

On a covariant version of Caianiello's Model

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Abstract

Caianiello's derivation of Quantum Geometry through an isometric embedding of the spacetime (\mathbf{M}, \tilde{g}) in the pseudo-Riemannian structure $(\mathbf{T}^*\mathbf{M}, g_{AB}^*)$ is reconsidered. In the new derivation, a non-linear connection and the bundle formalism induce a Lorentzian-type structure in the 4-dimensional manifold \mathbf{M} that is covariant under arbitrary local coordinate transformations in \mathbf{M} . If models with maximal acceleration are required to be non-trivial, gravity should be supplied with other interactions in a unification framework.

1 Introduction

The maximal proper acceleration of a massive particle has been introduced by Caianiello as a consequence of its re-interpretation of Quantum Mechanics in the contest of Information Theory and System Theory ([1]). In Caianiello's theory, the value of the maximal acceleration is given by the relation:

$$A_{max} := \frac{2mc^3}{\hbar}. \quad (1.1)$$

This value was obtained considering the evolution of a free particle in a flat, torsion-free phase-space and constitutes a notable element of his theory. Relevant too, the different interpretations can in principle be checked

experimentally and their refutation can be an striking test of their supporting frameworks.

There are two different kinds of interpretations of the formula (1.1). In the first one, m is the rest mass of the particle being accelerated. In the second interpretation, m is an universal mass scale, typically of the same order than Planck's mass M_p . Indeed, for $m \sim M_p$ the order of the maximal acceleration coincides with the value obtained from string theory ([2]). Another possibility was considered in ref. [7], where the maximal acceleration has an universal value, corresponding with m of the order the lightest neutrino's mass. It is important to note that in all these interpretations, the value of the maximal acceleration is given in terms of relativistic constants and that it is invariant under arbitrary local coordinate changes.

We state the index convention used in this note. Indices denoted by minor and greek letters run from 0 to 3, while capital indices run from 0 to 7. If the contrary is not stated, Einstein's convention should be understood. In Caianiello's Quantum Geometry the spacetime manifold \mathbf{M} is 4-dimensional, the tangent bundle \mathbf{TM} is 8-dimensional and the projection $\pi : \mathbf{TM} \rightarrow \mathbf{M}$ induces an effective 4-dimensional geometry different from the original metric geometry of \mathbf{M} . This can be achieved through an embedding procedure ([3]). As result, the metric of the space-time \mathbf{M} is modified from

$$ds_0^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad \mu, \nu = 0, 1, 2, 3$$

to the new line element

$$ds^2 = \left(1 + \frac{\dot{x}^\sigma(s_0)\ddot{x}_\sigma(s_0)}{A_{max}^2}\right) ds_0^2 = \lambda(\ddot{x}(s_0)) ds_0^2, \quad \sigma = 0, 1, 2, 3. \quad (1.2)$$

$g_{\mu\nu}(x)$ is the initial Lorentzian spacetime metric at the point $x \in \mathbf{M}$ and \ddot{x}^μ are the components of the acceleration at this point, $\ddot{x}^\mu(s_0) = \frac{d^2 x^\mu}{ds_0^2}$. When it is possible to invert the equation $\dot{x}^\mu(s_0) = y^\mu(s_0)$, the immersion procedure is an embedding and the metric (1.2) lives in \mathbf{M} , because the factor $\lambda(\ddot{x}(s_0)) \rightarrow \tilde{\lambda}(x(s_0))$ lives in \mathbf{M} . However, $\tilde{\lambda}(x(s_0))$ is not an invariant factor and therefore the line element (1.2) is not invariant under arbitrary coordinate transformations of the spacetime manifold \mathbf{M} .

The aim of this note is to provide a solution to this covariance problem using the minimal geometric content and standard methods from Differential Geometry. In this sense, the paper is a minimal extension of the initial model, compared with other attempts to describe the geometry of maximal acceleration ([16]). In addition to the original embedding procedure suggested by Caianiello et al. ([3]), we introduce an alternative approach,

where a higher order non-degenerate Finsler-Lagrange structure appears. These type of structures were extensively studied by Miron's school ([13], [20]), at least in the positive definite case.

