

# Slowly Rotating Homogeneous Stars and the Heun Equation

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**Abstract.** The scheme developed by Hartle for describing slowly rotating bodies in 1967 was applied to the simple model of constant density by Chandrasekhar and Miller in 1974. The pivotal equation one has to solve turns out to be one of Heun's equations. After a brief discussion of this equation and the chances of finding a closed form solution, a quickly converging series solution of it is presented. A comparison with numerical solutions of the full Einstein equations allows one to truncate the series at an order appropriate to the slow rotation approximation. The truncated solution is then used to provide explicit expressions for the metric.

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## 1. Introduction

Until now, no one has succeeded in finding an analytic solution to Einstein's equations that describes an isolated, rotating, three dimensional perfect fluid. If such a solution is to be found, then most likely for a homogeneous body in equilibrium. After all, three limiting cases are known analytically for this equation of state: the non-rotating limit (inner and outer Schwarzschild solution), the portion of the Newtonian limit made up of the Maclaurin spheroids and the disc limit (the relativistic disc of dust, see [1]).

Given such an analytic solution, it could then be expanded about various limits. An appropriate post-Newtonian expansion would yield the Maclaurin spheroids in the zeroth order, then the analytically known first post-Newtonian corrections to the next order (see [2, 3]) and so on. Such a post-Newtonian expansion can be expressed entirely in terms of elementary functions at least up to the fourth order [4]. Similarly, an expansion with respect to the angular velocity would result in the inner and outer Schwarzschild solution in the zeroth order. In this paper we shall consider the first order contribution to the slow rotation expansion using the formalism introduced by Hartle [5].

Chandrasekhar and Miller [6] already considered this problem in 1974 by solving the equations numerically. Since there now exist computer programs capable of solving the complete Einstein equations in the case of stationarity and axial symmetry to extremely high accuracy, a further numerical treatment of the slowly rotating approximation would only be useful inasmuch as it provides a fairly simple alternative for arriving at a good approximation to the full field equations. An analytic treatment,

however, can pinpoint where the stumbling blocks preventing further progress lie, and can augment the disc solution [1] in suggesting a “lower bound” for the complexity of the full-solution. A discussion of some of the analytic issues that arise in this context can be found in [7] and references therein.

We present the basic equations for a slowly rotating homogeneous perfect fluid in §2 and show that the equation of primary importance is one of Heun’s equations. In §3 we discuss some of the properties of Heun’s equations and consider what transformations could result in a simpler equation. Various series solutions are discussed and derived in §4, and use is made of the numerical solution of the full Einstein equations in order to determine an appropriate truncation order. Using these truncated series, approximate expressions for all metric functions are discussed in §5 and provided in Appendix A.

## 2. Basic Equations for Slow Rotation

As in [6], we devote this section to listing the fundamental equations derived by Hartle and then specialize them to the case of constant energy density.

The metric describing a slowly rotating, stationary and axisymmetric fluid is given by

$$ds^2 = -e^{2\nu_s} (1 + 2h) dt^2 + e^{2\lambda_s} \left[ 1 + \frac{e^{2\lambda_s}}{r} 2m \right] dr^2 + r^2 (1 + 2k) \left[ d\theta^2 + \sin^2 \theta (d\varphi - \omega dt)^2 \right] + \mathcal{O}(\Omega^3) \quad (1)$$

with

$$\begin{aligned} h &= h_0(r) + h_2(r) P_2(\cos \theta) \\ m &= m_0(r) + m_2(r) P_2(\cos \theta) \\ k &= k_2(r) P_2(\cos \theta) \\ \omega &= \omega(r). \end{aligned}$$

In the above equations,  $P_2(\cos \theta)$  denotes Legendre’s polynomial of order 2,  $\Omega$  is the angular velocity (see [5] for an account of what ‘slow rotation’ means in terms of  $\Omega$ ),  $\omega$  is of order  $\Omega$  and  $h$ ,  $m$  and  $k$  are of order  $\Omega^2$ . In the above expansion in spherical harmonics, use was made of a coordinate freedom in order to transform away the term  $k_0(r)$ .

The non-rotating metric, obtained when  $\Omega \rightarrow 0$ , is

$$ds^2 = -e^{2\nu_s} dt^2 + e^{2\lambda_s} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2).$$

For a non-rotating perfect fluid with pressure  $p_s$ , constant energy density  $\varepsilon$ , radius<sup>2</sup>  $r_s$  and mass

$$M_s = \frac{4}{3} \pi \varepsilon r_s^3, \quad (2)$$

the metric functions are<sup>3</sup> (see e.g. [8])

$$r \geq r_s : \quad e^{\lambda_s} = \frac{r}{r - 2M_s}, \quad e^{\nu_s} = \frac{r - 2M_s}{r} \quad (3)$$

$$r \leq r_s : \quad e^{\lambda_s} = \frac{1}{B(r)}, \quad e^{\nu_s} = \frac{1}{2} [A - B(r)] \quad (4)$$

<sup>2</sup> The coordinate radius of the star  $r_s$  is not to be confused with  $R_s$  in [6], which they use to denote the Schwarzschild radius.

<sup>3</sup> We adopt units in which  $c = G = 1$ .

