

Does the cosmological constant imply the existence of a minimum mass?

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Abstract

We show that in the framework of the classical general relativity the presence of a positive cosmological constant implies the existence of a minimal mass and of a minimal density in nature. These results rigorously follow from the generalized Buchdahl inequality in the presence of a cosmological constant.

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I. INTRODUCTION

One of the most important characteristics of compact relativistic astrophysical objects is their maximum allowed mass. The maximum mass is crucial for distinguishing between neutron stars and black holes in compact binaries and in determining the outcome of many astrophysical processes, including supernova collapse and the merger of binary neutron stars. The theoretical values of the maximum mass and radius for white dwarfs and neutron stars were found by Chandrasekhar and Landau and are given by $M_{\max} \approx \left[(\hbar c/G) m_B^{-4/3} \right]^{3/2}$ and $R_{\max} \leq (\hbar/mc) (\hbar c/Gm_B^2)^{1/2}$, where m_B is the mass of the baryons and m the mass of either electron or neutron [1]. Thus, with the exception of composition-dependent numerical factors, the maximum mass of a degenerate star depends only on fundamental physical constants. For non-rotating neutron stars with the central pressure at their center tending to the limiting value $\rho_c c^2$ an upper bound of around $3M_\odot$ has been found [2]. The maximum mass of different types of astrophysical objects (neutron stars, quark stars etc.) under different physical conditions, including rotation and magnetic fields, was considered by using both numerical and analytical methods ([3], and references therein).

With the use of the gravitational field equations for a static equilibrium configuration, Buchdahl [4] obtained an absolute limit of the mass-radius ratio of a stable compact object, given by $2GM/c^2R \leq 8/9$. This limit has been generalized in the case of scalar-tensor theories [5], for charged fluid spheres [6], and for the Schwarzschild-de Sitter geometries in the presence of a cosmological constant [7].

If the problem of the maximum mass of compact objects had been considered in great detail, the more fundamental question of the possible existence of a minimum mass in the framework of general relativity had been investigated at a much lesser extent. The minimum mass of neutron stars or of white dwarfs can be derived qualitatively from energy considerations [8]. A lower limit for the radius of the neutron stars of the form $R \geq (3.1125 - 0.44192x + 2.3089x^2 - 0.38698x^3)$, with $x = M/M_\odot$ and $1 \leq x \leq 2.5$ has been found in [9].

At a microscopic level two basic quantities, the Planck mass m_P and the Planck length l_P are supposed to play a fundamental physical role. The Planck mass is derived by equating the gravitational radius $2Gm/c^2$ of a Schwarzschild mass with its Compton wavelength \hbar/mc . The corresponding mass $m_{Pl} = (c\hbar/2G)^{1/2}$ is of the order $m_{Pl} \approx 1.5 \times 10^{-5}$ g. The

Planck length is given by $l_{Pl} = (\hbar G/c^3)^{1/2} \approx 1.6 \times 10^{-33}$ cm and at about this scale quantum gravity will become important for understanding physics. The Planck mass and length are the only parameters with dimension mass and length, respectively, which can be obtained from the fundamental constants c , G and \hbar .

The observations of high redshift supernovae [10] and the Boomerang/Maxima data [11], showing that the location of the first acoustic peak in the power spectrum of the microwave background radiation is consistent with the inflationary prediction $\Omega = 1$, have provided compelling evidence for a net equation of state of the cosmic fluid lying in the range $-1 \leq w = p/\rho < -1/3$. To explain these observations, two dark components are invoked: the pressure-less cold dark matter (CDM) and the dark energy (DE) with negative pressure. CDM contributes $\Omega_m \sim 0.25$, and is mainly motivated by the theoretical interpretation of the galactic rotation curves and large scale structure formation. DE provides $\Omega_{DE} \sim 0.7$ and is responsible for the acceleration of the distant type Ia supernovae. The best candidate for the dark energy is the cosmological constant Λ , which is usually interpreted physically as a vacuum energy. Its size is of the order $\Lambda \approx 3 \times 10^{-56}$ cm⁻² [12]. In some theoretical models it is assumed that the cosmological constant can be derived from the reduction to 4D of higher-dimensional unified theories [13]. Since at least 70% of the Universe consists of vacuum energy, it is natural to consider Λ as a fundamental constant. Therefore we can choose as the set of fundamental constants (c, G, \hbar, Λ) .

By using dimensional analysis Wesson [14] has found two different masses which can be constructed from this set of constants. The mass m_P relevant at the quantum scale is $m_P = (\hbar/c) \sqrt{\Lambda/3} \approx 3.5 \times 10^{-66}$ g while the mass m_{PE} relevant to the cosmological scale is $m_{PE} = (c^2/G) \sqrt{3/\Lambda} \approx 1 \times 10^{56}$ g.