The present paper is organized in the following way. First, we review in *section 2* the deduction of *equation (1.2)* in usual Quantum Geometry. Then, after the introduction of the notion of non-linear connection in *section 3*, we understand the covariance problem and we show how using the non-linear connection is possible to solve it. In *section 4* we re-derive the immersion interpretation of (1.2) but using the correct formalism introduced in *section 3*. We also discuss the usefulness of the standard embedding procedure and we describe an alternative interpretation of the original formalism. In order to provide a general framework for the new formulation, we briefly introduce the relation of maximal acceleration with Finslerian Deterministic Systems. Finally, a discussion of some implications of the new formulation and its connection with the old one is also presented in *section 5*.

2 Elements of Caianiello's Model

We review the formal procedure of the derivations of the Quantum Geometry in Caianiello's model. Let us consider the 8-dimensional tangent bundle **TM**. It is endowed with a pseudo-Riemannian metric defined by the expression

$$ds^2 = g_{AB}dX^A dX^B, \quad A, B = 0, \dots, 7 \quad (2.1)$$

and where the natural coordinates are defined by

$$X^A = (x^\mu, y^\mu) = \left(x^\mu, \frac{c^2}{A_{max}^2} \frac{dx^\mu}{ds_0} \right), \quad \mu = 0, \dots, 3.$$

The metric coefficients g_{AB} is given in terms of the space-time metric $g_{\mu\nu}$ by

$$g_{AB} = g_{\mu\nu} \oplus g_{\mu\nu}.$$

The associated line element in **TM** is expressed as

$$ds^2 = \left(dx^\mu dx^\nu + \frac{c^4}{A_{max}^2} d\dot{x}^\mu d\dot{x}^\nu \right) g_{\mu\nu}, \quad (2.2)$$

where it was supposed that the set $\{dx^\mu, d\dot{x}^\mu\} = \{dx^\mu, dy^\mu\}$ is a basis for the dual frame along a possible trajectory ([3]). The embedding procedure requires the introduction of a timelike congruence, associated with the trajectories of particles with fixed positive mass. Indeed, the original Caianiello's

argument introduces an embedding procedure depending on this congruence, jointly with a quantum mechanical probability density, in order to reproduce Quantum Mechanics in the spacetime (\mathbf{M}, \tilde{g}) induced from the phase space geometry $(\mathbf{T}^*\mathbf{M}, g_B^*)$, with $g_{AB}^* = g^{AB} = g_{AB}^{-1}$. Then, the Lorentzian-type structure that they induce on \mathbf{M} from the initial metric (2.1) depends on the particular timelike congruence and this is why is not a properly Lorentzian structure.

Let us consider in some detail the embedding procedure. It can be understood in a simple way if the vector field $\frac{dx^\mu}{ds_0}$, $\mu = 0, \dots, 3$ is given in terms of the coordinates of the particle along its physical trajectory. That means that trajectories are injective curves, which is very likely the case if these trajectories are in spacetime, where there are no physical loops due to causality. This was the method used by Caianiello and co-workers's papers: there is a *one to one* correspondence between the values of the parameter s_0 and the points $x(s_0)$. The embedding procedure consist on the inversion $x(s_0) \rightarrow s_0(x)$, where s_0 is the length of the physical trajectory, counting from a fixed point. This produce a non-local dependence on the effective metric, due to the intrinsic dependence on the whole trajectory, as discussed before.

In Caianiello's model, an associated metric structure is defined for the co-tangent bundle $\mathbf{T}^*\mathbf{M}$. Caianiello's model is based on the fact that the geometry of the associated pseudo-Riemannian cotangent bundle provides a geometric description of Quantum Mechanics where, for instance, the curvature tensor components obtained from the above metric are related with Heisenberg's indeterminacy relations. Maximal acceleration is obtained as a consequence of this approach to the foundations of Quantum Mechanics ([1]).