with

$$A := 3\sqrt{1 - \frac{2M_S}{r_S}}, \quad B(r) := \sqrt{1 - \frac{2M_S r^2}{r_S^3}} \quad (5)$$

and the normalized pressure is given by

$$\frac{p_S}{\varepsilon} = \frac{A/3 - B}{B - A}.$$

Integrating the equation for hydrostatic equilibrium, it is natural to define

$$\begin{aligned} p^* &= \ln(\varepsilon + p) - \int^\varepsilon \frac{d\varepsilon'}{\varepsilon' + p(\varepsilon')} \\ &= \ln(\varepsilon + p) \quad (\varepsilon = \text{constant}), \end{aligned} \quad (6)$$

which we expand as with the metric coefficients

$$p^* = p_S^* + \delta p_0^*(r) + \delta p_2^*(r) P_2(\cos \theta).$$

Hydrostatic equilibrium is then realised when

$$\begin{aligned} \delta p_0^* + h_0 - \frac{1}{3} r^2 e^{-2\nu_S} \tilde{\omega}^2 &= \text{constant} = \gamma \\ \delta p_2^* + h_2 + \frac{1}{3} r^2 e^{-2\nu_S} \tilde{\omega}^2 &= 0, \end{aligned} \quad (7)$$

where we have introduced<sup>4</sup>

$$\tilde{\omega} := \Omega - \omega.$$

Letting

$$\begin{aligned} j &:= e^{-(\lambda_S + \nu_S)} \\ v_2 &:= h_2 + k_2, \end{aligned}$$

and restricting ourselves to the case of constant density, we can write the field equations as

$$\frac{d^2 \tilde{\omega}}{dr^2} = -\frac{d\tilde{\omega}}{dr} \frac{d}{dr} \ln(r^4 j) - \frac{4\tilde{\omega}}{r} \frac{d}{dr} \ln(j) \quad (8a)$$

$$\frac{dm_0}{dr} = \frac{1}{12} r^4 j^2 \left( \frac{d\tilde{\omega}}{dr} \right)^2 - \frac{1}{3} r^3 \tilde{\omega}^2 \frac{d}{dr} j^2 \quad (8b)$$

$$\frac{dh_0}{dr} = \left[ m_0 e^{2\lambda_S} \left( \frac{1}{r^2} + 8\pi p_S \right) - \frac{1}{12} r^3 j^2 \left( \frac{d\tilde{\omega}}{dr} \right)^2 + 4\pi r (\varepsilon + p_S) \delta p_0^* \right] e^{2\lambda_S} \quad (8c)$$

$$\begin{aligned} \frac{dh_2}{dr} &= \left( \frac{-2e^{2\lambda_S}}{r^2} / \frac{d\nu_S}{dr} \right) v_2 + \left[ -2 \frac{d\nu_S}{dr} - \frac{1}{r} / \frac{d\nu_S}{dr} \left( \frac{1}{2j^2} \frac{d(j^2)}{dr} + \frac{d\lambda_S}{dr} \right) \right] h_2 \\ &+ \left[ \frac{r^2 j^2}{6} \left( \frac{d\tilde{\omega}}{dr} \right)^2 - \frac{r \tilde{\omega}^2}{3} \frac{d(j^2)}{dr} \right] \left( r^2 \frac{d\nu_S}{dr} - \frac{e^{2\lambda_S}}{2} / \frac{d\nu_S}{dr} \right) \end{aligned} \quad (8d)$$

$$\frac{dv_2}{dr} = -2 \frac{d\nu_S}{dr} h_2 + \left( \frac{1}{r} + \frac{d\nu_S}{dr} \right) \left[ \frac{1}{6} r^4 j^2 \left( \frac{d\tilde{\omega}}{dr} \right)^2 - \frac{1}{3} r^3 \tilde{\omega}^2 \frac{d(j^2)}{dr} \right] \quad (8e)$$

$$m_2 = r e^{-2\lambda_S} \left( -h_2 + \frac{1}{6} r^4 j^2 \left( \frac{d\tilde{\omega}}{dr} \right)^2 - \frac{1}{3} r^3 \tilde{\omega}^2 \frac{d(j^2)}{dr} \right). \quad (8f)$$

<sup>4</sup> Our notation  $\tilde{\omega} = \Omega - \omega$  corresponds to  $\bar{\omega}$  in [5] and  $\varpi$  in [6]. In this paper, a bar above a symbol will denote a dimensionless quantity.

In the vacuum region, where  $\varepsilon = p = 0$  and  $j = 1$ , the solutions to the above equations are

$$r \geq r_S : \\ \tilde{\omega} = \Omega - \frac{2J}{r^3} \quad (9a)$$

$$m_0 = \delta M - \frac{J^2}{r^3} \quad (9b)$$

$$h_0 = \frac{\delta M}{r - 2M_S} + \frac{J^2}{r^3(r - 2M_S)} \quad (9c)$$

$$h_2 = \frac{J^2}{r^3} \left( \frac{1}{M_S} + \frac{1}{r} \right) + K Q_2^2 \left( \frac{r}{M_S} - 1 \right) \quad (9d)$$

$$v_2 = -\frac{J^2}{r^4} - K \frac{2M_S}{\sqrt{r(r - 2M_S)}} Q_2^1 \left( \frac{r}{M_S} - 1 \right) \quad (9e)$$

$$m_2 = \frac{1}{9} r A^2 \left( -h_2 + \frac{6J^2}{r^4} \right), \quad (9f)$$

where  $Q_n^m$  is the associated Legendre function of the second kind<sup>5</sup> and  $J$ ,  $\delta M$  and  $K$  are constants,  $J$  being the angular momentum and  $\delta M$  the change in mass with respect to the non-rotating configuration.

Equations (8a) are listed in a hierarchical order, each equation being soluble once the preceding equations have been solved (note that (8d) and (8e) form a coupled system). Eq. (8a) plays a pivotal role, since the solution of the remaining equations relies on the solution of this equation. Hence we shall devote the next section entirely to this equation. Before doing so, it will be convenient to put (8a), which is a Heun equation, into the standard form (see e.g. [9]).

To bring it into this form, let us begin with a transformation of variables in (8a) from  $r$  to the  $B(r)$  of (5). Making use of the constant  $A$  from (5), we find

$$(B - A)(B^2 - 1) \frac{d^2 \tilde{\omega}}{dB^2} + (4B^2 - 5AB + 1) \frac{d\tilde{\omega}}{dB} - 4A\tilde{\omega} = 0. \quad (10)$$

With the further substitution (see [10] (2.329))<sup>6</sup>

$$z := \frac{1 - B}{2}$$

we arrive at

$$\frac{d^2 \tilde{\omega}}{dz^2} + \left( \frac{\gamma}{z} + \frac{\delta}{z - 1} + \frac{\epsilon}{z - a} \right) \frac{d\tilde{\omega}}{dz} + \frac{\alpha\beta z - q}{z(z - 1)(z - a)} \tilde{\omega} = 0 \quad (11)$$

with

$$a = \frac{1 - A}{2}, \quad q = -2A, \quad \alpha = 3, \quad \beta = 0, \quad \gamma = \frac{5}{2}, \\ \delta = \frac{5}{2}, \quad \epsilon = \alpha + \beta + 1 - \gamma - \delta \implies \epsilon = -1.$$

<sup>5</sup> The minus sign in front of the second term of (9e) has to do with the normalization of the Legendre function and the choice of the branch cut.

