s_{,\mu} \neq 0$ ). The fluid flow no longer follows lines of constant  $s$ , as it happens in the conventional hydrodynamics when no source is present ( $\psi = 0$ ). The projection perpendicular to the four-velocity gives ( $P_{\mu\alpha}T^{\mu\nu}{}_{;\nu} = 0$ , with  $P_\mu^\nu \equiv u^\nu u_\mu + \delta_\mu^\nu$ ):

$$(\rho + p)u_{\alpha;\nu}u^\nu = -p_{,\nu}P_\alpha^\nu,$$

which is the relativistic Euler equation. The last two equations are completely equivalent to the eqs. (2) and (3).

The variation rate of the number of particles  $N$  and the total entropy  $S$  of the fluid may still be written in a more suggestive way:

$$\frac{dN}{d\tau} = V\psi, \quad \frac{dS}{d\tau} = -\frac{\mu}{T} \frac{dN}{d\tau}, \quad (6)$$

where  $\mu = h - sT$  is the chemical potential. From the first of these two equations we see that the sign of the source term determines as to whether the particles are created or annihilated. The other equation describes the variation of the entropy, whose change is determined by both, the sign of the chemical potential and the source term.

## III. THE ACTION PRINCIPLE

The relativistic perfect fluid action functionals were developed by Taub [12] and Schutz [9]. Here we follow closely Schutz's velocity potential formalism [9]. In the case of the conventional hydrodynamics, where no particle creation takes place ( $\psi = 0$ ), one starts with the following action [9, 10, 11]:

$$S = \int d^4x \left\{ -\sqrt{-g}\rho(n, s) + J^\mu (\phi_{,\mu} + s\theta_{,\mu} + \beta_A \alpha^A_{,\mu}) \right\},$$

with  $A$  taking the values 1, 2 and 3. Here,  $\phi$  and  $\theta$  are Lagrange multipliers introduced to satisfy the particle number and the entropy conservation constraints respectively. One further assumes the existence of Lagrangian coordinates  $\alpha^A$  which label the flow lines, and consequently introduces the  $\beta_A$  potentials in form of Lagrange multipliers.  $J^\mu$  is the particle number current-density, defined as  $J^\mu \equiv \sqrt{-g}nu^\mu$ . The expression for the current permits to write the particle number density as  $n = |J|/\sqrt{-g}$ . To include the gravity as a dynamical field into the picture, one adds, as usual, the Einstein-Hilbert term to the above action. The variables in which the action is formulated are, therefore:  $g^{\mu\nu}$ ,  $J^\mu$ ,  $\phi$ ,  $s$ ,  $\theta$ ,  $\beta_A$  and  $\alpha^A$ . Starting with this action, one derives both the hydrodynamical equations of motion and the energy-momentum tensor for the fluid [11].

We now consider the action principle for the hydrodynamics described in the previous section. For this purpose, we put forward the following action [16]:

$$S = \int d^4x \left\{ -\sqrt{-g}\rho(n, s) + J^\mu (\phi_{,\mu} + \beta_A \alpha^A_{,\mu}) \right\},$$

where now the entropy per particle  $s$  is not an independent variable any more. We assume  $s = s(\phi)$ , so that the “shift symmetry”  $\phi \rightarrow \phi + \text{const.}$  present in the Schutz’s original action is globally broken. We have also suppressed the term  $J^\mu s \theta_{,\mu}$  in the action, since now we do not expect the entropy per particle  $s$  to conserve. To justify physically the functional dependence of the entropy on the velocity potential (apart from the fact that such an action leads to the equations of motion we expect) we suggest that since the non-conservation of the particles via the equation (5) leads to the entropy change, on one hand, and that we model the particle non-conservation by the symmetry breaking with the  $\phi$ -dependent term in the action on the other, it looks reasonable to introduce this dependence in the entropy term. One must bear in mind, however, that due to the particular parametrisation  $s(\phi)$ , the particle variation rate is neither arbitrary, nor generic, yet we find it sufficiently general for the purposes of the physics we are interested in. With this in mind, one may show [16] that the equations of hydrodynamics as well as the form of the energy momentum tensor of the section II can be recovered from the above action.