Following with the deduction of Caianiello's model, for a particle of mass m , the line element (2.2) is reduced to (1.2). The associated structure has a metric components given by

$$\tilde{g}_{\mu\nu} = \left(1 + \frac{\ddot{x}^\sigma \ddot{x}_\sigma}{A_{max}^2}\right) g_{\mu\nu}, \quad (2.3)$$

which depend on the squared length of the space-time 4-acceleration $\|\ddot{x}\|^2 := g_{\mu\nu} \ddot{x}^\mu \ddot{x}^\nu$. The term

$$h_{\mu\nu} = \frac{\ddot{x}^\sigma \ddot{x}_\sigma}{A_{max}^2} g_{\mu\nu}$$

is called the quantum correction, because it vanishes when \hbar goes to zero.

There is another possibility that is to consider (2.3) as a higher order Lagrange structure, investigated by the group Miron's school ([13], [20]). A

a modern approach to these structures, with treatment of spinor and other related geometric topics, is compiled in [12] and also ref. [14] is of interest at this point. In these references, the mathematical formalism of diverse non-Riemannian metric structures is exposed and formulated with generality.

3 The Non-Linear Connection

In this section we introduce the minimal mathematical notions and tools that we need in order to formulate in a covariant way Caianiello's Quantum Geometry Model. The references on basic Finsler Geometry that we use are [4] and [20], and for its higher order generalizations is [20], adapting definitions and notions to the formulation in the case of non-degenerate Finsler and higher order structure. The major changes with these references consists on using non-homogeneous tensors and structures, instead of the usual homogeneous of degree zero expressions found in standard treatments.

Definition 3.1 *A non-degenerate Finsler structure F defined on the n -dimensional manifold \mathbf{M} is a real function $F^2 : \mathbf{TM} \rightarrow [0, \infty[$ such that it is homogeneous of dimension 2 in y and it is smooth in the split tangent bundle $\mathbf{N} = \mathbf{TM} \setminus \{0\}$ and the hessian matrix*

$$g_{\mu\nu}(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^\mu \partial y^\nu} \quad (3.1)$$

is non-degenerate in \mathbf{N} . The particular case when the manifold is 4-dimensional and $g_{\mu\nu}$ have signature $(+, +, +, -)$ is called Finsler spacetime ([23]).

$g_{\mu\nu}(x, y)$ is the matrix of the fundamental tensor $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$. In general, a non-degenerate structure will be defined by a generalized fundamental tensor g that is non-degenerate, symmetric and some smoothness conditions hold.

We should take care of the singularities $F(x, y) = 0$. In the above definition and the principal notions that we develop below eludes this singularity, following the above definition, based on the treatment of J. Beem ([21]).

Definition 3.2 *Let (\mathbf{M}, F) be a non-degenerate Finsler structure and (x, y, \mathbf{U}) a local coordinate system on \mathbf{TM} . The Cartan tensor components are defined by the set of functions,*

$$A_{\mu\nu\rho} = \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial y^\rho}, \quad \mu, \nu, \rho = 0, \dots, 3. \quad (3.2)$$

The components of the Cartan tensor are zero if and only if the Finsler spacetime (\mathbf{M}, F) is a Lorentzian structure. It is an homogeneous tensor of order -1 in y .

One can introduce several non-linear connection in the manifold \mathbf{N} . First, the non-linear connection coefficients are defined by the formula

$$N_\nu^\mu = \gamma_{\nu\rho}^\mu y^\rho - A_{\nu\rho}^\mu \gamma_{rs}^\rho y^r y^s, \quad \mu, \nu, \rho, r, s = 0, \dots, 3. \quad (3.3)$$

The coefficients $\gamma_{\nu\rho}^\mu$ are defined in local coordinates by

$$\gamma_{\nu\rho}^\mu = \frac{1}{2} g^{\mu s} \left(\frac{\partial g_{s\nu}}{\partial x^\rho} - \frac{\partial g_{\rho\nu}}{\partial x^s} + \frac{\partial g_{s\rho}}{\partial x^\nu} \right), \quad \mu, \nu, \rho, s = 0, \dots, 3;$$

$A_{\nu\rho}^\mu = g^{\mu l} A_{l\nu\rho}$ and $g^{\mu l} g_{l\nu} = \delta_\nu^\mu$. As a consequence, N_ν^μ are not homogeneous coefficients.

