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Colliding black holes with linearized gravity

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We give a brief summary of results and ongoing research in the application of linearized theory to the study of black hole collisions in the limit in which the holes start close to each other. This approximation can be a valuable tool for comparison and code-checking of full numerical relativity computations. The approximation works quite well for the head-on case and this is motivation to pursue its use in other more interesting contexts. We summarize current efforts towards establishing the domain of validity of the approximation and its use in generation and evolution of initial data for more interesting physical cases.

I. INTRODUCTION

The intention of this talk is to summarize the application of linearized gravity, in the specific form of the theory of black hole perturbations, to the study of the collision of black holes. Most of the results are already present in the literature, and the rest of the work is still in progress so I present here only a brief survey.

The motivation for studying black hole collisions is quite clear. In the next few decades gravitational wave detectors will come online that will require "templates" of possible waveforms from different sources. The collision of black holes is one of the main candidates for observable sources of gravitational radiation. Although the initial and advanced LIGO detectors will not quite have the frequency range to detect the waves produced in the final moments of the most common collisions, it is expected that future detectors will, and knowing the waveform for the final moments can also lead to insights into the waveforms emitted earlier on.

The presence of this strong motivation from the experimental side has led to the formation of an alliance of numerical relativity groups (the "binary black hole grand challenge collaboration") with the goal of numerically simulating the collision of two black holes using supercomputers. The degree of difficulty of this project is reflected in the fact that several established numerical relativity groups have decided to team efforts in order to tackle it.

Here we will like to offer a much more modest approach, which is based on a simple idea: when a collision of two black holes starts with the holes so close to each other that they are surrounded by a common horizon, the problem looks from the point of view of an external observer as a single distorted black hole. It can therefore be treated with perturbation theory. Although one expects this approach to only yield results in a small range of initial separation, it provides —at least for that range— a benchmark against which one can calibrate numerical codes of the fully numerical approach. In reference [1] an explicit calculation was carried out using this idea. We took the initial data for the head-on collision of two black holes given by the Misner [2] solution and re-wrote it in such a way that in the case that the two black holes are close to each other it explicitly looks like "Schwarzschild plus something small". We took the "something small" and evolved it using the equations of linearized gravity (the Zerilli equation) and computed the radiated energy. The results are shown in figure 1, where we plot the energy radiated in the collision as a function of the initial separation and compare with the results of the NCSA group [3] using a numerical integration of the full Einstein equations. We see that the close approximation works very well until the holes are no longer surrounded by a common apparent horizon ($\mu_0 = 1.3$) and works within the correct order of magnitude up to when the holes are no longer surrounded by an event horizon ($\mu_0 = 2.0$). Also shown is a "far approximation" based on a particle-membrane paradigm [4]. Comparisons of waveforms have also been performed [4] and they also show very good agreement between the linearized theory and the full numerical simulations.

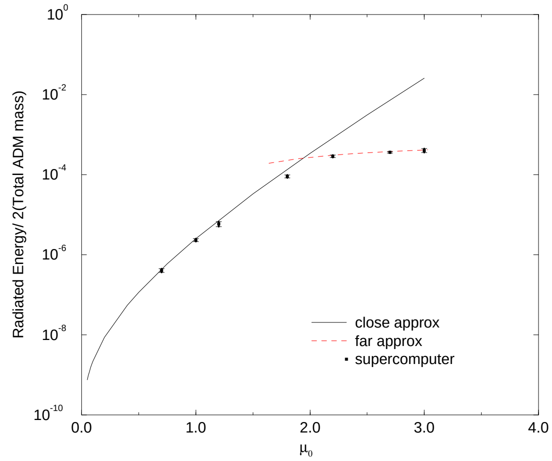

FIG. 1. Comparison of results for the radiated energy vs initial separation in a collision for the close approximation, the fully numerical results and the far approximation. Vertical scale is logarithmic.

All this shows that the use of linearized gravity in the close limit can be a valuable aid to full numerical evolutions of the two black hole problem. It is therefore quite tempting to apply the linearized treatment to more interesting situations, specifically the in-spiraling collision of two black holes with angular momentum. There are two main obstacles to doing this computation and we will detail them in the next two sections.

