

T-spheres as a limit of Lemaître-Tolman-Bondi solutions

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In the Tolman model there exist two quite different branches of solutions - generic Lemaître-Tolman-Bondi (LTB) ones and T-spheres as a special case. We show that, nonetheless, T-spheres can be obtained as a limit of the class of LTB solutions having no origin and extending to infinity with the areal radius approaching constant. It is shown that all singularities of T-models are inherited from those of corresponding LBT solutions. In doing so, the disc type singularity of a T-sphere is the analog of shell-crossing.

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The famous Tolman model [1] remains viable until now (see, for example, [2], [3] for its astrophysical applications). Its solutions split to two branches. The first one is the widely known Lemaître-Tolman-Bondi (LTB) solutions which describe an inhomogeneous collapse of dust (or its time reversal). There is also one more brunch that arises as a special solution of Einstein equations with the areal radius $R = R(t)$ not depending on a spatial coordinate [4], [5], [6] (recently such a type of solutions was discussed for rotating dust [7]). For such a kind of solution the 01 Einstein equation reduces to the identity in contrast to the LTB case where its integration is essential for finding the metric. The special solution under discussion (called "T-spheres" or "T-models" in [5], [6]) possesses a number of unusual properties. For instance, they realize an "ideal gravitational machine" in that an infinite amount of matter is bound to a finite mass, etc. It was stressed in [5], [6] that T-spheres cannot be obtained from the LTB solutions. As far as the structure of singularities is concerned, it was shown that some singularities of T-models are similar to those of the closed Friedmann solution (which is the important particular case of LTB solutions) but, in addition, there are also

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singularities of the disk type.

The aim of the present work is to show that, in spite of the fact that T-spheres are not contained in any LTB family of solutions, they can be obtained from LTB as their limiting case. In doing so, the LTB prototype that generates T-spheres is not the Friedmann solution but the solution that contains no origin from one side. We also discuss the relationship between singularities of both models and show that singularities of T-spheres (including those that are absent in Friedmann-like models) are actually the same as in the LTB prototype under discussion. Some time ago it was already found that, by a quite different limiting transition, one can obtain from the LTB family also the Vaidya one [8], [9], [10] and that the singularities of both models are the same. In this respect, our results extend the similar relationship to T-models.

In general, the LTB solution is characterized by three function of which only two (because of the freedom in rescaling a radial coordinate) can be chosen arbitrarily. The metric can be written in the form

$$ds^2 = -dt^2 + \exp(2\omega)dr^2 + R^2(t, r)d\Omega^2, \quad (1)$$

$$\exp(2\omega) = \frac{R'^2}{1 + f(r)}, \quad (2)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, prime and dot denote derivatives with respect to r and t , correspondingly. For simplicity, the cosmological term $\Lambda = 0$ (but all results are generalized easily to the case $\Lambda \neq 0$). Here

$$\dot{R}^2 = \frac{F(r)}{R} + f, \quad (3)$$

where $F(r) = 2m(r)$, the quantity $m(r)$ plays the role of an active gravitational mass. The function $f = f(r)$ can have any sign. For our purposes, as will be seen from what follows, it is sufficient to restrict ourselves to the case $f < 0$ (the so-called elliptic case). Then

$$R = \frac{F}{2(-f)}(1 - \cos\eta), \quad (4)$$

$$\eta - \sin\eta = \frac{2(-f)^{3/2}}{F}(t - a), \quad a = a(r), \quad (5)$$

$a(r)$ is the time of the big bang (if $t \geq a$) or big crunch (if $t \leq a$). The energy density

$$8\pi\rho = \frac{F'}{R'R^2}. \quad (6)$$

We will need for what follows a convenient representation of R' [11] that can be found from (4), (5):

$$R' = \left(\frac{F'}{F} - \frac{f'}{f}\right)R - \left[a' + \left(\frac{F'}{F} - \frac{3f'}{2f}\right)(t - a)\right]\dot{R}. \quad (7)$$

The solution for the T-model reads

$$ds^2 = -dt^2 + b^2(t)dz^2 + R(t)^2d\Omega^2, \quad (8)$$

$$R = \frac{R_0}{2}(1 - \cos \eta), \quad \eta - \sin \eta = 2\frac{(t - t_0)}{R_0}, \quad t_0 = \text{const}. \quad (9)$$

$$b(t) = \varepsilon \cot \frac{\eta}{2} + 2M'(r_0)\left(1 - \frac{\eta}{2} \cot \frac{\eta}{2}\right), \quad (10)$$

$$\rho = \frac{M'(r_0)}{4\pi R^2 b(t)}, \quad (11)$$

where $M(r_0)$ is the proper mass, $\varepsilon = 0, \pm 1$.