Using these coefficients one obtains a splitting of \mathbf{TN} . Let us consider the local coordinate system (x, y) of the manifold \mathbf{TN} and an open subset $\mathbf{U} \in \mathbf{N}$. The induced tangent basis for $\mathbf{T}_u\mathbf{N}$ is

$$\left\{ \frac{\delta}{\delta x^1} \Big|_u, \dots, \frac{\delta}{\delta x^n} \Big|_u, \frac{\partial}{\partial y^1} \Big|_u, \dots, \frac{\partial}{\partial y^n} \Big|_u \right\}, \quad \frac{\delta}{\delta x^\nu} \Big|_u = \frac{\partial}{\partial x^\nu} \Big|_u - N_\nu^\mu \frac{\partial}{\partial y^\mu} \Big|_u.$$

The set of local sections $\left\{ \frac{\delta}{\delta x^1} \Big|_u, \dots, \frac{\delta}{\delta x^n} \Big|_u, u \in \mathbf{U} \right\}$ generates the local horizontal distribution \mathcal{H}_U while $\left\{ \frac{\partial}{\partial y^1} \Big|_u, \dots, \frac{\partial}{\partial y^n} \Big|_u, u \in \mathbf{U} \right\}$ the local vertical distribution \mathcal{V}_U . The subspaces \mathcal{V}_u and \mathcal{H}_u are such that the splitting of $\mathbf{T}_u\mathbf{N}$ holds:

$$\mathbf{T}_u\mathbf{N} = \mathcal{V}_u \oplus \mathcal{H}_u, \quad \forall u \in \mathbf{U}.$$

This decomposition is invariant by the right action of $\mathbf{GL}(2n, \mathbf{R})$ and defines a connection in the bundle $\mathbf{TN} \rightarrow \mathbf{M}$. The corresponding dual basis in the dual vector bundle $\mathbf{T}^*\mathbf{N}$ is

$$\{dx^0, \dots, dx^3, \delta y^0, \dots, \delta y^3\}, \quad \delta y^\mu = (dy^\mu + N_\nu^\mu dx^\nu).$$

An extensive and general treatment of the notion of non-linear connection, allowing for connections in associated bundles that are g -compatible, can be found in the work of Vacaru et al. (for example [12] and [14]) in the contest of Lagrange spaces and other generalizations. In this general framework, the coefficients of an alternative non-linear connection are given by:

$$N_\nu^\mu = \frac{1}{2} \frac{\partial}{\partial y^\nu} \left[g^{\mu\rho} \left(\frac{\partial^2 F^2}{\partial y^\rho \partial y^\sigma} y^\sigma - \frac{\partial F}{\partial x^\rho} \right) \right], \quad \mu, \nu, \rho, \sigma = 0, \dots, 3. \quad (3.4)$$

These coefficients also define an splitting of \mathbf{TN} similar to the described before, such that the formal formulae is maintained. Note that in contrast with (3.3), this non-linear connection is homogeneous in y and does not have singularities at $F = 0$.

The covariant splitting is equivalent to the existence of a particular selection of the non-linear connection. These non-connections, that are also connections in the sense of Ehresmann ([5]), are associated with particular splitting of \mathbf{TN} due to associated non-degenerate metric structures in \mathbf{N} . In our case, the structure is given by

$$g_{AB} = g_{\mu\nu} dx^\mu \otimes dx^\nu + g_{\mu\nu} (\delta y^\mu \otimes \delta y^\nu). \quad (3.5)$$

The metric (3.5) is called a Sasaki-type pseudo-Riemannian metric in \mathbf{N} . A particular definition for the non-linear connection, with respect the corresponding structure (3.3), the horizontal sub-space spanned by the distributions $\{\frac{\delta}{\delta x^\mu}, \mu = 0, \dots, 3\}$ is orthogonal respect the distribution developed by $\{\frac{\partial}{\partial y^\mu}, \mu = 0, \dots, 3\}$.

