Rare Kaon Decays on the Lattice

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Abstract

We show that long distance contributions to the rare decays $K \to \pi \nu \bar{\nu}$ and $K \to \pi \ell^+ \ell^$ can be computed using lattice QCD. The proposed approach requires well established methods, successfully applied in the calculations of electromagnetic and semileptonic form factors. The extra power divergences, related to the use of weak four-fermion operators, can be eliminated using only the symmetries of the lattice action without ambiguities or complicated non-perturbative subtractions. We demonstrate that this is true even when a lattice action with explicit chiral symmetry breaking is employed. Our study opens the possibility of reducing the present uncertainty in the theoretical predictions for these decays.

1 Introduction

Rare decays mediated by flavour-changing neutral-currents (FCNC) are among the deepest probes to uncover the fundamental mechanism of quark flavour mixing. Within the Standard Model (SM), these rare decays are strongly suppressed both by the GIM mechanism [1] and by the hierarchy of the CKM matrix [2], and are often dominated by short-distance dynamics. As a result, FCNC processes are very sensitive to possible new sources of flavour mixing, even if these occur well above the electroweak scale. The sensitivity to physics beyond the SM of these rare processes is closely related to the theoretical accuracy on which we are able to compute their amplitudes within the SM.

Within the family of FCNC decays, long-distance effects are not always negligible and, in most cases, they represent the dominant source of theoretical uncertainty. Longdistance contributions are typically relevant in: i) amplitudes where the GIM mechanism is only logarithmic; ii) amplitudes where the power-like GIM suppression of the long-distance component is partially compensated by a large CKM coefficient. So far, the evaluation of these non-perturbative contributions has been performed by means of effective theories. These analytic tools require the introduction of additional parameters, the knowledge of which constitutes a source of sizable theoretical uncertainty.

In this paper we show that for a class of very interesting processes, such as $K^+ \to \pi^+ \nu \bar{\nu}$ and $K \to \pi \ell^+ \ell^-$, it is possible in principle to compute non-perturbatively the longdistance contribution to the physical amplitudes on the lattice. The physical information is encoded in the following *T*-products:

$$\mathcal{T}^{\mu}_{Q,J}(q^2) = \mathcal{N}_V \int d^4x \int d^4y \, e^{-i\,q\cdot y} \, \langle \pi | T[Q(x) \, J^{\mu}(y)] | K \rangle \,, \tag{1}$$

where Q denotes a generic four-quark operator of the effective weak Hamiltonian, J^{μ} is either the electromagnetic or the weak neutral current, and \mathcal{N}_V is an appropriate volume factor. If the invariant mass of the lepton pair (q^2) is smaller than any physical hadronic threshold, the calculation proceeds as in the case of semileptonic form factors (see e.g. refs. [3, 4]), and one obtains directly the relevant amplitude. When instead the leptonic invariant mass exceeds the pion threshold, the final state interaction induces problems similar to those encountered with non-leptonic kaon decays [5, 6]. However, the knowledge of the amplitude for $q^2 < m_{\pi}^2$ is sufficient to determine the leading unknown effective couplings of these amplitudes within the framework of chiral perturbation theory (CHPT) [7]–[9]. Therefore, the combination of lattice calculations and CHPT should allow to reach an unprecedented level of precision for these rare decays.

When using a lattice action with explicit chiral symmetry breaking, such as Wilson, Clover or twisted mass fermions, further problems arise because of additional ultraviolet (power) divergences which may appear in the operator matrix elements or in the relevant T-products.¹ We show that for the electromagnetic current, gauge invariance prevents

¹ Alternative formulations which guarantee chiral symmetry in the physical matrix elements, such as overlap fermions, do not have this problem [10]. However, these formulations are not mature yet to be used for unquenched calculations of these complicated matrix elements, for quark masses close to the physical values.

the appearance of these divergences even if the most popular lattice actions are used. Consequently, when J^{μ} is the electromagnetic current, the *T*-products in eq. (1) are finite provided that a renormalized weak effective Hamiltonian is used. The situation is slightly more complicated when J^{μ} is the weak neutral current. In this case, simple power counting, related to the behavior of the *T*-product at short distances, shows that both quadratic and linear divergences may appear. We show that the quadratic divergence, which is not a peculiarity of the lattice regularization, is canceled by the GIM mechanism. Concerning the linear divergence, which is present only if there is an explicit chiral symmetry breaking term in the lattice action, we demonstrate that it can be avoided by using the maximally twisted mass fermion action [11].

There is a further subtlety concerning the ambiguity in the renormalization of the effective weak Hamiltonian out of the chiral limit [12]. In ref. [12] it has been shown that this ambiguity does not affect the physical $K \to \pi\pi$ amplitudes, but is present in "non-physical" matrix elements, such as $\langle \pi | Q | K \rangle$. This problem is present also in our case and implies an ambiguity in the *T*-products of eq. (1). By means of appropriate Ward Identities, we show that the physical amplitude, the extraction of which requires a specific spectral analysis discussed in the following, is instead free of ambiguities.

