What does the Letelier-Gal'tsov metric describe?

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Abstract

Recently the structure of the Letelier-Gal'tsov spacetime has become a matter of some controversy. I show that the metric proposed in (Letelier and Gal'tsov 1993 *Class. Quantum Grav.* **10** L101) is defined only on a part of the whole manifold. In the case where it can be defined on the remainder by continuity, the resulting spacetime corresponds to a system of parallel cosmic strings at rest w.r.t. each other.

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Spacetimes with conical singularities, which presumably describe cosmic strings, often exhibit a rich and non-trivial structure even when they are flat. In studying such spacetimes it would be helpful to have a way of constructing them in a uniform analytical way without resorting to cut-and-glue surgery. One such a way was proposed some time ago by Letelier and Gal'tsov. Consider a manifold $M = \mathbb{R}^2 \times P$, where P is a plane, coordinatized by x and y, from which N points are cut out. Endow M with the metric

$$g: \qquad \mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}z^2 - \mathrm{d}Z\,\mathrm{d}\overline{Z}, \tag{1a}$$

where

$$Z(\zeta, t) \equiv \int_{\zeta_0}^{\zeta} \prod_{i=1}^{N} (\xi - \alpha_i(t,z))^{\mu_i} \mathrm{d}\xi, \qquad \zeta \equiv x + iy \tag{1b}$$

If $N = 1$ and $\alpha_1 = const$ the metric reduces to

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}z^2 - |\zeta - \alpha_1|^{2\mu_1}(\mathrm{d}x^2 + \mathrm{d}y^2),$$

which in the case $\mu_1 > -1$ is the metric of a static cosmic string parallel to the z-axis. So, Letelier and Gal'tsov assumed [1] that in the general case the spacetime (M, g) describes "a system of crossed straight infinite cosmic strings moving with arbitrary constant relative velocities" [2]. On the other hand, Anderson [3] calculated what he interprets as the distance between two strings and found that it is constant. This led him to the conclusion that (1) "is just the static parallel-string metric ... written in an obscure coordinate system". Recently, however, his calculations have been disputed in [2] where the opposite result was obtained. The goal of the present note is to resolve this controversy. I show that in the general case the metric (1) is defined only in *a part* of M and cannot be extended to the remainder unless the resulting spacetime corresponds to the set of parallel strings at rest.

I shall consider the case $N = 2$ (generalization to larger N is trivial) and for simplicity use the following notation

$$a \equiv \alpha_2(t_0), \quad v \equiv \dot\alpha_2(t_0).$$

I set

$$\alpha_1(t_0) = \dot\alpha_1(t_0) = 0, \qquad \operatorname{Im} a = \operatorname{Im} v = 0$$

(this always can be achieved by an appropriate coordinate transformation and thus involves no loss of generality). So, at $t = t_0$

$$Z(\zeta) = \int_{\zeta_0}^{\zeta} \xi^{\mu_1}(\xi - a)^{\mu_2} \mathrm{d}\xi, \quad Z_{,t}(\zeta) = -\mu_2 v \int_{\zeta_0}^{\zeta} \xi^{\mu_1}(\xi - a)^{\mu_2 - 1}\mathrm{d}\xi. \tag{2}$$

Finally, I require that

$$\mu_1, \mu_2 > -1, \tag{3}$$

because if $\mu_i \leqslant -1$, the corresponding singularity is infinitely far and does not represent a string.

Let us begin with the observation that

$$(\xi e^{2\pi i} - \alpha)^{\mu} \neq (\xi - \alpha)^{\mu}, \qquad \text{when} \quad \mu \notin \mathbb{Z}$$

and therefore the function Z and, correspondingly, the metric (1a) are generally not defined on the whole M. Let us choose the domain D_Z of $Z(\zeta)$ to be[1] $P - \mathbb{R}_+$. Then we can adopt the conventions that

$$\zeta = |\zeta| e^{i\phi}, \quad \zeta - a = |\zeta - a| e^{i\psi}, \qquad \forall\, \zeta \in D_Z,$$

where ϕ and ψ are the angles shown in figure 1, and that the integrals in (2) are taken along curves Γ that do not intersect $\mathbb{R}_+$. Substituting (2) in (1a) we find for any $\zeta \in D_Z$

$$g_{ty}(\zeta) = -\operatorname{Im}\{\overline{Z}_{,\zeta}\, Z_{,t}\} = -\operatorname{Im}\left\{\overline{\zeta^{\mu_1}(\zeta - a)^{\mu_2}} Z_{,t}\right\}.$$

[1] This choice is made for the sake of simplicity. Instead of $\mathbb{R}_+$ one could take, say, a pair of curves starting from 0 and α, respectively. The result would not change.