Let the model have no origin on the left side, so that R neither vanishes nor grows unbounded but, instead, tends to a constant value at some $r = r_0$. To obtain the geodesically complete spacetime, we also assume that the geometry approaches that of a semi-infinite throat, the function f attains its possible minimum value $f = -1$, so that near r_0

$$f = -1 + k(r - r_0)^2, \quad k > 0. \quad (12)$$

We also take F' to be finite at r_0 . It is convenient to rescale r in such a way that $|a'(r_0)| = 1$, if $a'(r_0) \neq 0$. Then it follows from (4) - (7) that

$$R(r_0, t) = \frac{F(r_0)}{2}(1 - \cos \eta), \quad \dot{R}(r_0) = \cot \frac{\eta}{2}, \quad \eta - \sin \eta = \frac{2(t - a(r_0))}{F(r_0)} \quad (13)$$

$$R'(r_0, t) = \varepsilon \cot \frac{\eta}{2} + F'(r_0)\left(1 - \frac{\eta}{2} \cot \frac{\eta}{2}\right). \quad (14)$$

Here $\varepsilon = 0, -1, 1$ if $a'(r_0) = 0, 1, -1$ correspondingly.

(The particular explicit example of such a kind can be found in Sec. 6.4 of Ref. [11] : $f = -1 + B^2 \exp(\frac{2r}{r_0})$, $F = A^3(1 + C \exp[\frac{r}{r_0}])^3$, $a = 0$. On the first glance, it looks different but we can make the substitution $\exp(\frac{r}{r_0}) = \tilde{r} - \tilde{r}_0$, where \tilde{r} is the analog of our r in previous formulas, whence it is clear that it belongs to our family of solutions with $\varepsilon = 0$ and f having the form (12) everywhere, not only in the vicinity of \tilde{r}_0 .)

The metric coefficient g_{11} is equal to

$$\exp[2\omega(r_0)] = \frac{R'^2(r_0, t)}{k(r - r_0)^2}. \quad (15)$$

Let us make the substitution $r - r_0 = A \exp(\alpha z)$, the constants are chosen in such a way that $A > 0$, $\alpha > 0$, $\alpha^2 = k$, the new variable $-\infty < z < \infty$. Then

$$ds^2 = -dt^2 + b^2(t)dz^2 + R^2(t)d\Omega^2, \quad (16)$$

with

$$b(t) = R'(r_0, t). \quad (17)$$

But this is nothing other than the metric of the T-model. Although we considered only the small vicinity of r_0 , by substitution

$$r = r_0 + \lambda\chi, \quad (18)$$

where $\lambda \rightarrow 0$, χ is the new radial variable, after repeating all steps we arrive at (8) - (10) in *all* space. We should identify $t_0 = a(r_0)$. It remains to be seen that $F'(r_0) = 2M'(r_0)$. Remembering that $F = 2m = 8\pi \int dr R^2 R' \rho$ and the proper mass $M = 4\pi \int dr R^2 \exp(\frac{\omega}{2}) \rho$, we have

$$M' = \frac{F' \exp(\omega)}{2R'}. \quad (19)$$

Taking into account that in our case $\lim_{r \rightarrow r_0} \exp(\omega) = b = R'(r_0, t)$, we see that indeed $F'(r_0) = 2M'(r_0)$ that completes our proof. Thus, T-models are indeed obtained as a limit of LTB solutions.