The paper is organized as follows: in sect. 2 we recall the basic ingredients of radiative decays in the framework of the effective Hamiltonian approach. In sect. 3 we describe the strategy for computing the relevant amplitudes from the Euclidean Green functions and discuss the structure of the divergences in both cases: when they cancel because of gauge invariance and when it is necessary to get rid of them using GIM mechanism and twisted mass. In sect. 4 we show how to extract the physical amplitude in spite of the ambiguity of the renormalized effective Hamiltonian. The results are summarized in the conclusions.

2 Effective Hamiltonian for $K \to \pi \ell^+ \ell^-(\nu \bar{\nu})$ decays

The dimension-six effective Hamiltonian relevant to evaluate $s \to d\ell^+\ell^-(\nu\bar{\nu})$ amplitudes at next-to-leading order accuracy, renormalized at a scale $M_W \gg \mu > m_c$, can be written as

$$\mathcal{H}_{eff} = \mathcal{H}_{eff}^{|\Delta S|=1} + \mathcal{H}_{eff}^{\text{FCNC}} + \frac{G_F}{\sqrt{2}} \sum_{q=u,d,s,c} Q_q^{\text{NC}} + \frac{G_F}{\sqrt{2}} \sum_{q=u,c \ q'=d,s} V_{ij} Q_{qq'}^{\text{CC}} + \text{h.c.} , \qquad (2)$$

where V_{ij} denote the elements of the CKM matrix,

$$\mathcal{H}_{eff}^{|\Delta S|=1} = \frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \left[\sum_{i=1,2} C_i \left(Q_i^u - Q_i^c \right) + \sum_{i=3\dots8} C_i Q_i + \mathcal{O}\left(\frac{V_{ts}^* V_{td}}{V_{us}^* V_{ud}} \right) \right], \quad (3)$$

is the usual $|\Delta S| = 1$ weak Hamiltonian, for which the Wilson coefficients are known at the NLO [13], and

$$\mathcal{H}_{eff}^{\text{FCNC}} = \frac{G_F}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \theta_W} V_{us}^* V_{ud} \left[\sum_{i=7V,7A,\nu} C_i Q_i + \mathcal{O}\left(\frac{V_{ts}^* V_{td}}{V_{us}^* V_{ud}}\right) \right] . \tag{4}$$

Here

$$Q_{qq'}^{CC} = \bar{q}\gamma^{\mu}(1-\gamma_5)q' \,\bar{\nu}\gamma_{\mu}(1-\gamma_5)\ell$$

$$Q_q^{NC} = \bar{q}\gamma^{\mu} \left[2\hat{T}(1-\gamma_5) - 4\hat{Q}\sin^2\theta_W\right]q$$

$$\times \left[\bar{\nu}\gamma_{\mu}(1-\gamma_5)\nu - \bar{\ell}\gamma_{\mu}(1-\gamma_5 - 4\sin^2\theta_W)\ell\right]$$
(5)

are the charged-current and neutral-current effective interactions obtained by the integration of the heavy W and Z fields,

$$Q_{7V} = \overline{s}\gamma^{\mu}(1-\gamma_5)d\,\bar{\ell}\gamma_{\mu}\ell\,\,,\tag{6}$$

$$Q_{7A} = \overline{s}\gamma^{\mu}(1-\gamma_5)d\,\bar{\ell}\gamma_{\mu}\gamma_5\ell\,\,,\tag{7}$$

$$Q_{\nu} = \overline{s}\gamma^{\mu}(1-\gamma_5)d\,\overline{\nu}\gamma_{\mu}(1-\gamma_5)\nu , \qquad (8)$$

are the leading FCNC operators, and

$$Q_1^q = \bar{s}_{\alpha} \gamma^{\mu} (1 - \gamma_5) q_{\beta} \bar{q}_{\beta} \gamma_{\mu} (1 - \gamma_5) d_{\alpha} , \qquad (9)$$

$$Q_2^q = \bar{s}_\alpha \gamma^\mu (1 - \gamma_5) q_\alpha \ \bar{q}_\beta \gamma_\mu (1 - \gamma_5) d_\beta \tag{10}$$

the leading four-quark operators. The four-quark operators originated by penguin contractions are denoted by $Q_{1...6}$, whereas Q_7 and Q_8 correspond to magnetic and chromomagnetic operators, respectively (see e.g. ref. [13]).

Thanks to both the GIM mechanism and the unitarity of the CKM matrix, the contributions to the FCNC amplitudes can be unambiguously decomposed into two parts, the first one proportional to the CKM combination $V_{us}^*V_{ud}$, the second one proportional to $V_{ts}^*V_{td}$. Since $|V_{ts}^*V_{td}| \ll |V_{us}^*V_{ud}|$, the contribution proportional to $V_{ts}^*V_{td}$ is negligible but for cases where it is enhanced by the large top-quark mass (i.e. for amplitudes which exhibit a power-like GIM mechanism). In these cases, the amplitudes are completely dominated by short distances (top-quark loops) and can be evaluated in perturbation theory to an excellent degree of approximation. In this paper instead we are interested only in the long-distance components of the amplitudes, therefore we can safely work in the limit $V_{td} = 0$.