<sup>6</sup> Both this substitution and the resulting (11) are not unique, since there is freedom as to how the original four singularities of the equation,  $(1, -1, A, \infty)$ , are mapped onto  $(0, 1, a, \infty)$ . This freedom will be considered amongst the transformation of §3.

### 3. Heun's Equation: Properties and Transformations

Using the terminology of [9], we consider three categories of solutions to Heun's equation: *local solutions*, *Heun functions* and *Heun polynomials*.

**Local solutions** to Heun's equation are valid in the neighbourhood of one singularity and are associated with one of the two exponents there. Since there are four singularities, there are a total of eight such solutions. **Heun functions** are solutions valid in an region containing two adjacent singularities. **Heun polynomials** are solutions to Heun's equation valid at three singularities. Despite the name, not all such solutions are polynomials, but are of the simple form

$$Hp(z) = z^{\sigma_1}(z-1)^{\sigma_2}(z-a)^{\sigma_3}p_n(z), \quad (12)$$

where  $p_n(z)$  is a polynomial of degree  $n$  and  $\sigma_{1,2,3}$  is one of the exponents associated with the singularity at  $z = 0, 1, a$  respectively.

In order to determine which type of solution we need to look for, we have to determine the domain of relevance for our particular Heun equation given by (11). The parameter  $A$  and the quantities that depend on it are defined on the intervals

$$A \in (1, 3), \quad a \in (0, -1), \quad q \in (0, -3/2).$$

The limit of infinite central pressure is given by  $A \rightarrow 1$  and the Newtonian limit by  $A \rightarrow 3$ . We exclude these limits in this analysis (as indicated by the open intervals), but note that Chandrasekhar and Miller include an interesting discussion of the limit of infinite central pressure in [6] and believe that consideration of such limits in the context of the confluent Heun equation could lead to interesting results.

The interior of the star, i.e. the region of validity of (11), is given by the interval

$$B \in [1, A/3] \implies z \in \left[0, \frac{3-A}{6}\right] = \left[0, \frac{a+1}{3}\right], \quad (13)$$

where  $z = 0$  represents the centre of the star and  $z = (3-A)/6$  its surface.

Since  $a$  is always strictly negative whereas  $z$  is always real and positive, we are interested in a solution including the singularity at zero and extending along the real axis toward the singularity at  $z = 1$  (though not reaching it). The function  $\tilde{\omega}$  must remain finite (and non-zero) at the point  $z = 0$ , whence we require at the very least the local solution there corresponding to the exponent  $\sigma_1 = 0$ .

The remainder of this section is devoted to a survey of attempts that can be made to find a closed form solution to our Heun equation satisfying these constraints. Although none of these attempts proved successful, the following subsections are intended to stimulate new ideas in this direction, explain where problems lie in what may seem promising solution strategies and help the reader avoid choosing fruitless paths of thought.

#### 3.1. Heun Polynomials

The most tractable solution to Heun's equation is one of the Heun polynomials, so that we begin by searching for such a solution even though the validity at the two 'additional' singularities is not necessary for our purposes. As was mentioned in footnote 6, the particular form of the Heun equation given in (11) is not unique since various transformations map a Heun equation onto a new Heun equation with other parameters and exponents. Indeed there are 192 mappings of Heun's equation onto itself, corresponding to the 192 local solutions that can be provided for it.

**Table 1.** The values of the exponents  $\sigma_i$  of (12) and the parameters  $\alpha$  and  $\beta$  corresponding to the four classes of Heun polynomials that would yield permissible solutions to our physical problem. The numbering of the classes follows the conventions in [9].

Class	$\sigma_1$	$\sigma_2$	$\sigma_3$	$\alpha$	$\beta$
I	0	0	0	$-n$	$\gamma + \delta + \epsilon + n - 1$
III	0	$1 - \delta$	0	$\delta - n - 1$	$\gamma + \epsilon + n$
V	0	0	$1 - \epsilon$	$\epsilon - n - 1$	$\gamma + \delta + n$
VII	0	$1 - \delta$	$1 - \epsilon$	$\delta + \epsilon - n - 2$	$\gamma + n + 1$

These are made up of 24 transformations of the independent variable, which map the four singularities onto themselves, and 8 elementary power transformations of the dependent variable. A chapter on these transformations is contained in [9] and a discussion of their group structure as well as a useful table containing all of them can be found in [11]. These transformations will not occupy us further in this paper, but the statements made and conclusions drawn are valid (modulo obvious modifications resulting from the transformation) for all mappings of (11) onto itself.

The exponents  $\sigma_i$  (see (12)) of the four Heun polynomials consistent with our physical problem as well as the parameters  $\alpha$  and  $\beta$  for the solutions are listed in Table 1. A comparison with (11) shows us that only a Heun polynomial of class I is consistent with the given parameters since  $n$ , the degree of the polynomial  $p_n(z)$  in (12), can of course only be a non-negative integer. The set of equations that must be solved in general in order to determine  $p_n(z)$  provides a polynomial equation of degree  $n + 1$  for the accessory parameter  $q$ . In our case, the equation is simply  $q = 0$ , which lies just outside the allowed range for  $q$  and thus there exists no Heun polynomial, which is a solution of the physical problem being considered.

### 3.2. Heun Functions

Since we require a solution valid at  $z = 0$  and extending along the real axis toward  $z = 1$ , a Heun Function seems to be the appropriate solution. It is related to the two-point connection problem treated in [12] and discussed further in [13]. The search for such solutions generally implies solving an eigenvalue problem and is valid only for a restricted set of values for the accessory parameter  $q$ . In our case,  $q$  is not independent of the singularity  $a$ , and no such solutions exist.