Introducing the equations of motion back into the action we obtain the on-shell expression

$$S_{on-shell} = \int d^4x \sqrt{-g} p,$$

which coincides with the on-shell expression of Schutz for conventional hydrodynamics. Thus, we have that the Lagrangian for the hydrodynamics with particle non-conservation may still be given by the pressure of the fluid.

#### IV. THE IRROTATIONAL FLOW

We now assume the fluid flow to be irrotational. We also note, that although until now we have expressed the action in terms of  $\rho(n, s)$ , it is often convenient to use, with the help of the usual thermodynamic relations, a different parametrisation of the action [11]. In the context of the irrotational flow it will be more convenient to work with the equation of state  $p = p(h, s)$ .

First, let us see what happens in the conventional case when the particle number is conserved. In this case we have an isentropic fluid ( $s = \text{const.}$ ) and the action may be expressed as

$$S = \int d^4x \sqrt{-g} \left\{ p(|V|) - \left( \frac{\partial p}{\partial h} \right)_s \left[ |V| - \frac{V^\mu \varphi_{,\mu}}{|V|} \right] \right\},$$

where we have defined the current  $V^\mu \equiv hu^\mu$ , and the subindex  $s$  refers to the fact that the partial derivative  $(\partial p / \partial h)$  is evaluated at constant  $s$ . The variables are  $g^{\mu\nu}$ ,  $V^\mu$  and  $\varphi$ , and the following equations of motion result:

$$u_\mu = -h^{-1} \varphi_{,\mu}, \quad (7)$$

$$(nu^\mu)_{;\mu} = 0, \quad (8)$$

with the energy-momentum tensor given by

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} = \left( \frac{\partial p}{\partial h} \right)_s hu^\mu u^\nu + pg^{\mu\nu}. \quad (9)$$

Comparing the last equation with the equation (4) allows to define the pressure and the energy density of the fluid as:  $p = p$  and  $\rho = nh - p$  (with  $n = (\partial p / \partial h)_s$ ). The pressure and the density defined via the stress-energy tensor coincide with their usual thermodynamical definitions. The equation (7) is the expression of the fact that the fluid flow is irrotational, whereas the equation (8) is the particle number conservation equation.

The identity  $u_\mu u^\mu = -1$  and the equation (7), lead to the following expression for the enthalpy:

$$h = \sqrt{-\varphi_{,\mu} \varphi^{,\mu}}. \quad (10)$$

To make contact with the now popular K-essence cosmology [14, 15], we write the action on-shell as

$$S_{on-shell} = \int d^4x \sqrt{-g} F(X), \quad (11)$$

where we define:

$$X \equiv -\frac{1}{2} \varphi_{,\mu} \varphi^{,\mu} = \frac{h^2}{2}, \quad (12)$$

$$p(h) = p\left(\sqrt{2X}\right) \equiv F(X), \quad (13)$$

Therefore, if one has an irrotational fluid where the number of particles is conserved, it is described by the hydrodynamics derived from the Lagrangian (11), which depends only on the derivatives of the velocity potential defined by the equation (7). Moreover, the conservation equation (8), is just the Euler-Lagrange equation derived from the action (11):

$$-(nu^\mu)_{;\mu} = [F'(X) \varphi^{,\mu}]_{;\mu} = 0,$$

where we have used  $n = (\partial p / \partial h)_s = hF'(X)$ , and the prime stands for the derivative of the function with respect to its argument. For completeness, we give the expression for the density and the pressure of the fluid in terms of the variable  $X$ :

$$p = F(X), \quad \rho = 2XF'(X) - F(X). \quad (14)$$

These expressions are known in the context of K-field as purely kinetic K-field [14, 15, 17].