We can seemingly neglect the matrix elements of $Q_{1...6}$ and Q_8 in the evaluation of $K \to \pi \ell^+ \ell^- (\nu \bar{\nu})$ amplitudes: these matrix elements vanish at the tree level and the corresponding Wilson coefficients are substantially smaller than those of $Q_{12}^{u,c}$. In this approximation, we only have to consider the contributions of the leading FCNC operators in eqs. (6)–(8) and the non-trivial contractions of $Q_{1,2}^{u,c}$ with the electromagnetic current and the currents defined by $Q_{qq'}^{CC}$ and Q_q^{NC} . The $K \to \pi$ matrix elements of the FCNC operators in eqs. (6)–(8) can be extracted from data on the leading $K_{\ell 3}$ modes using isospin symmetry [14], or even computed directly on the lattice, with high accuracy, as recently shown in [3].² Concerning the contractions of $Q_{1,2}^{u,c}$, those with a charged current receive

² In principle, in the $K \to \pi \ell^+ \ell^-$ case one should also consider the tree-level matrix element of the magnetic operator $Q_7 = m_s \bar{s} \sigma^{\mu\nu} (1 - \gamma_5) dF_{\mu\nu}$, which cannot be directly extracted from $K_{\ell 3}$ data. However, within the SM the smallness of the corresponding Wilson coefficient makes this contribution negligible for practical purposes. This matrix element can be computed on the lattice with standard techniques, as shown in [15].

very small non-perturbative contributions (estimated to be below 1% at the amplitude level in the $K^+ \to \pi^+ \nu \bar{\nu}$ case and even smaller in all the other channels), which can be reliably estimated within CHPT [9, 16]. Thus the main problem are the contractions of $Q_{1,2}^{u,c}$ with a neutral current, as outlined in eq. (1).

So far, this problem has been addressed with the following two-step procedure: i) integrating out the charm as dynamical degree of freedom; ii) constructing the chiral realization of the corresponding effective Hamiltonian with light quarks only. This procedure suffers from two sources of theoretical errors: slow convergence of perturbation theory because of the low renormalization scale of the effective Hamiltonian ($\mu < m_c$); uncertainties associated to the new low-energy couplings appearing in the effective theory. Both these sources of uncertainties are naturally reduced in the lattice approach, where the effective Hamiltonian is renormalized above the charm scale and the *T*-products are evaluated in full QCD.

We now discuss separately electromagnetic and neutrino amplitudes in more detail.

2.1 $K \rightarrow \pi \ell^+ \ell^-$

The main non-perturbative correlators relevant for these decays are those with the electromagnetic current. In particular, the relevant T-product in Minkowski space is [7, 8]

$$\left(\mathcal{T}_{i}^{j}\right)_{\mathrm{em}}^{\mu}(q^{2}) = -i \int d^{4}x \, e^{-i \, q \cdot x} \, \langle \pi^{j}(p) | T \left\{ J_{\mathrm{em}}^{\mu}(x) \left[Q_{i}^{u}(0) - Q_{i}^{c}(0) \right] \right\} | K^{j}(k) \rangle \,, \quad (11)$$

$$J_{\rm em}^{\mu} = \frac{2}{3} \sum_{q=u,c} \bar{q} \gamma^{\mu} q - \frac{1}{3} \sum_{q=d,s} \bar{q} \gamma^{\mu} q \qquad (12)$$

for i = 1, 2 and j = +, 0. Thanks to gauge invariance we can write

$$\left(\mathcal{T}_{i}^{j}\right)_{\mathrm{em}}^{\mu}\left(q^{2}\right) = \frac{w_{i}^{j}(q^{2})}{(4\pi)^{2}} \left[q^{2}(k+p)^{\mu} - (m_{k}^{2} - m_{\pi}^{2})q^{\mu}\right] .$$
(13)

The normalization of (13) is such that the O(1) scale-independent low-energy couplings $a_{+,0}$ defined in [8] can be expressed as

$$a_j = \frac{1}{\sqrt{2}} V_{us}^* V_{ud} \left[C_1 w_1^j(0) + C_2 w_2^j(0) + \frac{2N_j}{\sin^2 \theta_W} f_+(0) C_{7V} \right] .$$
(14)

where f_+ is the $K \to \pi$ vector form factor and $\{N_+, N_0\} = \{1, 2^{-1/2}\}$ [3]. To a good approximation, the decay rates of the CP-conserving transitions $K^+ \to \pi^+ \ell^+ \ell^-$ and $K_S \to \pi^0 \ell^+ \ell^-$ are proportional to the square of these effective couplings [8]:

$$\mathcal{B}(K^+ \to \pi^+ e^+ e^-) \approx 6.6 \ a_+^2 \times 10^{-7} , \qquad \mathcal{B}(K_S \to \pi^0 e^+ e^-) \approx 10.4 \ a_0^2 \times 10^{-9} .$$
(15)

At present, we are not able to predict $a_{+,0}$ with sufficient accuracy: we simply fit their $\mathcal{O}(1)$ values from the measured rates of the corresponding decay modes (an updated numerical analysis can be found in [17]). Being completely dominated by long distance contributions,

these two CP-conserving processes would provide an excellent testing ground for the lattice technique.