### 3.3. Rational and other Transformations

The next step we take in looking for a solution of our Heun equation is to consider a broader set of transformations. Since Heun's equations are closely related to the hypergeometric equations, one can hope to find a transformation relating the two equations. Kuiken [14] addressed the question as to when a hypergeometric equation can be transformed into a Heun equation via a rational substitution of the independent variable (excluding the trivial case  $\alpha\beta = q = 0$ ). Her work was reexamined and completed by Maier [15]. Using their results, we have found that no rational transformation of the independent variable maps our Heun equation (with

the specified parameter range) onto a hypergeometric equation<sup>7</sup>. Even had such a transformation been found, it would only have been valid for specific values (or one specific value) of the parameter  $A$ , because of the nature of the necessary conditions for such a transformation to exist.

Having ruled out rational transformations, we turn our attention briefly to integral transformations derived by Carlitz and Valent and described in [16], which impose no restriction on the accessory parameter. Such transformations have proved useful in finding previously unknown, closed form solutions to Heun's equation. In our case, these methods only allow for marginal progress. Consider first a transformation of the independent variable

$$B' = \frac{B - a}{B(1 - a)}.$$

Applying to this the transformation (9) of [16], and denoting the new quantities by a double prime, one finds that

$$\gamma'' = \varepsilon'' = \frac{1}{2} \quad \text{and} \quad \alpha'' + \beta'' = \delta''$$

and can apply the quadratic transformation discussed on pg. 59 of [17]. Here, however, one is not guaranteed that the solution will be regular at the centre of the star.

More modern ideas, such as relating the Heun equation to the Schrödinger equation through the “generalized associated Lamé” (GAL) potential seem tailor made to the situation being considered here. Indeed, the analogue of (32) in [18] with  $b = 3/2$  can be applied to (11), but the restriction to the accessory parameter is  $q = 0$  as it was in the case of a Heun polynomial.

We know in fact that no meromorphic solution to our Heun equation exists (for arbitrary  $q$ ) since the necessary condition  $\epsilon = 1/2 - m$ ,  $m \in \mathbb{Z}$  [19] is not met.

#### 4. Series Solutions

The power series solution (Frobenius solution) is the most obvious local solution to Heun's equation. It could be used to generate a solution regular at the point  $z = 0$  and corresponding to the exponent zero,

$$S := \sum_{m=0}^{\infty} \tilde{c}_m z^m,$$

which converges, in general, within a circle extending out to the next singularity. It is also possible to generate a series solution in terms of hypergeometric functions as described in [20]

$$H := \sum_{m=0}^{\infty} c_m \phi_m, \quad \text{with} \tag{14}$$

$$\phi_m = \frac{\Gamma(\alpha - \delta + m + 1)\Gamma(\beta - \delta + m + 1)}{\Gamma(\alpha + \beta - \delta + 2m + 1)} z^m F(\alpha + m, \beta + m; \alpha + \beta - \delta + 2m + 1; z),$$

where  $\Gamma(x)$  is the gamma function and  $F(a, b; c; z)$  the hypergeometric function. The coefficients  $c_m$  are given by the three term recurrence relation

$$\begin{aligned} L_0 c_0 + M_0 c_1 &= 0 \\ K_m c_{m-1} + L_m c_m + M_m c_{m+1} &= 0 \quad (m = 1, 2, 3, \dots) \end{aligned} \tag{15}$$

<sup>7</sup> Many of the computations in this paper made use of the computer algebra program Maple<sup>TM</sup>. Maple is a trademark of Waterloo Maple Inc.

**Table 2.** The error of the partial sums  $S_n$  and  $H_n$  is shown at the point on the star's surface  $z = (3 - A)/6$  for the value  $A = 5/2$ .

$n$	$ S - S_n $	$ H - H_n $	$ H - \frac{1}{1+z}(zH_n + H_{n+1}) $
0	$2 \times 10^{-1}$	$2 \times 10^{-1}$	$2 \times 10^{-2}$
1	$2 \times 10^{-2}$	$4 \times 10^{-3}$	$2 \times 10^{-4}$
2	$2 \times 10^{-3}$	$1 \times 10^{-4}$	$3 \times 10^{-6}$
3	$2 \times 10^{-4}$	$7 \times 10^{-6}$	$6 \times 10^{-8}$
4	$2 \times 10^{-5}$	$5 \times 10^{-7}$	$1 \times 10^{-9}$
5	$2 \times 10^{-6}$	$5 \times 10^{-8}$	$5 \times 10^{-10}$

where

$$\begin{aligned}
 K_{m+1} &= a \frac{(\alpha + m)(\beta + m)(\epsilon + m)(\alpha + \beta - \delta + m)}{(\alpha + \beta - \delta + 2m)(\alpha + \beta - \delta + 2m + 1)} \\
 L_m &= am(\gamma + m - 1) \left[ \frac{(\alpha + m)(\alpha - \delta + m + 1) + (\beta + m)(\beta - \delta + m + 1)}{(\alpha + \beta - \delta + 2m - 1)(\alpha + \beta - \delta + 2m + 1)} \right. \\
 &\quad \left. - \frac{1}{\alpha + \beta - \delta + 2m - 1} \right] - m(\alpha + \beta - \delta + m) - q \\
 &\quad + a \frac{\alpha\beta(\gamma - 2m) - \epsilon m(\delta - m - 1)}{\alpha + \beta - \delta + 2m + 1} \\
 M_{m-1} &= a \frac{(\alpha - \delta + m)(\beta - \delta + m)m(\gamma + m - 1)}{(\alpha + \beta - \delta + 2m - 1)(\alpha + \beta - \delta + 2m)}.
 \end{aligned}$$