Now we compare the singularities in both models. The non-vanishing components of the Riemann tensor in the orthonormal frame are

$$R_{\hat{\theta}\hat{\theta}\hat{\theta}\hat{\theta}} = \frac{\ddot{R}}{R}, \quad (20)$$

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = \ddot{\omega} + \dot{\omega}^2, \quad (21)$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{1}{R^2} [1 - R'^2 \exp(-2\omega) + \dot{R}^2], \quad (22)$$

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = \frac{\dot{\omega}\dot{R}}{R} + \frac{\exp(-2\omega)}{R} (R'\omega' - R''). \quad (23)$$

To simplify comparison, let us use the variable z instead of r both in T-models and LBT-solutions, $r - r_0 = A \exp(\alpha z)$. Then (subscript "LTB" and "T" refers to LTB and T-models, respectively) $\omega_{LTB} \rightarrow \omega_T$, $R_{LTB} \rightarrow R_T$, $(\frac{\partial R}{\partial z})_{LTB} \rightarrow 0 = (\frac{\partial R}{\partial z})_T$, $(\frac{\partial^2 R}{\partial z^2})_{LTB} = (\frac{\partial R}{\partial r})_{r_0} A \alpha^2 \exp(\alpha z) + (\frac{\partial^2 R}{\partial z^2})_{r_0} A^2 \alpha^2 \exp(\alpha z) \rightarrow 0 = (\frac{\partial^2 R}{\partial z^2})_T$. Thus, all curvature components

go smoothly to those of T-models. As a result, the values of Kretsmann scalars coincide and the singularities in both cases are the same.

It was observed by Ruban [5], [6] that in T-models there are singularities of the disc type ($b \rightarrow 0$, R is finite). Now it is clear that this type of singularities in the T-sphere solutions is inherited from shell-crossing in LTB ones although the standard definition of shell-crossing cannot be applied directly to T-spheres. Indeed, usually shell-crossing occurs at some isolated values of r where $R' = 0$, $F' \neq 0$. In the T-case $R' \equiv 0$, the coordinate r becomes degenerate and all "shells" share the same value of R . If, instead, one labels shells in the T-case by values of z , the condition $b = 0$ means just crossing of shells with different z .

Thus, the Vaidya spacetime and T-spheres can be obtained from the Tolman models by two different transitions. In the first case $f \rightarrow +\infty$ [8], [9], [10], in the second one $f \rightarrow -1$ according to (12). In both cases the singularities of the LTB prototype and its limiting counterpart coincide. From the general viewpoint, such limiting transitions represent examples of the "limits of spacetime" [12]. It would be interesting to elucidate, whether such a relationship between singularities is inherent to this procedure as such or it is rather the property of some particular models.

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- [1] R. C. Tolman, 1934, Proc. Nat. Acad. Sci. USA, 20, 169; reprinted in 1997, Gen. Relativ. Gravitation, 29, 935
 - [2] K. Bolejko, A. Krasinski and C.Hellaby, Formation of voids in the Universe within the Lemaitre-Tolman model, gr-qc/0411126.
 - [3] A. Krasinski and C.Hellaby, Phys.Rev. D69 (2004) 043502.
 - [4] Datt, B. (1938). Z. Physik 108, 314 [Reprinted: Gen. Rel. Grav. 31, 1619 (1999)].
 - [5] V. A. Ruban, Sov. Phys. JETP (Letters) 8, 414 (1968). [Reprinted: Gen. Rel. Grav. 33, 369 (2001)].
 - [6] V. A. Ruban, Sov. Phys. JETP 29, 1027 (1969). [Reprinted: Gen. Rel. Grav. 33, 375 (2001)].
 - [7] J. R. Gair, Class. Quant. Grav. 19, 6345 (2002).
 - [8] J. P. S. Lemos, Phys. Rev. Lett. 68 (1992) 1447.
 - [9] C. Hellaby, Phys. Rev. D 49 (1994) 6484.

- [10] S. Gao and J. P. S. Lemos, Phys. Rev. D 71 (2005) 084022.
- [11] C. Hellaby, Some properties of singularities in the Tolman model, Ph. D. Theses, 1985. Queen's University at Kingston, Ontario, Canada.
- [12] Geroch R Commun. Math. Phys. 13 (1969) 180.