Typically, in cosmology, one uses an equation of state  $p = f(\rho)$  to describe an isentropic fluid. To obtain an action in the form (11) describing such a fluid, one only has to express the energy density as  $\rho = f^{-1}(p)$ , insert the latter expression into the equation of energy density (14), and obtain the differential equation for  $F$ ,  $f^{-1}(F) = 2XF' - F$ . This gives then for  $F(X)$ :

$$\int^F \frac{dF^*}{f^{-1}(F^*) + F^*} = \ln [CX^{1/2}], \quad (15)$$

where  $C$  is an arbitrary integration constant. The equation (15) establishes how to pass from the standard hydrodynamical description of an isentropic irrotational perfect fluid ( $p = f(\rho)$ ), to the language of an action principle (11). Put differently, the purely kinetic K-field, is interpretable in terms of an isentropic perfect fluid with an equation of state which can be easily put into the form  $p = p(\rho)$ . Thus, *any solution to the Einstein's field equations with the energy momentum tensor of the irrotational perfect fluid with the equation of state  $p = p(\rho)$  is by default interpretable as a solution for the purely kinetic K-fluid.*

We now consider the irrotational flow where the number of particles is not conserved. In this case, the action can be expressed as [16]:

$$S = \int d^4x \sqrt{-g} \left\{ p(|V|, s) - \left( \frac{\partial p}{\partial h} \right)_n \left[ |V| - \frac{V^\mu \phi_{,\mu}}{|V|} \right] \right\}$$

along with  $s = s(\phi)$ . The variables still are  $g^{\mu\nu}$ ,  $V^\mu$  and  $\phi$ , and the equations of motion that follow are

$$u_\mu = -h^{-1} \phi_{,\mu}, \quad (16)$$

$$(nu^\mu)_{;\mu} = -nT \frac{ds}{d\phi}. \quad (17)$$

In the last equation we have used  $-nT = (\partial p / \partial s)_h$ . The form of the energy-momentum tensor is left unchanged and is given by the equation (9). The equation (16) expresses again the fact that the flow is irrotational, and the continuity equation (17), if we define the particle creation rate as

$$\psi \equiv -nT \frac{ds}{d\phi}, \quad (18)$$

is the balance equation (3). We hence model the creation of particles through the function  $s = s(\phi)$ . Using the property  $u_\mu u^\mu = -1$  and the equation (16), we obtain the equation (10), and now the on-shell action becomes:

$$S_{on-shell} = \int d^4x \sqrt{-g} L(\phi, X), \quad (19)$$

where we have used the equation (12) and have defined

$$p(h, s) = p(\sqrt{2X}, s(\phi)) \equiv L(\phi, X). \quad (20)$$

We thus have succeeded in giving the action for the irrotational fluid flow where number of particles is not conserved in terms of the scalar velocity potential and its derivatives. Moreover, the continuity equation of the fluid (17), using  $n = (\partial p / \partial h)_s = h \partial L / \partial X$ ,  $u^\mu =$

$-h^{-1} \phi^{,\mu}$  and  $\psi = -nT ds / d\phi = \partial L / \partial \phi$  becomes the Euler-Lagrange equation for the action (19):

$$\left[ \frac{\partial L}{\partial X} \phi^{,\mu} \right]_{;\mu} + \frac{\partial L}{\partial \phi} = 0. \quad (21)$$

We finally express the pressure and the density of the fluid in terms of the scalar field:

$$p = L(\phi, X), \quad \rho = 2X \frac{\partial L(\phi, X)}{\partial X} - L(\phi, X). \quad (22)$$

## V. K-FLUID

A special case arises when the fluid has a separable equation of state  $p(h, s) = f(s)g(h)$ . In this case, the action takes the form

$$S = \int d^4x \sqrt{-g} K(\phi) F(X), \quad (23)$$

with definitions  $F(X) \equiv g(\sqrt{2X})$  and  $K(\phi) \equiv f(s(\phi))$ . The entropy per particle  $s$  can be then expressed as a function of the potential term  $K(\phi)$ :

$$s = f^{-1}[K(\phi)], \quad (24)$$

and the equation (22) permits to express the pressure and energy as:

$$p = K(\phi)F(X), \quad \rho = K(\phi)[2XF'(X) - F(X)]. \quad (25)$$

The above expressions are analogous to factorisable K-field theories [14, 15], and we therefore refer to these fluids as K-fluids. The case in which there is no particle creation (purely kinetic K-fluid) is obtained with  $K(\phi) = \text{const}$ .