The analysis in [20] shows that  $H$  converges in the domain

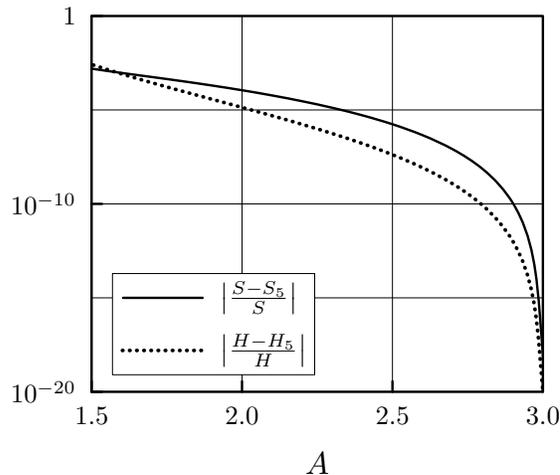
$$\left| \frac{1 - (1 - z)^{1/2}}{1 - (1 + z)^{1/2}} \right| < \left| \frac{1 - (1 - a)^{1/2}}{1 - (1 + a)^{1/2}} \right|. \quad (16)$$

Taking into account the range of  $z$  (Eq. 13), it turns out that both of these series will converge for  $A \in (3/2, 3)$ , but not in the range  $A \in (1, 3/2]$ . If one chooses to use this approximation as a rough model for neutron stars, one is unlikely to be interested in  $A < 1.5$  however.<sup>8</sup>

There are two reasons for opting for the series of hypergeometric functions  $H$  in favour of the power series  $S$ . For one thing, it generally converges more quickly than the corresponding power series. In addition, it is an alternating series in which successive partial sums straddle the limiting value of the sum, so that they provide both an upper and lower bound. A comparison of the two series can be found in Table 2 for the value of the function  $\tilde{\omega}$  at the star's surface  $z = (3 - A)/6$  for  $A = 5/2$ . The partial sums  $H_n$  tend to be about one order of magnitude more accurate than the corresponding  $S_n$ . Taking into account that  $H$  is an alternating series, a weighted average such as the one presented in the last column of the table, utilizes  $H_n$  and  $H_{n+1}$  to produce a somewhat more accurate estimate for  $H$  than would be given by  $H_{n+1}$  alone. The weight chosen in the table is based on the fact that each term in the series (14) is multiplied by  $z^m$ .

Whereas the table shows that  $H$  converges faster than  $S$  for  $A = 5/2$ , Fig. 1 shows how close  $S_5$  and  $H_5$  have come to their asymptotic values for the full range of convergent  $A$  values. We see that both series converge very quickly for  $A \rightarrow 3$  and

<sup>8</sup> For a density of  $\epsilon = 10^{18}$  kg/m<sup>3</sup>,  $A = 3/2$  corresponds in the static model to  $M_S \approx 5.5 \times 10^{30}$  kg =  $2.8M_\odot$ .



**Figure 1.**  $|(S - S_5)/S|$  and  $|(H - H_5)/H|$  on the surface of the star  $z = (3 - A)/6$  are depicted over the range  $A \in (3/2, 3)$ .

that the hypergeometric series converges faster than the power series for most of the convergent interval.

## 5. The Approximate Metric

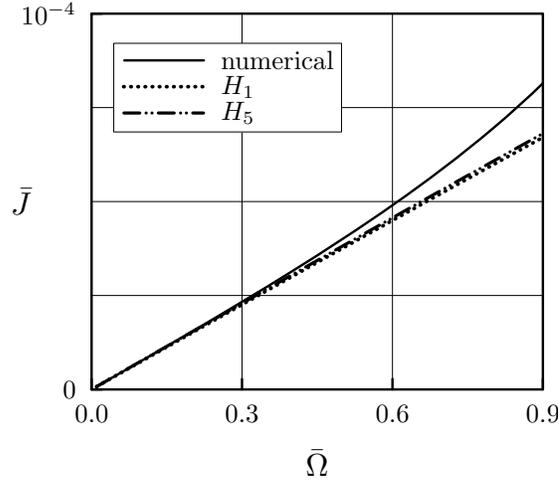
### 5.1. Determining the Truncation Order

Having found a convergent series solution to Heun's equation (for a certain range of  $A$ ), we can now compare the slow rotation approximation with the solution of the full Einstein equations for axially symmetric, stationary, homogeneous, uniformly rotating fluids to motivate how to choose the truncation order of the partial sum  $H_n$ . The solution of Einstein's equations can be found numerically, and using the spectral program described in [21], we can reach machine accuracy and thus have an absolute measure for the accuracy of the Hartle approximation. We choose to compare configurations with the same total mass<sup>9</sup>  $\bar{M}$  and angular velocity  $\bar{\Omega}$ , where the bar indicates that the quantity is expressed in terms of units of the energy density  $\varepsilon$ .

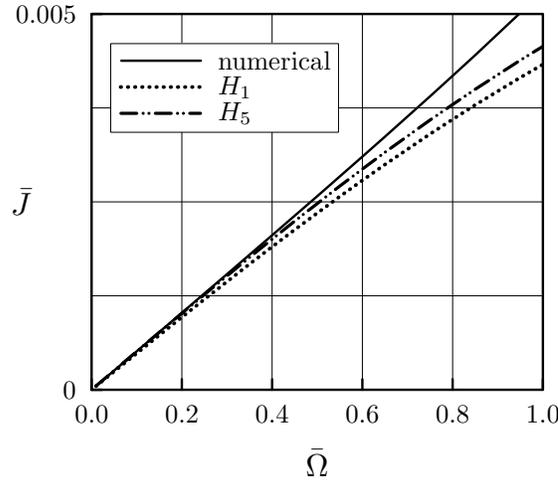
We are interested in solutions of Einstein's equations that are connected to the non-rotating limit via a continuous parameter transition. Such solutions were studied in [22] and termed the 'generalized Schwarzschild' class of solutions. A sequence of stars from this class with constant mass  $\bar{M} < 4/9\bar{R}_S \approx 0.145$  can be followed from the non-rotating limit to the mass-shedding limit, at which a cusp forms along the equatorial rim. Along any such sequence,  $\bar{\Omega}$  reaches a global maximum somewhere between the two limiting points. In the approximation considered here, however, the mass is monotonic in  $A \in (3/2, 3)$  for a given  $\bar{\Omega}$ . Therefore it is not possible to find more than one configuration for given  $\bar{M}$  and  $\bar{\Omega}$  and we are forced to restrict our attention to the portion of the sequence between the static limit and the maximum value for  $\bar{\Omega}$ .