II. SECOND ORDER PERTURBATIONS: GIVING THE FORMALISM ERROR BARS

Assuming initial data for a black hole collision is given, we can rather easily evolve and compute energies in linearized theory. Why therefore not do it for the inspiraling collision? The main reason is that for that case there are no numerical results with which to compare and the linearized formalism does not have a measure of error in it: it therefore has little predictive power. There is no consistent way to say when the close approximation breaks down. In fact, this example teaches us a valuable lesson about perturbation theory: when is linearized perturbation valid? The obvious answer "when perturbations are small" is clearly naive. To begin with, "small" should be characterized in a coordinate invariant way. Moreover, as this example shows, perturbations can be "large" and perturbation theory can still be valid: it just needs to happen that the perturbations be large in regions of spacetime that do not contribute in a significant way to the physics of interest. In the two black hole example, such a region is the interior of the horizon and regions close to it, in which perturbations mostly fall into the black hole.

How is one to characterize when to trust the approximation? The answer is simple: work out the second order perturbations, compute the physical quantities of interest and use how much the first and second order calculation differ as a measure of the accuracy of the first order results. The advantage of this answer is that it is phrased in terms of what one is exactly interested in: the physical quantities. In the case of the collision of two black holes these are the radiated waveforms and energies.

The formalism for second order perturbations of black holes has not been worked out in the past. It can be studied in detail as we do in reference [5]. Here I just sketch some of the outstanding points. It turns out that all the information can be coded into a single variable, exactly as in the first order perturbation case and that that variable satisfies a "Zerilli equation",

$$-\frac{\partial\psi^{(2)}}{\partial t}+\frac{\partial\psi^{(2)}}{\partial r_*}+V(r)\psi^{(2)}=S \tag{1}$$

where $r_* = r + \log(r/2M - 1)$ and the Zerilli function $\psi^{(2)}$ is a coordinate invariant combination of the perturbed metric coefficients. This equation is exactly the same as the one satisfied by the first order perturbations (including the "potential" V(r), which can be seen in reference [1]). However, there is an important difference: the right-hand side is not zero but a "source" term S, which is listed explicitly in reference [5] and which is a complicated function quadratic in the first order perturbations and their derivatives. The way in which we derived this equation is to compute a particular combination of the Einstein equations, writing the perturbed metric in a particular coordinate system, the so called "Regge-Wheeler" gauge. This, in turn is a way of deriving the original Zerilli equation. The expression we get for $\psi^{(2)}$ is therefore a representation in that gauge of a gauge invariant quantity. The explicitly gauge invariant form of $\psi^{(2)}$ can also be computed.

We therefore are in a position to evolve to second order the problem of black hole collisions and therefore to endow the first order predictions with "error bars". This will be crucial for the inspiralling case, where numerical results are not expected for some time.

III. INITIAL DATA IN THE CLOSE APPROXIMATION

In the head-on collision case we were lucky to have an exact solution to the initial value problem that we could evolve. For the more realistic cases there are no exact solutions available at present and it is unlikely that they will be easily found in the future. There is an immediate alternative at hand. There exist already well tested numerical codes [7] for solving the initial value problem in general relativity in the context of black hole collisions. One could simply take these initial data evaluated for the case in which the black holes are close and "read off" from them the departures from Schwarzschild to be evolved using the linearized theory. This is certainly possible and has already been illustrated for Brill-Lindquist-type initial data by Abrahams and Price [6].

Apart from the possibility of using numerical initial data for realistic collisions it is interesting to notice that one can, up to a certain extent, solve the initial value problem analytically if one is only interested in initial data for the close approximation. The idea is simple: in the close approximation the initial data for a black hole collision departs a small amount from the initial data for a Schwarzschild spacetime for a single black hole with mass equal to the sum of the masses of the colliding holes. Therefore one can develop an approximation technique for the initial data starting from the initial data of Schwarzschild and adding small corrections proportional to the separation of the holes. We illustrate here only the zeroth order results, details will be given in a forthcoming paper in collaboration with John Baker.

The initial value problem of general relativity can be conveniently cast in the conformal formalism [8]. One is interested in solving the momentum and Hamiltonian constraints

$$\nabla^a(K_{ab}-g_{ab}K)=0 \tag{2}$$
$$^3R-K_{ab}K^{ab}+K^2=0 \tag{3}$$

where g_{ab} is the spatial metric, K_{ab} is the extrinsic curvature and ^3R is the scalar curvature of the three metric. One proposes a three metric that is conformally flat $g_{ab}=\psi^4\delta_{ab}$, with ψ^4 the conformal factor and a decomposition of the extrinsic curvature $\widehat{K}_{ab}=\psi^{-2}K_{ab}$.