One usually assumes without loss of generality that  $K(\phi) > 0$  ( $f(s) > 0$ ), while  $F(X)$  may be either positive or negative, allowing for tensions instead of pressure. Yet, we want to have a positive energy density, we therefore must have

$$2XF'(X) - F(X) \geq 0. \quad (26)$$

In addition, the particle number density  $n$  must also be positive, so we need

$$F'(X) \geq 0 \quad (27)$$

and

$$\text{sgn}[f'(s)] = -\text{sgn}[p] \quad (28)$$

to have positive temperature. One may further define the sound speed in a usual way:

$$c_s^2 = \left( \frac{\partial p}{\partial \rho} \right)_s = \frac{F'(X)}{2XF''(X) + F'(X)}$$

(cf [18]). For ordinary fluids one usually also imposes  $0 \leq c_s^2 \leq 1$ , and therefore using (27), we have:

$$F''(X) \geq 0. \quad (29)$$

We therefore refer to K-hydrodynamics, or K-fluid for short, as to an irrotational fluid with an equation of state  $p(h, s) = f(s)g(h)$  and a particle variation rate given by  $\psi(h, s) = k(s)g(h)$ . The particle variation rate is further parametrised by  $s = s(\phi)$ , where  $\phi$  is the velocity potential of this irrotational fluid, and the peculiar functional form of the particle production rate  $\psi(h, s)$  is a consequence of this parametrisation choice. The action for the K-fluid is given by the equation (23). We impose positivity of the energy density (26), the particle number density (27) and the temperature (28), but are less stringent with the pressure, though one may always impose the positivity of the pressure as well. The fluid flow is stable as long as (29) holds.

We can now express the fluid parameters in terms of the scalar field. The particle number density and particle production rate take the form:

$$n = K(\phi)\sqrt{2X}F'(X), \quad (30)$$

$$\psi = F(X)K'(\phi), \quad (31)$$

and consequently the sign of the derivative of  $K$  defines as to whether the creation or annihilation of particles takes place. When  $K(\phi) = \text{const.}$ , the expression  $N = Vn$  is the Noether's charge associated with the "shift symmetry"  $\phi \rightarrow \phi + \text{const.}$  of the action. These expressions above can be written in terms of the action without the explicit knowledge of the function  $f(s)$ . However, to evaluate the entropy per particle (24), total entropy  $S = Vns$  and temperature

$$T = \frac{-f'(s)}{n}F(X), \quad (32)$$

one must know the form of  $f(s)$ . Some examples will be given in the following section.

With the above hydrodynamical interpretations, let us look for a moment at the K-fluids where the number of particles is conserved. The entropy per particle is then a constant, say  $s_0$ , and therefore the equation of state has the form  $p = p(h)$ . This is an isentropic fluid characterised by the function  $F(X)$  and the constant  $f(s_0)$ . The action for the fluid becomes:

$$S = \int d^4x \sqrt{-g} f(s_0) F(X), \quad (33)$$

where the function  $F(X)$  is given by (15) subject to the conditions (26), (27) and (29), while  $f(s)$  must verify (28). The Lagrangian (33), up to a non-essential multiplicative constant, is the Lagrangian for the purely kinetic K-field [17], for which we have defined the pressure, the energy density (25), the entropy per particle (24)

$s = s_0$ , the particle number density (30) and the temperature (32).

To close this section we give the dynamical equation (21) in the case of the factorisable K-fluid theory:

$$\nabla_\mu [K(\phi)F'(X)\phi'^\mu] + K'(\phi)F(X) = 0. \quad (34)$$

## VI. PARTICULAR EXAMPLES

Let us consider some particular examples. We start by specifying the following equation of state:

$$p(h, s) = e^{\mp s} g(h), \quad (35)$$

where we have  $-$  for  $p > 0$  and  $+$  for  $p < 0$ , in accordance with (28). We see that this equation of state is of the form described in the previous section. One must further specify the function  $s(\phi)$  in terms of the particle creation rate, so that the entropy per particle (24) can be expressed as a function of the potential:

$$s = \mp \ln [K(\phi)]. \quad (36)$$

For the entropy per particle to be positive, one should impose  $0 < K(\phi) < 1$  ( $K(\phi) > 1$ ) for  $p > 0$  ( $p < 0$ ). With the equation of state (35), the temperature of the fluid becomes

$$T = \frac{-1}{n} \frac{\partial p}{\partial s} = \frac{|p|}{n}.$$

Note, that this expression for the temperature coincides with the expression one would have for a typical fluid composed of non-interacting physical particles (generalized to negative pressures), and is a consequence of the choice we made for the equation of state (35). In terms of the field we have the particle number density (30), particle rate production (31), and with this choice of  $f(s)$  we can compute the entropy per particle (36) and the temperature (32)

$$T = \frac{|F(X)|}{\sqrt{2X}F'(X)}. \quad (37)$$

If we consider the case where the particle number remains constant, the action for the fluid becomes

$$S = \int d^4x \sqrt{-g} e^{\mp s_0} F(X), \quad (38)$$

where the function  $F(X)$  is evaluated from (15).