The interpretation of the mass m_{PE} is straightforward: it is the mass of the observable part of the universe, equivalent to 10^{80} baryons of mass 10^{-24} g each. The interpretation of the mass m_P is more difficult. By using the dimensional reduction from higher dimensional relativity and by assuming that the Compton wavelength of a particle cannot take any value, Wesson [14] proposed that the mass is quantized according to the rule $m = (n\hbar/c) \sqrt{\Lambda/3}$. Hence m_P is the minimum mass corresponding to the ground state $n = 1$.

These results about the fundamental mass have been obtained by using a phenomenological approach. It is the purpose of the present Letter to give a rigorous proof on the existence of a minimum mass in general relativity. The existence of such a mass is a di-

rect consequence of the presence of a non-zero cosmological constant in the gravitational field equations. Therefore these two quantities are strongly inter-related. In order to prove the existence of a minimum mass we follow the approach introduced by Buchdahl [4] and generalized to the case of a non-zero Λ in [7].

The present Letter is organized as follows. The limiting density and mass for a general relativistic object is derived in the next Section. We conclude our results in the last section.

II. LOWER MASS AND DENSITY BOUNDS FOR STATIC GENERAL RELATIVISTIC SPHERES

We assume that the spherically symmetric general relativistic mass distribution is described by the metric (in the present Section we set $c = 1$):

$$ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1)$$

Static and spherically symmetric perfect fluids in general relativity are described by three independent field equations (for four unknown functions) that imply conservation of energy-momentum. Eliminating the function $\nu(r)$ from the field equations yields the well known Tolman-Oppenheimer-Volkoff equation in the presence of a cosmological constant Λ [7].

Let us introduce Buchdahl variables, defined by [4]

$$y^2 = e^{-\lambda} = 1 - 2w(r)r^2 - \frac{\Lambda}{3}r^2, \quad \zeta = e^{\nu/2}, \quad x = r^2, \quad (2)$$

where $w(r)$ is the mean density up to r , $w(r) = m(r)/r^3$ and $m(r)$ is the mass inside radius r , $m(r) = 4\pi \int_0^r \rho(r') r'^2 dr'$, with ρ the mass density of the compact object with radius R .

Eliminating the pressure function from the field equations, one obtains the following differential equation [4, 7]

$$(y\zeta_{,x})_{,x} - \frac{1}{2} \frac{w_{,x}\zeta}{y} = 0. \quad (3)$$

Eq. (3) can be used to compare solutions with decreasing energy density with ones having constant density, for which the second term in Eq. (3) vanishes. In the latter case one can integrate Eq. (3) and compare it with a decreasing solution, which then yields the generalized Buchdahl inequality in the presence of the cosmological constant [7]:

$$\sqrt{1 - \frac{2GM}{R} - \frac{\Lambda}{3}R^2} \geq \frac{1}{3} - \frac{\Lambda}{12\pi G\rho}. \quad (4)$$

Eq. (4) provides a lower bound for the mass and density of general relativistic objects. To prove this result, we start by squaring Eq. (4), multiplying it by $G^2 M^2$, eliminating the density on the right-hand side by $\rho = 3M/(4\pi R^3)$ and taking all terms to the left-hand side. Then we obtain the following expression

$$-2G^3 M^3 + \frac{8}{9}G^2 M^2 R - \frac{\Lambda}{3}G^2 M^2 R^3 + \frac{2\Lambda}{27}GMR^4 - \left(\frac{\Lambda}{9}R^3\right)^2 R \geq 0, \quad (5)$$

which can be written as a product of three terms

$$-2 \left(GM + \frac{\Lambda}{6}R^3 \right) \left[GM - \frac{2R}{9} \left(1 - \sqrt{1 - \frac{3\Lambda}{4}R^2} \right) \right] \left[GM - \frac{2R}{9} \left(1 + \sqrt{1 - \frac{3\Lambda}{4}R^2} \right) \right] \geq 0. \quad (6)$$

Dividing by the factor (-2) reverses the inequality sign. Hence either one or all three terms of the product must have a negative sign in order to fulfill the latter equation.

With $\Lambda = 0$ we can easily find the correct signs. The first term is strictly positive for $\Lambda = 0$ (it reads GM), hence only one of the remaining terms is negative. The second term for $\Lambda = 0$ also yields GM . Therefore, the last term must be negative, which for vanishing Λ gives $2GM \leq 8R/9$, which is nothing but the standard Buchdahl inequality [4].

Since the signs of the three terms are now known, let us analyze Eq. (6) with a non-zero cosmological constant. We shall consider separately the cases of a positive ($\Lambda > 0$) and of a negative ($\Lambda < 0$) cosmological constant. For $\Lambda > 0$ the analysis of the signs of Eq. (6) gives the following algebraic conditions to be satisfied by the mass and radius of the matter distribution and by the cosmological constant.

(i) Positivity of the first term of (6) implies

$$GM \geq -\frac{\Lambda}{6}R^3. \quad (7)$$

For positive Λ this is trivially fulfilled.

(ii) Positivity of the second term yields

$$GM \geq \frac{2R}{9} \left(1 - \sqrt{1 - \frac{3\Lambda}{4}R^2} \right), \quad (8)$$

which as before gives a lower bound on the mass.