For the case of timelike trajectories, we can also introduce the treatment of Asanov for non-degenerate Finsler spacetimes ([22]), which we will extend to non-degenerate Lagrange and generalized Finsler-Lagrange structures. Asanov considered the same notion as in *definition 3.1*, but restricting the smoothness condition of the hessian $g_{\mu\nu}$ to the vectors $y \in \mathbf{T}_x\mathbf{M}$ with $F(x, y) > 0$. In particular, let us denote the sets of admissible vectors by

$$\tilde{\mathbf{N}}_x := \{y \in \mathbf{T}_x\mathbf{M} \mid F(x, y) > 0\}, \quad \tilde{\mathbf{N}} = \cup_{x \in \mathbf{M}} \tilde{\mathbf{N}}_x.$$

In the case of non-degenerate Finsler structures, the treatment of Asanov starts with the definition (3.1) for Finsler spacetimes, but restricted to the set of admissible vectors $\tilde{\mathbf{N}}$. Under these restrictions homogeneous definitions of Cartan, non-linear connection and Sasaki-type metric can be introduced (for standard definitions see [4] or [20]). In particular, the Sasaki type metric is defined by

$$g_{AB} = g_{\mu\nu} dx^\mu \otimes dx^\nu + g_{\mu\nu} \left(\frac{\delta y^\mu}{F} \otimes \frac{\delta y^\nu}{F} \right). \quad (3.6)$$

It defines a non-degenerate metric in $\tilde{\mathbf{N}}$ and has associated an arc-length function that is re-parametric invariant (the main difference with the metric (3.5)). However, Asanov's treatment excludes the null trajectories $F = 0$. Although this is not too expensive for the discussion of this paper, about maximal acceleration, it is incomplete as a whole picture.

4 Covariant Quantum Geometry

In this section we will follow Beem's convention. Since using Asanov's treatment, the effective non-degenerate structure in the spacetime manifold \mathbf{M} is also non-homogeneous, no advantage is obtained using this approach to avoid the singularities at $F = 0$. Indeed in this contest, Beem's treatment appears less restrictive than Asanov's ones.

In order to investigate the properties of the pseudo-Riemannian structure $(\tilde{\mathbf{N}}, g_{AB})$, we have to note first that the distributions $\{(dx^\mu, dy^\mu), \mu = 0, \dots, 3\}$ do not form a consistent basis of $\mathbf{T}_u^* \tilde{\mathbf{N}}$ for a general non-flat manifold. The problem is localized in the set of 1-forms $\{dy^\mu, \mu = 0, \dots, 3\}$. Under local coordinate transformations of \mathbf{M} , the induced transformation rules are

$$d\tilde{x}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\rho} dx^\rho, \quad d\tilde{y}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} dy^\nu + \frac{\partial^2 \tilde{x}^\mu}{\partial x^\nu \partial x^\rho} y^\nu dx^\rho.$$

Therefore, the non-covariance problem of the covariance of eq. (1.2) is at the begin of the Caianiello's construction: the distributions $\{(dx^\mu, dy^\mu), \mu = 0, \dots, 3\}$ is not convenient to describe differential forms over \mathbf{N} and produces non-covariant results.

In order to solve this problem, we propose to consider the analogous construction as in Caianiello's model but using the basis (3.2). In a similar way as in Caianiello's model, we consider the Sasaki-type metric in \mathbf{N}

$$dl^2 = g_{\mu\nu} dx^\mu dx^\nu + \left(\frac{1}{A_{\max}}\right)^2 g_{\mu\nu} \delta y^\mu \delta y^\nu,$$

It is because the existence of the non-linear connection, represented by the above splitting of \mathbf{TN} , that this construction have invariant meaning. This metric can be expressed at the point $u \in \mathbf{N}$ as

$$dl^2|_u = ds^2 + \left(\frac{1}{A_{\max}}\right)^2 g_{\mu\nu} (N_\rho^\mu N_\xi^\nu dx^\rho dx^\xi + dy^\mu N_\rho^\nu dx^\rho + dy^\nu N_\rho^\mu dx^\rho).$$