**Example 1:** Fluid with constant adiabatic index  $p = w\rho$  ( $w = \text{const.}$ ). We have  $\rho = f^{-1}(F) = F/w$ . From (15) we obtain:

$$F(X) = \pm X^{\frac{1+w}{2w}}, \quad (39)$$

where the sign  $+$  corresponds to  $w > 0$ , while the sign  $-$  corresponds to the case  $-1 \leq w < 0$ , after the constraints (26) and (27) have been applied [23]. In the case of the

stable flow, the constraint (29) imposes the positivity of the pressure together with  $0 < w \leq 1$ . Such a fluid is then described by the action

$$S = \int d^4x \sqrt{-g} e^{-s_0} X^{\frac{1+w}{2w}},$$

with  $0 < w \leq 1$ , and where

$$p = e^{-s_0} X^{\frac{1+w}{2w}}, \quad \rho = \frac{e^{-s_0}}{w} X^{\frac{1+w}{2w}},$$

$$n = e^{-s_0} \frac{(1+w)}{\sqrt{2}w} X^{\frac{1}{2w}}, \quad T = \frac{\sqrt{2}w}{1+w} X^{\frac{1}{2}}.$$

In a spatially flat FRW universe ( $ds^2 = -dt^2 + a^2(t) [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)]$ ), solving the dynamical (34) and Friedmann ( $3H^2 = \rho$ ) equations, we easily recover  $X(a) \propto a^{-6w}$  and  $a(t) \propto t^{2/3(1+w)}$ .

Now, let us turn to the theories where the number of particles is not conserved. The kind of action we have is (23), sticking to the factorisable theories. A typical potential to use would be the well studied  $K(\phi) \propto 1/\phi^2$  [14, 15, 19], due to the fact that it leaves one with solutions with constant enthalpy per particle in spatially flat isotropic universes. Yet, if we would use the equation of state (35) we would certainly run into trouble because of the restrictions on the function  $K(\phi)$ . There are two ways to circumvent this problem: either consider a different equation of state, or a different potential. If we stick to the above equation of state, then for example the following potentials  $K(\phi) = A \cosh \phi$  with  $A \geq 1$  and  $K(\phi) = A \exp(-\phi^2) + B$ , with  $A, B > 0$  and  $A + B < 1$ , will do. Now, the only advantage of using the equation of state (35) is that the temperature is given as the ratio of the pressure to the particle number density. The next simplest choice for an equation of state would be the one for which the entropy function  $f(s)$  is a power-law. We thus take

$$p(h, s) = s^b g(h), \quad (40)$$

where  $b$  is an arbitrary constant such that  $\text{sgn}(b) = -\text{sgn}(p)$  to satisfy (28). The entropy then, according to (24), is

$$s = [K(\phi)]^{\frac{1}{b}}, \quad (41)$$

and is compatible with the potentials of the form  $K(\phi) \propto 1/\phi^2$ . The particle number density and the particle rate production in terms of the field are given by (30) and (31), whereas the temperature now takes the form

$$T = \frac{|bF(X)|}{\sqrt{2X}F'(X)} [K(\phi)]^{-\frac{1}{b}}. \quad (42)$$

We will now look at two ‘‘similar’’ fluids in a spatially flat FRW universe, but with different properties with respect to the particle number conservation. In the first

case, the fluid is isentropic with a conserved particle number and provokes interest in both field theory [4] and cosmology [20]. The second case represents the same fluid where the number of particles is not conserved and has a Lagrangian of the form of Sen’s tachyon condensate [13], which has recently become of considerable interest in cosmology [19, 22].