In Figs 2 and 3 we see how  $\bar{J}$  depends on  $\bar{\Omega}$  for one sequence close to the Newtonian limit ( $\bar{M} = 0.01$ ) and one highly relativistic configuration ( $\bar{M} = 0.1$ ). We see verified

<sup>9</sup> A discussion as to how one can integrate (8a) to determine the mass follows in § 5.2.



**Figure 2.** The angular momentum  $\bar{J}$  is shown as a function of angular velocity  $\bar{\Omega}$  for a sequence of star's with the constant mass  $\bar{M} = 0.01$ . The solid line depicts numerical results, which are accurate to better than 10 digits and thus act as an absolute standard of reference.



**Figure 3.** As in Fig. 2, but for a sequence with the mass  $\bar{M} = 0.1$ .

in each plot that the approximation becomes arbitrarily good for  $\Omega \rightarrow 0$ . Furthermore, we see in agreement with Fig. 1 that  $H$  converges quickly for large  $A$ , and indeed  $H_5$  is barely distinguishable from  $H_1$  in this figure ( $A \approx 2.8$  over the whole range of the plot). This does not mean that the slow rotation approximation is good however. The discrepancy between the correct and approximated values for  $\bar{J}$  are quite noticeable for  $\bar{\Omega} = 0.6$ , although its maximal value along this sequence is  $\bar{\Omega} \approx 1.23$ . The inaccuracies due to the slow rotation approximation are evident in Fig. 3 as well. Here, however, the improvement brought about in going from  $H_1$  to  $H_5$  is more pronounced ( $A$  varies from about 2.7 to 2.8 over the range of the plot).

To have more quantitative information as to when to truncate the series  $H$ ,

**Table 3.** The relative error in the angular momentum as a function of  $H_n$  is shown for various values of  $\bar{\Omega}$  for a sequence with  $M = 0.01$ 

$\bar{\Omega}$	$\left  \frac{J-J(H_1)}{J} \right $	$\left  \frac{J-J(H_2)}{J} \right $	$\left  \frac{J-J(H_3)}{J} \right $	$\left  \frac{J-J(H_\infty)}{J} \right $
0.01	$1.5 \times 10^{-2}$	$2.5 \times 10^{-4}$	$2.6 \times 10^{-5}$	$1.8 \times 10^{-5}$
0.1	$1.7 \times 10^{-2}$	$1.6 \times 10^{-3}$	$1.8 \times 10^{-3}$	$1.8 \times 10^{-3}$
0.2	$2.3 \times 10^{-2}$	$7.1 \times 10^{-3}$	$7.4 \times 10^{-3}$	$7.3 \times 10^{-3}$
0.4	$4.5 \times 10^{-2}$	$2.9 \times 10^{-2}$	$3.0 \times 10^{-2}$	$3.0 \times 10^{-2}$
0.8	$1.4 \times 10^{-1}$	$1.3 \times 10^{-1}$	$1.3 \times 10^{-1}$	$1.3 \times 10^{-1}$
1.23	$4.7 \times 10^{-1}$	$4.6 \times 10^{-1}$	$4.6 \times 10^{-1}$	$4.6 \times 10^{-1}$

**Table 4.** As in Table 3, but for a sequence with  $M = 0.1$ .

$\bar{\Omega}$	$\left  \frac{J-J(H_1)}{J} \right $	$\left  \frac{J-J(H_2)}{J} \right $	$\left  \frac{J-J(H_3)}{J} \right $	$\left  \frac{J-J(H_\infty)}{J} \right $
0.01	$5.2 \times 10^{-2}$	$1.3 \times 10^{-2}$	$5.8 \times 10^{-3}$	$1.5 \times 10^{-5}$
0.1	$5.3 \times 10^{-2}$	$1.1 \times 10^{-2}$	$7.3 \times 10^{-3}$	$1.5 \times 10^{-3}$
0.2	$5.8 \times 10^{-2}$	$6.7 \times 10^{-3}$	$1.2 \times 10^{-2}$	$6.0 \times 10^{-3}$
0.4	$7.5 \times 10^{-2}$	$1.1 \times 10^{-2}$	$2.9 \times 10^{-2}$	$2.4 \times 10^{-2}$
0.8	$1.4 \times 10^{-1}$	$7.9 \times 10^{-2}$	$9.5 \times 10^{-2}$	$9.0 \times 10^{-2}$
1.45	$4.0 \times 10^{-1}$	$3.6 \times 10^{-1}$	$3.7 \times 10^{-1}$	$3.7 \times 10^{-1}$

Tables 3 and 4 consider the same sequences and show the relative error in the angular momentum as a function of  $H_n$  for various values of  $\bar{\Omega}$ . Each table shows a range of  $\bar{\Omega}$  values extending up to its maximum for the respective sequence. The extremely large errors emphasize how poor the slow rotation approximation is for homogeneous bodies near the mass-shedding limit. One should keep in mind however that these errors are not astrophysically relevant since the fastest known pulsars have an angular velocity of  $\bar{\Omega} \approx 0.4$ , meaning that the approximation holds to within about 3%. If one chooses the order of truncation, by requiring that the relative error in  $\bar{J}$  be in the vicinity of a few percent whenever the slow rotation approximation allows for such accuracy, then  $H_2$  seems an appropriate choice. When providing explicit expressions for metric functions in the next section, we shall choose this truncation order.

### 5.2. Deriving Explicit Expressions

If we label the arguments in the hypergeometric functions of  $H$  (see (14)) by  $a, b$  and  $c$ , then we find  $a + b - c = -3/2$ . By applying Gauss' relations for contiguous hypergeometric functions (see e.g. § 2.8 in [23]), we can convert this into two functions with  $c = -1/2$  and  $c = 1/2$ . Quadratic transformations are known to exist for such functions so that they can be converted into associated Legendre functions of the form  $P_\nu^{\pm n+1/2}$ ,  $n \in \mathbb{Z}$  (see e.g. §7.3.1 Eqs (36) and 41 in [24]). Associated Legendre functions of this form can, in turn, be represented with a finite number of terms, § 3.6.1 in [23]. The explicit expression that results for  $H_2$  can be found in the appendix.