On the other hand, the calculation of a_0 from first principles would have a very interesting phenomenological application in the $K_L \to \pi^0 \ell^+ \ell^-$ case, which proceeds via a CP-violating amplitude: the calculation of a_0 would allow to determine in a modelindependent way the sign of the interference between the (long-distance) indirect-CPviolating component of the amplitude and the interesting (short-distance) direct-CPviolating term [17]. This result would allow to perform a very precise test of direct-CP-violation in the kaon sector.

2.2 $K \rightarrow \pi \nu \bar{\nu}$

The power-like GIM mechanism of the leading electroweak amplitude, implies a severe suppression of long-distance effects in these modes. In the CP-violating channel, $K_L \rightarrow \pi^0 \nu \bar{\nu}$, long-distance contributions are negligible well below the 1% level [16]. However, this is not the case for the charged channel, $K^+ \rightarrow \pi^+ \nu \bar{\nu}$, where the suppression of long-distance effects is partially compensated by a large CKM coefficient. The *T*-product which determines the size of non-perturbative effects in this mode is [18, 19]

$$\left(\mathcal{T}_{i}^{+}\right)_{Z}^{\mu}(q^{2}) = -i \int d^{4}x \, e^{-iq \cdot x} \, \langle \pi^{+}(p) | T \left\{ J_{Z}^{\mu}(x) \left[Q_{i}^{u}(0) - Q_{i}^{c}(0) \right] \right\} | K^{+}(k) \rangle \,, \tag{16}$$

where $J_Z^{\mu} = \bar{q}\gamma^{\mu}(2\hat{T}(1-\gamma_5)-4\hat{Q}\sin^2\theta_W)q$ is the neutral current defined by Q_q^{NC} in (5). Separating the electromagnetic component, we can write $(\mathcal{T}_i^+)_Z^{\mu} = (\mathcal{T}_i^+)_L^{\mu} - 4\sin^2\theta_W (\mathcal{T}_i^+)_{\text{em}}^{\mu}$, where

$$\left(\mathcal{T}_{i}^{+}\right)_{L}^{\mu}(q^{2}) = -i \int d^{4}x \, e^{-i \, q \cdot x} \, \langle \pi^{+}(p) | T \left\{ J_{L}^{\mu}(x) \left[Q_{i}^{u}(0) - Q_{i}^{c}(0) \right] \right\} | K^{+}(k) \rangle \,, \quad (17)$$

$$J_L^{\mu} = \sum_{q=u,c} \bar{q} \gamma^{\mu} (1-\gamma_5) q - \sum_{q=d,s} \bar{q} \gamma^{\mu} (1-\gamma_5) q .$$
 (18)

Contrary to $(\mathcal{T}_i^+)_{em}^{\mu}$, the structure of $(\mathcal{T}_i^+)_L^{\mu}$ is not protected by gauge invariance and we can decompose it as

$$\left(\mathcal{T}_{i}^{+}\right)_{\mathrm{L}}^{\mu}(q^{2}) = \frac{m_{K}^{2}}{\pi^{2}} \left[z_{i}^{+}(q^{2})(k+p)^{\mu} + \mathcal{O}(q^{\mu}) \right] , \qquad (19)$$

where the normalization is such that the $z_i^+(q^2)$ are expected to be $\mathcal{O}(1)$ [9]. The value of these form factors at $q^2 = 0$ is sufficient to control the long-distance contributions to the $K^+ \to \pi^+ \nu \bar{\nu}$ amplitude down to the 1% level of precision [9].

Charm and, more generally, long-distance contributions to the $K^+ \to \pi^+ \nu \bar{\nu}$ amplitude, are usually parametrized in terms of a scale-independent coefficient P_c [18]. According to the decomposition (19), this can be be written as

$$P_c = \frac{1}{|V_{us}|^4} \left\{ \frac{m_K^2}{M_W^2} \left[C_1 z_1^+(0) + C_2 z_2^+(0) \right] + f_+(0) C_\nu \right\} .$$
⁽²⁰⁾

The coefficient P_c expresses the relative weight of the subleading terms relative to the top-quark amplitude, which is the leading contribution and is precisely determined in perturbation theory [18]. As can be noted, the non-perturbative parameters $z_i^+(0)$ appear in (20) multiplied by a very small coefficient: $m_K^2/M_W^2/|V_{us}|^4 \approx 0.015$. Thus even a determination of these matrix elements at the 30–50% level from lattice QCD would be sufficient to reduce the overall error on the $K^+ \to \pi^+ \nu \bar{\nu}$ rate around or below the 1–2% level.

3 *T*-products at short-distances on the lattice

In this section we discuss the properties of the Euclidean Green functions necessary to extract the physical amplitudes defined in eqs. (11) and (17) in a numerical simulation. Since the ultraviolet behavior is quite different in the two cases, we discuss them separately, starting from the *T*-product which involves the electromagnetic current. In both cases, we assume that the operators of the effective weak Hamiltonian have been renormalized, namely that all their physical matrix elements are finite as the lattice spacing goes to zero $(a \rightarrow 0)$. The renormalization of the effective Hamiltonian is discussed in the next section.