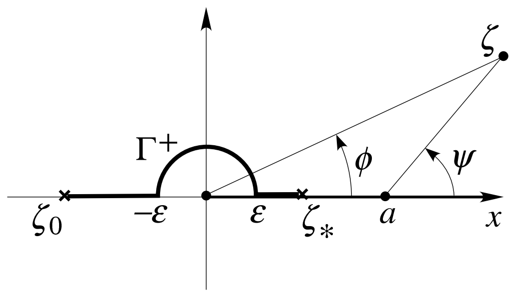

Figure 1: Γ_+ goes along the real axis from ζ_0, then along the upper half-circle of the radius ε, and then along the upper bank of the cut. Γ_- is obtained from Γ_+ by reflection w.r.t. the real axis.

In particular, picking $\zeta_* \in (0, a)$ we have

$$g_{ty}(\zeta_* \pm i0) = -\zeta_*^{\mu_1}(a - \zeta_*)^{\mu_2} \operatorname{Im}\Big\{e^{-i\pi[\mu_2+\mu_1(1\mp1)]} Z_{,t}\,(\zeta_* \pm i0)\Big\}. \qquad (4)$$

Let us choose Γ to be those of figure 1:

$$Z_{,t}\,(\zeta_* \pm i0) = -\mu_2 v \int_{\Gamma^\pm} \xi^{\mu_1}(\xi - a)^{\mu_2-1}\mathrm{d}\xi.$$

The value of the integral does not depend on ε, while at $\varepsilon \to 0$ the contribution of the half-circle vanishes by (3). Hence

$$Z_{,t}\,(\zeta_* \pm i0) = e^{i\pi(\mu_1+\mu_2)}\mu_2 v \int_0^{|\zeta_0|} r^{\mu_1}(r + a)^{\mu_2-1}\mathrm{d}r + e^{i\pi[\mu_2+\mu_1(1\mp1)]}\mu_2 v \int_0^{\zeta_*} r^{\mu_1}(a - r)^{\mu_2-1}\mathrm{d}r,$$

which being substituted in (4) gives

$$g_{ty}(\zeta_* \pm i0) = -\mu_2 v\zeta_*^{\mu_1}(a - \zeta_*)^{\mu_2} \operatorname{Im}\Big\{e^{\pm i\pi\mu_1} \int_0^{|\zeta_0|} r^{\mu_1}(r + a)^{\mu_2-1}\mathrm{d}r + \int_0^{\zeta_*} r^{\mu_1}(a - r)^{\mu_2-1}\mathrm{d}r\Big\}$$

and as a result

$$g_{ty}(\zeta_*+i0)-g_{ty}(\zeta_*-i0) = -2\mu_2 v\zeta_*^{\mu_1}(a-\zeta_*)^{\mu_2}\sin(\pi\mu_1)\int_0^{|\zeta_0|} r^{\mu_1}(r+a)^{\mu_2-1}\mathrm{d}r.$$

Since the metric must be continuous, for a negative μ_1 it follows $v = 0$, i. e. the strings are at rest with respect to each other. Repeating exactly the same reasoning with t changed to z one finds that $\partial_z \alpha_2(t_0) = 0$ and so, the strings are also parallel.

Remark. If one allows μ_1 to be positive, an exception appears: the metric may be smooth for $v \neq 0$ if $\mu_1 = n$. The singularities of this type are interesting and useful [4], but can hardly be called 'strings' (in particular such singularities cannot be 'smoothed out' without violation of the weak energy condition).

References

[1] P. S. Letelier and D. V. Gal'tsov *Class. Quantum Grav.* **10** (1993) L101.

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