From this pseudo-Riemannian structure in \mathbf{N} we obtain a Lorentzian-type structure in the spacetime manifold \mathbf{M} . Let us recall that $ds_0^2 = g_{\mu\nu}(x) dx_x^\mu dx_x^\nu$. Then,

$$ds_0^2 \left(\frac{dy^\mu}{ds_0} \frac{dy^\nu}{ds_0} \right) = dy^\mu dy^\nu.$$

When the constraint $y^\mu = \frac{dx^\mu}{ds_0} = \dot{x}^\mu$ is imposed, replacing $dx^\mu|_u$ by $dx^\mu|_x$ and using a particular timelike congruence to produce the inversion $x^\mu(s_0) \rightarrow s_0(x)$, the metric (3.3) induces an embedding in \mathbf{M} given by

$$dl_x^2 = ds_0^2 \left(1 + \left(\frac{1}{A_{max}}\right)^2 \ddot{x}^\sigma \ddot{x}_\sigma \right) + g_{\mu\nu}(x) \left(\frac{1}{A_{max}}\right)^2 (N_\rho^\mu N_\xi^\nu dx^\rho dx^\xi +$$

$$+d\dot{x}^\mu N_\rho^\nu dx^\rho + d\dot{x}^\nu N_\rho^\mu dx^\rho)).$$

Therefore the new space-time metric can be written as

$$dl_x^2 = ds_0^2 \left(1 + \left(\frac{1}{A_{max}} \right)^2 (\ddot{x}^\sigma \ddot{x}_\sigma + g_{\mu\nu} (N_\rho^\mu N_\xi^\nu \dot{x}^\rho \dot{x}^\xi + \ddot{x}^\mu N_\rho^\nu \dot{x}^\rho + \ddot{x}^\nu N_\rho^\mu \dot{x}^\rho)) \right). \quad (4.1)$$

We note that (4.1) is a Lorentzian-type structure, due to the use of the inversion procedure, except for the fact that it is not re-parametrization invariant in its dependence on the trajectories. The above procedure requires to introduce a timelike congruence, describing the particular evolution of the test particles. This implies that the Lorentzian-type metric (4.1) depends on the particular congruence. Although it is not a contradiction, this dependence is problematic because we do not know a priori the motion of the test particles and the initial configuration, which constitutes a rather uncomfortable geometric model and eventually a probability density function must be introduced.

An alternative treatment to obtain the embedding consists on consider that (4.1) defines a higher order Lagrange structure (for a definition of these structures see ref [20]), that being not homogeneous in y does not define a re-parametrization invariant arc-length in the tangent space (or via duality, in the phase space). We interpret the conditions $y^\mu = \frac{dx^\mu}{ds}$ and $dy^\mu = d\dot{x}^\mu$ in a more liberal way, in the sense that they are not directly differential equations associated to the evolution of a point particle, although we can accommodate any particular physical configuration in this formalism.

Either considering a Lorentzian-type structure associated with a congruence or as the line element corresponding to a generalized Lagrange structure, let us denote the quantum contributions by

$$h_1 = ds_0^2 \left(\frac{1}{A_{max}} \right)^2 (\ddot{x}^\sigma \ddot{x}_\sigma), \quad h_2 = ds_0^2 \left(\frac{1}{A_{max}} \right)^2 g_{\mu\nu} (N_\rho^\mu N_\xi^\nu \dot{x}^\rho \dot{x}^\xi + \ddot{x}^\mu N_\rho^\nu \dot{x}^\rho + \ddot{x}^\nu N_\rho^\mu \dot{x}^\rho).$$

Note that our treatment depends on the particular non-linear connection: different non-linear connections provides different quantum geometries. Although these contributions can be formally the same, since they depend on the non-linear connection, different non-linear connection will promote different competing theories.