The requirement that the metric functions be continuous at the star's boundary leads via (9a) to

$$\bar{\omega}|_{r=\bar{R}_S} = \bar{\Omega} - \frac{2\bar{J}}{\bar{R}_S^3}, \quad (17)$$

where we have chosen to express the relation in terms of dimensionless quantities. Taking into account

$$\bar{M}_S = \frac{4}{3}\pi\bar{R}_S^3 \iff \bar{R}_S^2 = \frac{3}{8\pi}(1 - A^2/9), \quad (18)$$

(cf. (2)), we thus know  $\bar{J}$  as a function of  $c_0$  and  $A$  for a given  $\bar{\Omega}$ . Since the normal derivative of  $\bar{\omega}$  must be continuous on this surface (see e.g. § 30.5 in [8]),

$$\left. \frac{d}{dr}\bar{\omega} \right|_{r=r_S} = \frac{6\bar{J}}{\bar{R}_S^4} \quad (19)$$

must hold as well. If we were to prescribe  $A$  as in [6], then this would suffice to determine  $c_0$  and  $\bar{J}$ .

If on the other hand, we choose to prescribe  $\bar{\Omega}$  and  $\bar{M}$  as in the previous section, then we have to solve (8b) before being able to determine both  $A$  and  $c_0$ .

Eq. (8b) can then be integrated from the centre out to the surface of the star to give an expression for  $\delta\bar{M} = \bar{m}_0|_{r=\bar{R}_S} + \bar{J}^2/\bar{R}_S^3$  from (9b). The constant of integration must be chosen such that  $m_0(0) = 0$  so that  $g_{rr}$  remains finite at the star's centre. The total mass of the system is then  $\bar{M} = \bar{M}_S + \delta\bar{M} = 4/3\pi\bar{R}_S^3 + \delta\bar{M}$ . Taking into account our choice of normalization (cf. (2))

$$\bar{M}_S = \frac{4}{3}\pi\bar{R}_S^3 \iff \bar{R}_S^2 = \frac{3}{8\pi}(1 - A^2/9),$$

we have arrived at expressions containing the unknown variables  $c_0$  and  $A$ , which can then be determined by prescribing  $\bar{M}$  and  $\bar{\Omega}$ .

Because the integrand in the integral to determine  $m_0$  contains quadratic terms in  $\bar{\omega}$  and its derivative, it does not seem possible to obtain an analytic expression for  $m_0$  based on the hypergeometric representation discussed above. In the appendix, expressions are thus provided for the metric functions in terms of a series in  $z$ .

Eq. (8c) for determining  $h_0$  is simply a first order, linear differential equation, where (7) is used to replace  $\delta p_0^*$  by an expression in  $h_0$ . The constant of integration can be chosen arbitrarily<sup>10</sup> and is chosen in the appendix such that  $h_0(0) = \gamma$ . This choice amounts to identifying a rotating body with a non-rotating one of the same central pressure since the choice implies through (7) that  $\delta p_0^*(0) = 0$ . The nature of such choices of identification and some of the effects different choices can have was discussed in the context of the post-Newtonian approximation in [4]. The interior solution for  $h_0$  must join continuously to its exterior solution, thus fixing the constant  $\gamma$  of (7).

The system of equations (8d) and (8e) for obtaining  $h_2$  and  $v_2$  can be solved as described in [6]. Both functions must vanish at the origin so that  $h$  and  $k$  have unique values there. The remaining constant as well as the constant  $K$  in (9d) and (9e) can then be determined by requiring that these two functions be continuous at the star's boundary.

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<sup>10</sup>The comment on pg. 67 of [6] that  $h_0$  must vanish at the origin seems to be an oversight. In that work, as here,  $\delta p_0^*$  was chosen to vanish at the origin, which then implies through (7) that  $h_0(0) = \gamma$ , which is not zero in general.

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### Appendix A. Explicit Representation of the Truncated Metric

The formulæ for the metric functions are collected in this appendix. They are valid over the range  $A \in (3/2, 3)$  and make use of the variable

$$z = \frac{1}{2} \left( 1 - \sqrt{1 - r^2/k^2} \right), \quad k = \frac{r_S}{\sqrt{1 - A^2/9}}.$$

The symbol  $F(a, b; c; z)$  refers to the hypergeometric function. The metric depends on three paramters (one scaling parameter and two ‘‘physical’’ parameters) that were here chosen to be  $A = 3\sqrt{1 - 2M_S/r_S}$  (see (5)),  $r_S$  and  $c_0$ .  $M_S$  and  $r_S$  are the mass and radius (in Schwarzschild coordinates) of a spherical star and  $c_0$  is an integration constant (see (14) and (15)) giving the value of  $\tilde{\omega}$  at the star’s centre  $\tilde{\omega}(0) = 4\sqrt{\pi}/3c_0$ . The expressions below do not diverge as one approaches the Newtonian limit  $A \rightarrow 3$  since  $c_0 \rightarrow 0$  sufficiently quickly in that limit.

$$\tilde{\omega} = \frac{4\sqrt{\pi}c_0}{3} \left[ 1 + \frac{8Az}{5(A-1)} F\left(1, 4; \frac{7}{2}; z\right) + \frac{8(Az)^2}{35(A-1)^2} F\left(2, 5; \frac{11}{2}; z\right) \right] + \dots \quad (1.1a)$$

which can be evaluated to yield

$$\begin{aligned} \tilde{\omega} = & \frac{\sqrt{\pi}c_0}{((A-1)z)^2(z-1)} \left[ \left( -1/24 \frac{(88z-63)A^2}{\sqrt{1-z}\sqrt{z}} + 2/3 \frac{\sqrt{z}A}{\sqrt{1-z}} \right) \arcsin(\sqrt{z}) \right. \\ & + \left( -\frac{2}{45}z^3 - \frac{13}{45}z^2 + \frac{23}{12}z - \frac{21}{8} \right) A^2 \\ & \left. - 2/9z(4z^2 - 10z + 3)A + 4/3z^2(z-1) \right] + \dots \end{aligned} \quad (1.1b)$$

or expanding as a power series and dropping higher terms

$$\tilde{\omega} = \frac{4\sqrt{\pi}c_0}{3} \left[ 1 + \frac{8A}{5(A-1)}z + \frac{8A(9A-8)}{35(A-1)^2}z^2 \right] + \dots \quad (1.1c)$$