The starting point to extract the physical matrix elements is the following Euclidean Green function

$$(\mathcal{T}_{i})_{X}^{\mu}(q^{2}, t_{\pi}, t_{K}) = \int d^{4}x \left\langle \Phi_{\pi}(t_{\pi}, \vec{p}) J_{X}^{\mu}(0) \left[Q_{i}^{u}(x) - Q_{i}^{c}(x) \right] \Phi_{K}^{\dagger}(t_{K}, \vec{k}) \right\rangle ,$$

$$t_{\pi} > 0 \,, \quad t_{K} < 0 \,, \qquad (21)$$

where the source (sink) for creating (annihilating) the pseudoscalar mesons at fixed space momentum are defined as

$$\Phi_i(t_i, \vec{q_i}) = \int d^3 z \, e^{-i \vec{q_i} \cdot \vec{z}} \, \Phi_i(t_i, \vec{z}) \,, \qquad (22)$$

and $\Phi_i(t_i, \vec{z})$ is a suitable local operator with the quantum numbers of the pion or kaon, respectively. Note that, in order to simplify the notation and the comparison between continuum and lattice formulae, we use the symbol of integral also to indicate sums over the lattice sites.

If not for the presence of the weak four-fermion operator, the calculation would proceed as for the standard weak and electromagnetic form factors, by studying the behavior of the Green functions at large t_{π} and $|t_K|$ [3]. This would give the form factors computed at momentum transfer $\vec{q} = \vec{k} - \vec{p}$ and with energy transfer $q_0 = E_K - E_{\pi}$. Since Q_i is summed over the whole lattice volume and hence it carries zero momentum, this general strategy remains valid also for the Green function in eq. (21). As explained in the previous section, in order to extract the relevant low energy couplings, we are interested only to study the correlation function for $q^2 < m_{\pi}^2$. In this range no rescattering of intermediate states is possible and thus we do not have problems in relating the Minkowskian T product to the Euclidean one.



Figure 1: One-loop topology which can originate power-like singularities to the Green function (21) for $x \to 0$. The dotted line denotes the generic insertion of $Q_i^{u,c}$, with possible Fierz re-arrangements.

The additional problem which arises in this case is the possibility that the Green function itself diverges because of the short distance behavior when $x \to 0$. By dimensional arguments, this divergence can at most be quadratic. At fixed lattice spacing a, this would imply potential contributions to the Green function of $\mathcal{O}(1/a^2)$. Fortunately this never happens, since the strongest divergence associated to the diagram in figure 1 is independent of the quark masses and is canceled by the GIM mechanism. However, this cancellation does not guarantee the absence of linear divergences, which are naturally present when using lattice actions which break explicitly chiral invariance.

3.1 The electromagnetic current

Even if the chirality of the fermion action is explicitly broken, we are still able to define a conserved vector current on the lattice, which we can identify with the electromagnetic one. For example, with Wilson fermions we have

$$\hat{J}_{V}^{\mu} = \frac{1}{2} \left[\bar{q}(x+\mu) U^{\mu\dagger}(x) (r+\gamma^{\mu}) q(x) - \bar{q}(x) U^{\mu}(x) (r-\gamma^{\mu}) q(x+\mu) \right] , \qquad (23)$$

where U^{μ} is the link variable. With a conserved current, gauge invariance is strong enough to protect the Green functions from the appearance of both quadratic and linear divergences. This remains true even when the Wick contractions correspond to a vacuum polarization diagram of the type in figure 1, where only one of the two currents is the lattice conserved one, and the other is a local vector current originating from the weak fourfermion operator. We have verified this argument by an explicit perturbative calculation using Wilson, Clover and twisted mass fermions. Since the results of this calculation (more precisely of the subdiagram in figure 2) could be useful for other applications, we give them below for the Wilson and Clover cases.



Figure 2: Subdiagram of figure 1 associated to the $x \to 0$ singularity.

The amplitude we have considered is

$$\Pi_{\mu\nu}(p) = \int \frac{d^4q}{(2\pi)^4} \operatorname{Tr} \left[\Gamma_{\nu}^{(1)}(q; p+q) \Delta(q) \gamma_{\mu} \Delta(q+p) \right] \\ = \frac{8}{(4\pi)^2} (\delta_{\mu\nu} p^2 - p_{\mu} p_{\nu}) \left\{ \mathcal{I}(p^2 a^2, m^2 a^2) + L \right\} , \qquad (24)$$

where $\Gamma_{\nu}^{(1)}(q; p+q)$ is the vertex derived from eq. (23) and $\Delta(q)$ the fermion propagator [20]. Both in Wilson and Clover cases we can identify a universal infrared term, given by

$$\mathcal{I}(p^2 a^2, m^2 a^2) = \int_0^1 dx \ x(1-x) \log\left[m^2 a^2 + p^2 a^2 x(1-x)\right] , \qquad (25)$$