If both quantum contributions are comparable, one expects $|h_1| \approx \frac{1}{3}|h_2|$. In the case of Riemannian structure, the Cartan tensor is zero and the non-linear connection is reduced to $N_\nu^\mu = \gamma_{\nu\rho}^\mu y^\rho$. Then the condition $|h_1| \approx \frac{1}{3}|h_2|$

can be expressed as the set of ordinary differential equations

$$\ddot{x}^\mu + k\gamma_{\nu\rho}^\mu \dot{x}^\nu \dot{x}^\rho = 0, \quad \mu, \nu, \rho = 0, \dots, 3.$$

The factor k is of order 1 when $h_1 \approx -\frac{1}{3}h_2$. Conversely, if the evolution is classical in the spacetime \mathbf{M} , both quantum corrections are related by $h_1 = -\frac{1}{3}h_2$ and the metric is the initial one g . In this case $k = 1$. We obtain that, for semi-classical particles, the new correction $|h_2|$ is as large as $3|h_1|$, inducing a natural almost cancelation of the quantum geometry corrections. Nevertheless, for pure quantum particles this is not necessary and a strong difference can appear between them.

For flat phase-space models ([1]), there is a global coordinate system where the connection coefficients are zero, recovering the original Caianiello's flat model:

$$dl^2 = ds_0^2(1 + h_1 + h_2) \longrightarrow dl^2 = ds_0^2(1 + h_1)$$

with metric coefficients

$$\tilde{g}_{\mu\nu} = \left(1 + \left(\frac{1}{A_{max}}\right)^2 \ddot{x}^\sigma \ddot{x}_\sigma\right) \eta_{\mu\nu},$$

being $\eta_{\mu\nu}$ the Minkowski metric. This is the typical metric tensor appearing in flat Quantum Geometry. Therefore, the predictions and corrections coming from the original flat model are also maintained in our revision.

On the other hand, from the expression (4.1), one obtains the general formula:

$$ds^2 = \left(1 + \frac{(D\dot{x})^\sigma (D\dot{x})_\sigma}{A_{max}^2}\right) ds_0^2. \quad (4.2)$$

For Berwald spaces with a linear connection D , the geodesic is just given by

$$(D\dot{x})^\sigma = \ddot{x}^\sigma + N_\rho^\sigma \dot{x}^\rho = \ddot{x}^\sigma + \gamma_{\nu\rho}^\sigma \dot{x}^\rho \dot{x}^\nu = 0.$$

This geodesic interpretation is general for curved Berwald spaces and it is covariant on the base manifolds.

Theorem 4.1 *Consider a Berwald spacetime (\mathbf{M}, F) such that the geodesics of an associated non linear connection are defined by the expression*

$$\ddot{x}^\sigma + N_\rho^\sigma \dot{x}^\rho = 0, \quad \sigma, \rho = 0, \dots, 3.$$

Then, the particles follow a non-geodesic evolution or the geometry induced from the Quantum Geometry is the same as the initial geometry.

In the case of a Berwald structure the geodesic equation of the Berwald connection is just given by equation (4.39) (a Berwald space is a Finsler structure where one can define a g -compatible connection that lives in the base manifold ([19])). Berwald structures have the benefice that preserve the Equivalence Principle and this is the reason why the *theorem* is stated in this category. In the general case the theorem should be

Theorem 4.2 *Consider a Finsler spacetime (\mathbf{M}, F) and suppose that the classical evolution of point particles is governed in the spacetime by the expression*

$$\ddot{x}^\sigma + N_\rho^\sigma \dot{x}^\rho = 0, \quad \sigma, \rho = 0, \dots, 3.$$

Then the effects of the Quantum Geometry on the initial space-time geometry are trivial.

Theorem 4.1 suggests that a non-trivial Quantum Geometry in Berwald spaces, where the Equivalence Principle holds, will imply departures from pure gravity. Unification and conjunction with other non-geometric interactions, implying deviation from classical geodesic evolution, are required. This conclusion follows if the induced structure in the spacetime \mathbf{M} is of the Berwald type, because $D_{\dot{x}}\dot{x} = 0$ is the geodesic equation and the Equivalence Principle holds. However, following a classical result of Einstein et al. ([15]), in General relativity pointlike particles follow geodesic evolution. Then, non-trivial Caianiello's models will depart from Einstein's gravity. Nevertheless, these models can be in principle compatible with the Equivalence Principle and with deviation from the geodesic motion for the physical geodesics.