We now leave off the ellipsis at the end of each expression.

$$\begin{aligned} m_0 = & \frac{512\pi c_0^2 r_S^3 A z^{5/2}}{525(9-A^2)^{3/2}(A-1)^5} \left( 840(A-1)^2 + 12(A-1)(A-125)z \right. \\ & \left. + (2975 + 602A - 441A^2)z^2 \right) \end{aligned} \quad (1.2)$$

$$\begin{aligned} \delta M = & \frac{16\pi c_0^2 r_S^3 (3-A)^{5/2}}{127575(9-A^2)^{3/2}(A-1)^5} \times \\ & \left( \sqrt{6}A(84015 - 109128A + 34922A^2 + 3176A^3 - 441A^4) \right. \\ & \left. + \frac{8}{63}(9-A^2)(A-1)(A+3)^{5/2}(45-56A+9A^2)^2 \right) \end{aligned} \quad (1.3)$$

$$h_0 = \gamma - \frac{1792A\pi c_0^2 r_S^2}{15(A-1)^3(A^2-9)}z^2 \quad (1.4)$$

with

$$\begin{aligned} \gamma &= \frac{-9\delta M}{A^2 r_S} + \frac{64\pi c_0^2 r_S^2 (A-3)}{893025 A^2 (A+3)(A-1)^4} \\ &\quad \times (81 A^8 - 522 A^7 - 2102 A^6 + 14262 A^5 + 65961 A^4 \\ &\quad - 165591 A^3 - 47466 A^2 + 298890 A - 164025) \\ h_2 &= f_0 z + \left( \frac{(3A-25)f_0}{7(A-1)} + \frac{512\pi c_0^2 r_S^2 A}{3(A^2-9)(A-1)^3} \right) z^2 \\ &\quad + \left( \frac{2(5A^2-17A+104)f_0}{21(A-1)^2} + \frac{512\pi c_0^2 r_S^2 A(129A-575)}{225(A^2-9)(A-1)^4} \right) z^3 \end{aligned} \quad (1.5)$$

$$\begin{aligned} v_2 &= \left( \frac{-2f_0}{A-1} - \frac{512\pi c_0^2 r_S^2 A}{3(A^2-9)(A-1)^3} \right) z^2 \\ &\quad - \left( \frac{4(A-13)f_0}{7(A-1)^2} + \frac{1024\pi c_0^2 r_S^2 A(7A+25)}{75(A^2-9)(A-1)^4} \right) z^3 \\ &\quad + \left( \frac{-2(5A^2-26A+221)f_0}{21(A-1)^3} \right. \\ &\quad \left. + \frac{512\pi c_0^2 r_S^2 A(51A^2-2262A+9275)}{1575(A^2-9)(A-1)^5} \right) z^4 \end{aligned} \quad (1.6)$$

where the continuity of  $h_2$  and  $v_2$  at the star's boundary then give for the constant  $f_0$  and  $K$  of (9d) and (9e)

$$\begin{aligned} f_0 &= \pi c_0^2 R_S^2 \times \left[ -512(A-1)(A^2-9) \left( 81 A^8 - 522 A^7 - 3560 A^6 + 23658 A^5 \right. \right. \\ &\quad \left. \left. + 44370 A^4 - 120771 A^3 - 2046924 A^2 + 3774195 A - 492075 \right) Q_2^1 \left( \frac{r_S}{M_S} - 1 \right) \right. \\ &\quad \left. - 768 A \left( 162 A^9 - 1206 A^8 - 3160 A^7 + 32728 A^6 + 14001 A^5 - 328536 A^4 \right. \right. \\ &\quad \left. \left. + 373590 A^3 - 2968596 A^2 + 4333095 A + 328050 \right) Q_2^2 \left( \frac{r_S}{M_S} - 1 \right) \right] / \\ &\quad \left[ 28350 (A-3)(A+3)^2 (A-1)^3 \right. \\ &\quad \left. \times (1989 - 2514 A + 962 A^2 - 74 A^3 + 5 A^4) Q_2^1 \left( \frac{r_S}{M_S} - 1 \right) \right. \\ &\quad \left. - 14175 (A-3)(A+3)(A-1)^2 A \right. \\ &\quad \left. (5 A^4 - 92 A^3 + 1790 A^2 - 5052 A + 4149) Q_2^2 \left( \frac{r_S}{M_S} - 1 \right) \right] \end{aligned} \quad (1.7)$$

$$\begin{aligned}
K = & \frac{\pi c_0^2 r_S^3 A}{8037225 r_S (A-1)^4 (A+3)} \times \left[ 64 \left( 2835 A^{13} - 65331 A^{12} + 893264 A^{11} \right. \right. \\
& - 4933304 A^{10} - 8773582 A^9 + 172596150 A^8 - 361180413 A^7 - 958944555 A^6 \\
& + 1549090818 A^5 + 11997038790 A^4 - 41313070791 A^3 + 54020240631 A^2 \\
& \left. \left. - 33095488275 A + 8082331875 \right) \right] / \\
& \left[ 2 (A-1) (A+3) (5 A^4 - 74 A^3 + 962 A^2 - 2514 A + 1989) Q_2^1 \left( \frac{r_S}{M_S} - 1 \right) \right. \\
& \left. - A (5 A^4 - 92 A^3 + 1790 A^2 - 5052 A + 4149) Q_2^2 \left( \frac{r_S}{M_S} - 1 \right) \right] \quad (1.8)
\end{aligned}$$

$m_2$  can then be calculated algebraically using (8f).

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