The constraints become,

$$\widehat{\nabla}^a\widehat{K}_{ab}=0 \tag{4}$$
$$\widehat{\nabla}^2\psi=\psi^{-7}\widehat{K}_{ab}\widehat{K}^{ab}. \tag{5}$$

where $\widehat{\nabla}$ is a derivative with respect to the flat spacetime. Since the momentum constraint is linear, one can propose as a solution for it for the case of two black holes the sum of the solutions for the case of individual holes[1] with momentum P_a,

$$\hat{K}_{ab}=\frac{3}{2r^2}\left[P_{(a}n_{b)}-(\delta_{ab}-n_an_b)P^cn_c\right] \tag{6}$$

where n_b is a unit normal in the direction of $\vec{r}$ and all vector fields are defined in the flat background spacetime.

One now can put this solution in the Hamiltonian constraint and one is left with an elliptic, highly non-linear equation for ψ. This is the equation that is usually solved

[1] The particular solution chosen depends on the boundary conditions imposed. This may add other terms to the simple ones we list here for brevity, but they all behave in a similar fashion with respect to the approximations we will consider.

numerically. There exist situations, however, where one can make some progress analytically. Consider the case in which the momenta of the holes is small [8]. In that case one can neglect the right-hand side of the Hamiltonian constraint and one only needs to solve a vacuum Laplace equation for ψ. The solution can therefore be very simply found, the difficulty depending on the boundary conditions one chooses for the problem (typically a "symmetrized" boundary condition is imposed, which complicates calculation quite a bit in certain cases, see [7] for details).

Another situation in which one can obtain an approximate solution is in the "close approximation". In that case one has two black holes of momenta equal and opposite $P_a^{(1)} = -P_a^{(2)}$, and since the black holes are close, the unit normals appearing in the form for the extrinsic curvature for each hole are approximately equal. That implies that the extrinsic curvature for the problem is approximately zero (as it should, since in the close limit the problem looks like a Schwarzschild black hole at rest.) Therefore one can again neglect the right-hand side of the Hamiltonian constraint and one is again left with a Laplace equation. Let us compare this approximation with the full numerical results. In order to do this we will compare the ADM energy of initial data for a collision of two holes of momentum P. The ADM energy in the conformal formalism is given by

$$E = -\frac{1}{2\pi} \oint_\infty \nabla_i \psi \, d^2 S^i \tag{7}$$

and we notice that it does not depend explicitly on the extrinsic curvature (it does implicitly via the constraints). Therefore at the approximation we are working, in which the constraints do not couple the conformal factor and the extrinsic curvature, the energy is independent of the extrinsic curvature and therefore independent of the momenta of the holes. We compare this prediction with the full numerical results of Cook in figure 2.

An interesting aspect is that one can advance this approximation one step further. One can input the extrinsic curvature and the conformal factor found as a fixed "source" in the equation determining the conformal factor and one can obtain a correction through the integration of a Poisson equation. Comparison of this approximation with the numerical data is currently in progress. Details are complicated by the particular boundary conditions that are usually chosen in the numerical computations.

It is evident that the "close approximation" can work in many other cases, apart from the head-on, equal momenta holes we considered here. The only changes will be that the solution one obtains in the "close limit" rather than being a slice of Schwarzschild will be a slice of Kerr or boosted Schwarzschild if the net result of the collision has angular momentum or linear momentum.

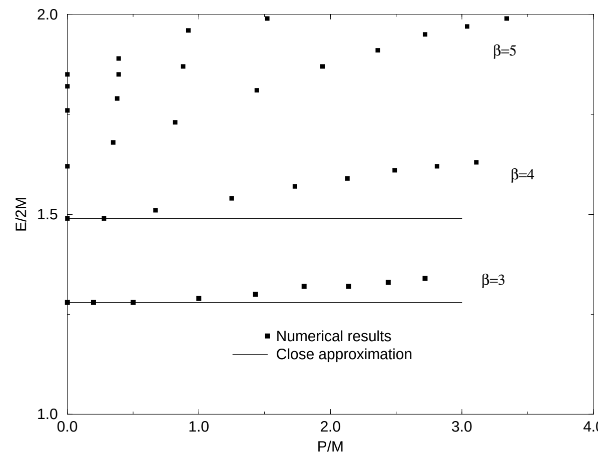

FIG. 2. The ADM energy of initial data for collisions of black holes of momenta P. The dots are the full numerical results of Cook, for different values of the initial separation β. We see that for small separations, the energy is approximately independent of the holes momenta, which coincides with the close approximation prediction, depicted by the solid line.

IV. SUMMARY

We have seen that the use of the "close approximation" can be a valuable aid to full numerical computations of the collision of two black holes. With the introduction of a second order scheme we are now in a position of offering reliable estimates of energies and waveforms that we expect people working on the full numerical simulations will find of use to calibrate codes and design strategies for better integrating the Einstein equations in this problem of great current physical interest.

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