Deviation from geodesic motion naturally occurs and then new field equations should be contemplated. A possible mathematical framework was developed in *section 2.3* of [12], where generalized Einstein's field equations for Finsler-Lagrange geometries were developed. Indeed, in the same reference but in *section 3*, particular examples were discussed in diverse general frameworks. These solutions will be of interest to our problem if they are free of singularities in the sense that geometric invariants are finite. This free-singularity condition is equivalent to the existence of maximal acceleration. Another possible general scheme for maximal acceleration is Deterministic Finslerian Models ([6]). In these models, maximal acceleration is contained in the geometric formulation of some dynamical systems, using the geometry of the phase space $\mathbf{T}^*\mathbf{TM}$.

Theorem 4.1 also suggests the need of consider the value of the maximal

acceleration in the sense of Brandt ([17]),

$$\frac{(D\dot{x})^\mu(D\dot{x})_\mu}{A_{max}^2} < 1. \quad (4.3)$$

This definition of proper maximal acceleration is covariant. In general, the connection used in our approach is not necessarily the same as in [17]. There is another argument for the inequality (4.3). If we impose that the quantum corrections leave invariant the sing of the spacetime interval, it is required the inequality (4.3) holds. This requirement can be argued from fundamental principles like locality and causality of the interactions: any interaction can be so strong to change the interval sing and only locality properties will be related with the interaction. The argument was explained in [6] and [7] in the contest of Deterministic Finslerian Models.

On the other hand, *theorem 4.2* implies that if we define the classical evolution by the equation $D_{\dot{x}}\dot{x} = 0$, then departure from it implies non-trivial quantum effects. If we wish to interpret this equation as equivalent to have the extremal of the classical finslerian arc-length functional, the non-linear connection should be the Cartan non-linear connection [20] or the Berwald non-linear connection ([19]) and similar conclusions as from *theorem 4.1* can be obtained, although allowing possible violations of the equivalence Principle.

5 Conclusions

The approach advocated in this note to Caianiello's Quantum Geometry has shown that the non-linear connection, in both non-degenerate Finslerian and pseudo-Riemannian frameworks (\mathbf{M}, g) , can play an important role on the foundations of the theory and in some of the predictions of the Quantum Geometry model in the case of non-zero curvature.

These deviations from standard Quantum Geometry are not negligible and are in principle so large as the original corrections of Caianiello's model. For instance, semi-classical regimen requires small deviations from classical evolution and therefore $h_1 \approx -\frac{1}{3}h_2$. Corrections coming from the non-linear connection terms could be also important for the modified Schwarzschild's geometry and for the study of neutrinos oscillations in this framework and in Kerr spaces([8],[9], [18]). Important changes can also be obtained in some astrophysical and cosmological consequences of maximal acceleration ([10],[11]). However, in the case of flat space models, the correction h_2 is zero and no new corrections have to be added to the original Quantum Geometry.

Covariant Caianiello's Quantum Geometry should be considered as a phenomenological description with origin in a deeper unified framework, because the existence of a maximal acceleration is not natural only with gravitational interaction. However with other interactions playing the game, maximal acceleration could be in harmony with the absence of singularities in geometric invariants.

In the case of violation of the Equivalence Principle, the approach indicated by *theorem 4.2* could be useful. If the particles follow the classical geodesic $D_{\dot{x}}\dot{x} = 0$, then Quantum Geometry will be trivial. But departure from this geodesic evolution will indicate the need of consider Caianiello's Quantum Geometry in a unification scheme. In this contest, Cartan and Berwald non-linear connections become an important tool for the study of the geometry associated with these models, linear connections in associated bundles are less relevant in our construction of Covariant Caianiello's models. Similar conclusions follows from *theorem 4.1* in the case that Equivalence Principle holds.

Finally, we wish to stress that independently of the kind of field theory behind the ideas expressed in this note, it seems that Covariant Caianiello's Model is an example where diverse types of structures like Finsler, Lagrange and generalized Lagrange as well as their non-degenerate analogous, appear in a natural way. In this sense, it is a particular realization of higher order Finsler-Lagrange geometries. This point is particularly important if we interpret (4.1) as inducing a higher non-degenerate metric structure in \mathbf{M} .

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