(iii) Finally, negativity of the last term of the product (6) reads

$$GM \leq \frac{2R}{9} \left(1 + \sqrt{1 - \frac{3\Lambda}{4}R^2} \right). \quad (9)$$

Putting the three above conditions (i)–(iii) together, leads to

$$\frac{2R}{9} \left(1 + \sqrt{1 - \frac{3\Lambda}{4} R^2} \right) \geq GM \geq \frac{2R}{9} \left(1 - \sqrt{1 - \frac{3\Lambda}{4} R^2} \right), \text{ for } \Lambda > 0. \quad (10)$$

We may Taylor expand the lower bound which then reads $(2R/9) \left(1 - \sqrt{1 - 3\Lambda R^2/4} \right) \approx \Lambda R^3/12$.

Therefore for a positive cosmological constant one obtains a lower bound for the mass and the density of a general relativistic object, given by

$$2GM \geq \frac{\Lambda}{6} R^3, \quad \rho = \frac{3M}{4\pi R^3} \geq \frac{\Lambda}{16\pi G} =: \rho_{\min}, \Lambda \geq 0. \quad (11)$$

In the case of a negative cosmological constant, $\Lambda < 0$, by repeating the previous analysis of the signs in Eq. (6) we obtain the condition

$$\frac{2R}{9} \left(1 + \sqrt{1 - \frac{3\Lambda}{4} R^2} \right) \leq GM \leq \frac{2R}{9} \left(1 - \sqrt{1 - \frac{3\Lambda}{4} R^2} \right), \text{ for } \Lambda < 0. \quad (12)$$

By performing a small Λ Taylor expansion we find

$$4\frac{R}{9} - \frac{\Lambda}{12} R^3 \leq GM \leq -\frac{\Lambda}{6} R^3, \text{ for } \Lambda < 0. \quad (13)$$

The original Buchdahl inequality [4], with $\Lambda = 0$ requires that $4R/9 \geq GM$. Since $\Lambda < 0$ we may write Eq. (13) as

$$4\frac{R}{9} + \frac{|\Lambda|}{12} R^3 \leq GM \leq +\frac{|\Lambda|}{6} R^3, \text{ for } \Lambda < 0. \quad (14)$$

Eq. (14), derived by assuming a negative cosmological constant, obviously violates in the limit $\Lambda \rightarrow 0$ the Buchdahl bound. The physical consequence of this fact is that we could have massive fluid balls which are surrounded by a horizon.

Therefore the requirement of the absence of a regular solution contained in the horizon rules out the possibility of the existence of a minimum bound for the mass in the presence of a negative cosmological constant. Moreover, the right hand side of Eq. (14) gives $GM \leq |\Lambda|R^3/6$, which would imply the un-physical condition that the numerical value of the minimal mass derived for $\Lambda > 0$ is actually the maximal allowed mass in nature for $\Lambda > 0$.

The same results on the non-existence of a minimum mass for a negative Λ can be obtained by considering the inequality $2R \left(1 + \sqrt{1 - 3\Lambda R^2/4}\right) / 9 \leq GM \leq -\Lambda R^3/6, \Lambda < 0$, which can be obtained from Eq. (6), by assuming that the first bracket is negative, the second is always positive (for a negative Λ) and that the last bracket is positive.

III. CONCLUSIONS

For $\Lambda = 0$ Eqs. (11) expresses the positivity of the total mass and of the total energy density of a compact general relativistic matter distribution, $M \geq 0$ and $\rho \geq 0$. For $\Lambda > 0$ we conclude that no object present in classical general relativity can have a density that is smaller than ρ_{\min} . In the derivation of this result we have also assumed that $R < 2/(\sqrt{3}\Lambda)$, that is, R is smaller than the size of the event horizon.

From Eq. (11) one can estimate the numerical value of the minimal density for a positive Λ as $\rho_{\min} = \Lambda c^2/16\pi G = 8.0 \times 10^{-30}$ g cm $^{-3}$ (in the present Section we shall restore c in all equations).

By assuming that the minimum mass in nature is $m_P = (\hbar/c) \sqrt{\Lambda/3}$ [14], it follows that the radius corresponding to m_P is given by

$$R_P = 48^{1/6} \left(\frac{\hbar G}{c^3}\right)^{1/3} \Lambda^{-1/6} \approx 1.9 l_{Pl}^{2/3} \Lambda^{-1/6}, \quad (15)$$

with the numerical value $R_P = 4.7 \times 10^{-13}$ cm = 4.7 fm. Hence, the radius R_P is of the same order of magnitude as the classical radius of the electron $r_e = e^2/m_e c^2 = 2.81 \times 10^{-13}$ cm.

Therefore from the previous analysis we conclude that in the framework of classical general relativity the possible existence of a minimal mass and of a minimal density in nature is strictly related to the presence of a positive cosmological constant. On the other hand, the positivity of Λ is confirmed by the cosmological observations [12].

If an absolute minimum length does exist, then, via the first of Eqs. (11), a positive cosmological constant also implies the existence of an absolute minimum mass in nature.

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