**Example 2:** Fluid with equation of state  $p = -A/\rho$ ,  $A = \text{const.} > 0$  (Chaplygin gas). In this case we have  $\rho = f^{-1}(F) = -1/F$ . Inserting this in (15), we obtain, up to some unessential constants:

$$F(X) = \pm \sqrt{1 \pm X}.$$

We assume the constraints (26) and (27) hold, and we are left therefore with negative pressure:

$$F(X) = -\sqrt{1 - X} \quad (43)$$

with  $0 \leq X \leq 1$ , and there is no problem with the constraint (29), indicating that the flow is stable. We can think of such a fluid as a fluid with the equation of state

$$p(h, s) = -s^b \sqrt{1 - \frac{h^2}{2}} \quad (44)$$

in which the number of particles is conserved ( $b = \text{const.} > 0$ ). The action is then

$$S = - \int d^4x \sqrt{-g} (s_0)^b \sqrt{1 - X},$$

and therefore

$$p = -(s_0)^b \sqrt{1 - X}, \quad \rho = \frac{(s_0)^b}{\sqrt{1 - X}},$$

$$n = (s_0)^b \sqrt{\frac{X}{2(1 - X)}}, \quad T = \frac{b}{s_0} \sqrt{\frac{2}{X}} (1 - X).$$

Solving the field equation (34) in a spatially flat FRW model, one obtains

$$X(a) = \frac{1}{1 + Ba^6},$$

where  $B$  is an integration constant. From here one may evaluate all the hydrodynamical parameters in terms of the scale factor, arriving to the unusual result that the temperature of the Chaplygin gas rises with the expansion. This basically happens due to the negative pressure of the fluid [21]. One can further solve, as well, the Friedmann equation to find the behaviour of the scale factor as a function of time [20].

**Example 3:** Tachyon condensate. The possibility of fluid description of tachyon condensate in bosonic and supersymmetric string theories discovered by Sen [13] has motivated a considerable amount of work studying the consequences of the rolling tachyon in cosmology [19, 22].

Here we are interested to look at tachyon condensate action in the light of the formalism developed above as a fluid where the number of particles is not conserved. The action for the K-fluid with the form of the tachyon condensate is [13, 19]

$$S = - \int d^4x \sqrt{-g} K(\phi) \sqrt{1-X}.$$

We can think of the above action as one describing a fluid with equation of state (44) in which the particle rate production is modeled by (41). From this action we can read off:

$$p = -K(\phi)\sqrt{1-X}, \quad \rho = \frac{K(\phi)}{\sqrt{1-X}},$$

$$n = K(\phi) \sqrt{\frac{X}{2(1-X)}}, \quad T = \frac{b}{[K(\phi)]^{\frac{1}{b}}} \sqrt{\frac{2}{X}} (1-X).$$

It is simple to obtain particular cosmological solutions for such a fluid if one assumes a spatially flat isotropic cosmology and a potential of the form  $K(\phi) = \beta/\phi^2$  with  $\beta > 0$  [19]. Solving Einstein's equations one finds for the velocity potential  $\phi(t)$  and the scale factor of the universe  $a(t)$ :

$$\phi(t) = \sqrt{\frac{4}{3n}} t, \quad a(t) = t^n, \quad (45)$$

where  $n$  is a constant given in terms of the parameter of the potential  $n = n(\beta)$ . Therefore the parameter  $\beta$ , the slope of the potential, defines the particle creation rate as well as different expansion rate. For these particular solutions the enthalpy per particle of the fluid (10) remains constant. This is in contrast with the case of Chaplygin gas, where the entropy per particle was constant.

We can use the expressions (6) to evaluate the increase of the number of particles and entropy of the system in a time interval  $\Delta t$ . For the toy models with equation of state of the form (40), and a particle creation rate modeled by  $K(\phi) = \beta/\phi^2$ , we obtain the following expressions in a spatially flat FRW universe:

$$\Delta N(t_1, t_2) = -\frac{8\pi\beta}{3} F(X) \int_{t_1}^{t_2} \left(\frac{a}{\phi}\right)^3 dt,$$

$$\Delta S(t_1, t_2) = -\frac{8\pi\beta^{\frac{1+b}{b}}}{3} F(X) \left[ 1 + \frac{2XF'(X)}{bF(X)} \right] \int_{t_2}^{t_1} \frac{a^3}{\phi^{\frac{2+3b}{b}}} dt.$$