while the finite constant L depends from the details of the regularization. In the Wilson case we find

$$L_W = -\frac{1}{6} \log \left(m^2 a^2 \right) + (1 - \delta_{\rho\sigma}) \int_{-\pi}^{\pi} \frac{d^4 q}{2 \pi^2} \frac{-\frac{1}{2} \cos q_{\rho} \sin^2 q_{\sigma} - \frac{1}{3} \cos q_{\rho} \cos 2 q_{\sigma}}{\Delta_F^2(q;m)} + \frac{r^2 \left(\frac{1}{3} \cos q_{\rho} \cos q_{\sigma} \sum_{\tau} \sin^2 \frac{q_{\tau}}{2} - \frac{1}{6} \cos q_{\sigma} \sin^2 q_{\rho} - \frac{1}{3} \cos q_{\rho} \sin^2 q_{\sigma} \right)}{\Delta_F^2(q;m)}$$

while in the Clover case

$$L_{\rm Cl} = L_{\rm W} + \frac{r^2}{2\,\pi^2} (1 - \delta_{\rho\sigma}) \int_{-\pi}^{\pi} d^4q \, \frac{\frac{1}{2}\,\sin^2 q_{\rho} - \cos q_{\sigma} \sum_{\tau} \sin^2 \left(\frac{q_{\tau}}{2}\right)}{\Delta_F^2(q)} \,. \tag{26}$$

Note that in both cases the absence of power divergences holds independently from the GIM mechanism. Using the above results we could then match the lattice calculation with the continuum one even in an effective theory where the charm quark is integrated out. The comparison of the results obtained with or without dynamical charm quarks would provide a useful insight about the validity of the standard effective theory obtained by renormalizing \mathcal{H}_{eff} below the charm mass. On the other hand, when the calculation is performed with a dynamical charm, the logarithmic divergence (and even the finite

coefficient) in eq. (24) is cancelled by the GIM mechanism. For this reason no matching lattice to continuum is needed in this case.

Beside the possible singularities for $x \to 0$, further divergences may arise from contact terms of Q_i with the external sources, namely for $x \to x_{\pi}$ or $x \to x_K$. However, it is easy to show that these contact terms do not contribute to the physical amplitudes. Let us consider the Minkowski *T*-product

$$(\mathcal{T}_{i}^{\mu})_{X}(q^{2},t_{\pi},t_{K}) = -i \int d^{4}x \left\langle 0|T\left\{\Phi_{\pi}(t_{\pi},\vec{p})J_{X}^{\mu}(0)\left[Q_{i}^{u}(x)-Q_{i}^{c}(x)\right]\Phi_{K}(t_{K},\vec{k})\right\}|0\rangle,$$
(27)

corresponding to the Euclidean Green function of eq. (21). The contact terms are proportional to the following pole terms: $(p^2 - m_{\pi}^2)^{-1}(k^2 - m_{\pi}^2)^{-1}$ or $(p^2 - m_K^2)^{-1}(k^2 - m_K^2)^{-1}$, while the on-shell amplitudes are obtained form the coefficient of $(p^2 - m_{\pi}^2)^{-1}(k^2 - m_K^2)^{-1}$. As we shall discuss in more detail in the next section, these different pole structures in the Minkowski space correspond to a different $t_{\pi} \to \infty$ and $t_K \to -\infty$ behavior in the Euclidean case. As a result, we can eliminate the contact terms by an appropriate spectral analysis of the Green function computed in the numerical simulation.

3.2 The axial current

With the axial current appearing in the *T*-product (17), which is relevant for $K \to \pi \nu \bar{\nu}$ decays, we cannot invoke gauge invariance: it remains true that the quadratic divergence is canceled by GIM, but we must face the problem of the linear one. With power divergences, any subtraction procedure, though non-perturbative, would produce an irreducible (and thus unacceptable) ambiguity in the final result. This implies that the linear divergence can only be an artifact of the regularization procedure. This divergence is indeed absent in regularizations which preserve chirality.

With Wilson fermions the explicit breaking of chiral invariance leads to the appearence of such linear divergence. Since this problem is associated only to the contact term of the integrand (21) for $x \to 0$, we can in principle obtain a finite subtracted *T*-product, with the correct chiral behaviour of the Green function, by an integration which avoids the region close to $x = 0.^3$ Otherwise, one could introduce an appropriate set of counterterms and fix their values by imposing an appropriate set of Ward identities, to recover the correct chiral behaviour. However, both these procedures are technically very complicated to be implemented.

A much simpler and technically feasible solution is obtained by means of maximally twisted mass terms [11]. In this case, the additional symmetries of the action imply that the amplitude we are interested in is even in the Wilson parameter (r). This, in turn, implies the absence of the linear divergence which can only be odd in r, being associated to the breaking of chirality. We have verified this statement by an explicit perturbative calculation at the one loop level. As expected, the structure of the divergent terms is the same as in the continuum and the result is free from ambiguities. The discussion of the

³We thank Massimo Testa for discussions on this point.

axial current can be repeated for a "non-conserved" vector current, such as the lattice local electromagnetic current, or the vector component of the weak left-handed current in eq. (17).

At this point we wish to comment about the possibility to determine the physical $K \to \pi\pi$ amplitudes by using information about the following *T*-product

$$\int d^4x \, e^{-i p_1 \cdot x} \left\langle \pi(p_2) | T[\mathcal{H}_{eff}(0) A^{\mu}(x)] | K(q_1) \right\rangle, \qquad (28)$$

where the axial current A^{μ} has the quantum numbers of the pion, but the kinematical configuration does not correspond to an on-shell pion field [22]. The Wick contractions for this *T*-product are similar to those considered for $K \to \pi \nu \bar{\nu}$. In particular, the quadratic divergence generated when $x \to 0$ is present also in this case. However, the situation is worse than the case discussed in this work, since there is no GIM mechanism to cancel the leading singularity. Of course we can define a renormalized *T*^{*}-product, but this would entail a finite ambiguity. The practical problems which need to be faced in order to avoid this ambiguity make this calculation very difficult (if not practically impossible) with Wilson-type fermions. For this reason, we do not believe that a lattice study of this *T*-product can provide a useful tool to simplify the problem of determining $K \to \pi\pi$ amplitudes.

$4 \quad \mathcal{H}_{eff} \text{ ambiguities} \\$

In this section we address the problems arising by the renormalization of the lattice operators of the effective weak Hamiltonian. We first note that only the parity-even or parity-odd terms of the operators contribute to the vector or axial-vector cases, respectively. This observation is relevant since parity-even and parity-odd parts of the operators renormalize in a different way under regularizations which break chiral symmetry. On general grounds, whether chirality is broken or not, the mixing with operators of dimension five or six, in the presence of the GIM mechanism, does not introduce any ambiguity and the corresponding mixing coefficients can be computed in lattice perturbation theory. The problem arises from the mixing of the standard dimension-six operators with the scalar and pseudoscalar densities, which we now consider separately for the two cases.

Schematically, we can write the renormalized operator as

$$\hat{Q}^{\pm} = Z^{\pm}(\mu a) \left[Q^{\pm} + C_P \left(m_c - m_u \right) (m_s - m_d) \, \bar{s} \gamma_5 d + C_S \left(m_c - m_u \right) \, \bar{s} d \right] \,, \quad (29)$$

where $Q^{\pm} = (Q_1 \pm Q_2)/2 + \ldots$ represents the ensemble of all the dimension six and five operators with mixing coefficients computed in perturbation theory. By dimensional arguments, it follows that the coefficients C_S and C_P are power divergent in the limit $a \rightarrow 0$:

$$C_P \sim \frac{1}{a} , \qquad C_S \sim \frac{1}{a^2} .$$
 (30)

Using suitable Ward identities (subtraction conditions), we can cancel the divergent parts of C_P and C_S ; however, this leaves an ambiguity in their finite values out of the chiral limit [12, 21]. For physical $K \to \pi\pi$ amplitudes this ambiguity turns out to be irrelevant: the pseudoscalar density is proportional to the four divergence of the axial current and its matrix element vanishes for the on-shell $K \to \pi\pi$ transition [12]. We stress that this conclusion does not hold for the $K \to \pi$ case: the off-shell matrix element $\langle \pi | \bar{s} d | K \rangle$ is different from zero thus, in general, the $\langle \pi | \hat{Q} | K \rangle$ matrix element does suffer from this ambiguity.

Since we are interested in physical amplitudes, we must be able to demonstrate that also in the case of radiative decays the matrix elements of scalar and pseudoscalar densities do not contribute to the on-shell amplitudes. This can be done by means of suitable Ward identities and the spectral analysis of the relevant Euclidean Green functions.

In the vector case we can use the following Ward identity

$$\int d^4x \left\{ \langle \Phi_{\pi}(x_{\pi}) \left[\nabla_{\mu} \hat{V}^{sd}_{\mu}(x) + (m_d - m_s) \bar{s} d(x) \right] \hat{J}^{\nu}_{V}(y) \Phi^{\dagger}_{K}(x_K) \rangle \right\} = = - \langle \Phi_{K}(x_{\pi}) \hat{J}^{\nu}_{V}(y) \Phi^{\dagger}_{K}(x_K) \rangle + \langle \Phi_{\pi}(x_{\pi}) \hat{J}^{\nu}_{V}(y) \Phi^{\dagger}_{\pi}(x_K) \rangle , \qquad (31)$$

where the term between square bracket is the rotation of the lattice action,

$$\hat{V}^{sd}_{\mu} = -\frac{1}{2} [\bar{s}(x)U_{\mu}(x)(r-\gamma_{\mu})d(x+\mu) - \bar{s}(x+\mu)U^{\dagger}_{\mu}(x)(r+\gamma_{\mu})d(x)], \qquad (32)$$

and the last two terms in (31) correspond to the rotation of the pion sink and the kaon source, respectively. The term with the four divergence of $\hat{V}^{sd}_{\mu}(x)$, integrated over all space, vanishes. Thus on the left-hand side we are left with the term we are looking for, up to overall factors, namely the contribution of the scalar density to the Euclidean Green function (21), which enters when the bare weak operators are replaced with the renormalized ones. We need to show that this term does not contribute to the physical amplitude.