For the tachyon-like model, taking into consideration (45), we have

$$\Delta N \propto t^{3n-2}, \quad \Delta S \propto t^{\frac{3nb-2(b+1)}{b}}.$$

Since we must impose  $X < 1$  for the action to be well-defined, one has  $n > 2/3$ , and, interestingly enough, this

implies that the particles are created in such a universe. The creation rate is best visualised by the expression

$$\frac{1}{N} \frac{dN}{dt} = \frac{1}{\sqrt{2X}} \frac{\frac{d}{d\phi} [\ln K(\phi)]}{\frac{d}{dX} [\ln F(X)]}.$$

For the above tachyon example we readily find that the creation rate fades with time as  $t^{-1}$ .

We see that the fluid we have is the same as in Chaplygin gas (has the same equation of state), but the production of particles changes the evolution of the universe. Changing the particle creation rate one changes the expansion rate of the model.

## VII. CONCLUSIONS

In this paper we have considered a Lagrangian approach to a Relativistic Hydrodynamics in which the number of particles is not conserved. The particle number non-conservation is modeled by introducing an explicit velocity potential dependent term into the fluid Lagrangian. In doing so, the usual shift symmetry of the action is broken, resulting in the appearance of a source term in the continuity equation. The conservation equation derived from the stress-energy tensor indicates that the particle number non-conservation must be balanced by an entropy flow. Both the entropy flow and the change in the particle number are expressed as function of the velocity potential. Although such a description is valid for a general flow, we concentrate on the purely potential fluid motion without vorticity, to make contact with some modern theories used for the description of matter in the universe.

By identifying the K-essence field variable  $2X$  with the *square* of the enthalpy per particle  $h$  we identify the K-field theory and the hydrodynamical Lagrangians we look at. In the case of purely kinetic K-essence, we observe that this theory is identical to the isentropic perfect fluid, and give a 'dictionary' (15) as to how to pass from the usual description in cosmology in terms of the equation of state  $p = p(\rho)$  to the K-theory Lagrangians of the form  $F(X)$ . On a formal level, therefore, the purely kinetic K-essence is no 'big news', but rather a simple conventional hydrodynamics in a disguise.

The non-conventional hydrodynamics (K-hydrodynamics), the one analogous to the K-essence with the potential term, is rather more involved. First, one must interpret such a hydrodynamics as a flow where the number of particles is not conserved. This, in turn, leads to a change in the entropy per particle, as well as to a global entropy flow. The fluid now is not isentropic and to give an hydrodynamical description the two equations of state  $p = f(s)g(h)$  and  $\psi = \psi(s, h)$  must be specified. We have found [16] that our parametrisation works for source terms of the form  $\psi(s, h) = k(s)g(h)$ , i.e. the source term must be separable in functions of entropy and enthalpy, and the enthalpy function must be

the same as the one which appears in the pressure. This restricts the generality of the approach, nevertheless, it is of direct application to the K-essence-like cosmologies.

We have finally considered several examples of fluids with both conserved and non-conserved number of particles in the context of spatially flat isotropic universe. The telling example is the comparison of Chaplygin gas on one hand and a K-fluid with the form of tachyon condensate on the other. In the first case one deals with an isentropic perfect fluid where the number of particles is conserved. The peculiarity of this example is that the temperature of the gas rises up with the expansion. The second example represents a fluid with the same equation of state, but with the number of particles (entropy per particle) not conserved. It is interesting, however, that for special creation rates, those with the potential  $K = \beta/\phi^2$  with  $\beta > 0$ , the enthalpy per particle rather than the entropy remains constant in the course of the expansion. We also observe that in such a universe creation rather than destruction of particles takes place.

From a technical point of view it appears that the velocity potential/ $X$ -variable formalism is quite useful to study the dynamics of the cosmological models. It would be interesting in the future to obtain and study some of the K-fluid type Lagrangians obtained as effective theories derived from fundamental interactions. The work in this direction is in progress and will be presented elsewhere.

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separately. In this case it is convenient to work with the equation of state  $\rho = \rho(n)$ .