In the Minkowski space, the physical amplitude is identified by the coefficient of the physical pole for $p^2 \to m_{\pi}^2$ and $k^2 \to m_K^2$. In the Euclidean space, this corresponds to a well-defined dependence on t_K and t_{π} (for $t_{\pi} \to \infty$ and $t_K \to -\infty$), namely

$$\frac{1}{(p^2 - m_\pi^2)(k^2 - m_K^2)} \quad \leftrightarrow \quad e^{-E_K |t_K|} \times e^{-E_\pi t_\pi} .$$
(33)

The Ward identity (31) tell us that the contribution of the scalar density give rise to a different pole structure:

$$\frac{1}{(p^2 - m_\pi^2)(k^2 - m_\pi^2)} \qquad \leftrightarrow \qquad e^{-E_\pi |t_K|} \times e^{-E_\pi t_\pi}$$

$$\frac{1}{(p^2 - m_K^2)(k^2 - m_K^2)} \qquad \leftrightarrow \qquad e^{-E_K |t_K|} \times e^{-E_K t_\pi}$$
(34)

Thus the scalar density contribution can simply be eliminated by a study of the time dependence of the appropriate Green function. Incidentally, this procedure eliminates also the divergent contact terms mentioned at the end of the section 3.1.

In the axial case, we have a similar situation, up to terms which vanish (linearly or quadratically in the lattice spacing) and inessential numerical factors. In particular, we can use the following Ward identity

$$\int d^4x \left\{ \langle \Phi_{\pi}(x_{\pi}) \left[-\nabla_{\mu} Z_A \hat{A}^{sd}_{\mu}(x) + (m_d + m_s) \bar{s} \gamma_5 d(x) + \mathcal{O}(a) \right] J^{\nu}_A(y) \Phi^{\dagger}_K(x_K) \rangle \right\} = -\langle \Sigma_K(x_{\pi}) J^{\nu}_A(y) \Phi^{\dagger}_K(x_K) \rangle + \langle \Phi_{\pi}(x_{\pi}) J^{\nu}_A(y) \Sigma^{\dagger}_{\pi}(x_K) \rangle ,$$
(35)

where again the term between square bracket is the rotation of the lattice action (for the explicit expressions of Z_A and the weak renormalized axial current see [23]) and Σ_i is a scalar particle source. This immediately shows that also the pseudoscalar density give rise to a time dependence different from the one in eq. (33) and thus does not contribute to the on-shell amplitude.

5 Conclusions

The potential of rare K decays in performing precise tests of the SM and setting stringent bounds on physics beyond the SM depends, to a large extent, from our ability compute their amplitudes within the SM. In this paper we have shown that for a class of very interesting processes, such as $K^+ \to \pi^+ \nu \bar{\nu}$ and $K \to \pi \ell^+ \ell^-$, the theoretical error associated to non-perturbative effects could be reduced by means of lattice calculations. In particular, the numerical study of the Euclidean Green functions in eq. (21), combined with CHPT, should allow to reach an unprecedented level of precision for these rare decays.

The main problem which needs to be addressed before starting a lattice calculation of these Euclidean Green functions is the absence of power divergences in the extraction of the physical amplitudes. These may originate from contact terms between the weak four-fermion operators and the external fields (π , K and the lepton current), or from the mixing of the four fermion operators with operators of lower dimensionality. In this paper we have shown that both these problems can be solved.

As demonstrated in section 4, the spectral analysis necessary to extract the physical amplitudes eliminates both the power divergences due to the operator mixing and the contact terms with the external π and K fields. The only remaining issue is then the ultraviolet behavior associated to the contact terms between the weak operators and the lepton current. This point is different for weak and electromagnetic currents.

In the electromagnetic case, relevant for $K \to \pi \ell^+ \ell^-$ decays, gauge invariance prevents the appearance of power divergences for all the popular Wilson-type actions. The cancellation of power divergences is also independent of the GIM mechanism. We can thus match the lattice calculation with the continuum one also in an effective theory where the charm quark is integrated out. The perturbative expressions necessary for this matching at the one-loop level have been presented both for Wilson and Clover fermions. The situation is slightly more complicated for the weak (axial or vector) current, relevant for $K^+ \to \pi^+ \nu \bar{\nu}$ decays, where we cannot invoke anymore gauge invariance. One can cancel power divergences also in this case with Wilson-type fermions, but only using maximally twisted mass terms and taking advantage of the GIM mechanism.

In summary, our analysis shows that the numerical study of the Green functions relevant for $K \to \pi \ell^+ \ell^-$ decays can be performed with any Wilson-type action, independently of the GIM mechanism. On the other hand, the study of $K^+ \to \pi^+ \nu \bar{\nu}$ decays on the lattice requires a more sophisticated action: with Wilson-type fermions the only possibility is to use maximally twisted mass terms. We believe that these results opens a new field of interesting physical applications to the lattice community.

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